THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1884 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI

1924

PUBLISHED BY THE ASSOCIATION

LANCASTER, PA., AND ANN ARBOR, MICH.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924
NUMBER 1, JANUARY

PUBLISHED BY THE ASSOCIATION
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others

LANCASTER PRESS, INC.
LANCASTER, PA.

AMERICAN MATHEMATICAL MONTHLY

MATHEMATICIANS AND MUSIC.

By R. C. ARCHIBALD, Brown University.

Presidential Address[1] *delivered*[2] *before the Mathematical Association of America, September 6, 1923.*

I.

"Mathematics and Music, the most sharply contrasted fields of intellectual activity which one can discover, and yet bound together, supporting one another as if they would demonstrate the hidden bond which draws together all activities of our mind, and which also in the revelations of artistic genius leads us to surmise unconscious expressions of a mysteriously active intelligence." In such wise wrote one[3] supremely competent to represent both musicians and mathematicians, the author of that monumental work, *On the Sensations of Tone as a physiological basis for the Theory of Music.*

"Bound together?" Yes! in regularity of vibrations, in relations of tones to one another in melodies and harmonies, in tone-color, in rhythm, in the many varieties of musical form, in Fourier's series arising in discussion of vibrating strings and development of arbitrary functions, and in modern discussions of acoustics.

This suggests that the famous affirmation of Leibniz, "Music is a hidden exercise in arithmetic, of a mind unconscious of dealing with numbers,"[4] must

---

[1] To the address as delivered a number of footnotes, mainly with a few references to the vast literature of the subject, have been added. A fundamental work in this connection is H. L. F. Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, and the best edition is the second English edition translated with many additions from the fourth (last) German edition by A. J. Ellis, London, 1885; the third edition was reprinted from the second in 1895, and the fourth from the third in 1912. While some mathematical discussion occurs in this work, the standard treatise on the mathematical theory is Rayleigh, *Theory of Sound*, 2 vols., second ed., London, 1894. Another work of high order is H. v. Helmholtz, *Vorlesungen über die mathematischen Akustik*, 1898, vol. 3 of *Vorlesungen über theoretische Physik*, Leipsic. H. Lamb, *The Dynamical Theory of Sound*, London, 1910, was intended as a stepping stone to the writings of Helmholtz and Raleigh. Between 1898 and 1915, 38 papers by various authors appeared at Leipsic in 8 Hefte of *Beiträge zur Akustik und Musikwissenschaft* herausgegeben von C. Stumpf. For the most part, they are reprints of articles in *Zeitschrift für Psychologie, Zeitschrift für Psychologie und Physiologie der Sinnesorgane*, and *6. Kongress der Gesellschaft für experimentelle Psychologie*. Another very valuable general work, discussing the writings of mathematicians on musical matters, is F. J. Fétis, *Biographie Universelle des Musiciens et Bibliographie générale de la Musique*, 8 vols., second ed., Paris, 1 (1873), 2 (1867), 3–4 (1869), 5–8 (1870). R. Eitner's *Biographisch-bibliographisches Quellen-Lexikon der Musiker der christlichen Zeitrechnung bis zur Mitte des 19. Jahrhunderts*, 10 vols., Leipsic, 1900–1904, is also sometimes useful.

[2] At a joint session of the Mathematical Association of America and of the American Mathematical Society, Vassar College, Poughkeepsie, N. Y.

[3] H. v. Helmholtz, *Vorträge und Reden*, Braunschweig, vol. 1, 1884, p. 82. See also Helmholtz, *Popular Lectures on Scientific Subjects*, London, 1873, p. 62.

[4] "Musica est exercitium arithmeticæ occultum nescientis se numerare animi," which occurs in a letter dated April 17, 1712, and addressed to Goldbach. It is letter 154 in Leibniz, *Epistolæ ad diversos*, vol. 1, Leipsic, 1734, p. 241. The quotation is in the section dealing with the question "Vnde oritur ex musica voluptas?" This is preceded and followed by two other sections on

be far from true if taken literally. But, in a very general conception of art and science, its verity may well be granted; for, in creating as in listening to music, there is no realization possible except by immediate and spontaneous appreciation of a multitude of relations of sound.

Other modes of expression and points of view were suggested by that great enthusiast to whom America owed much, him who called himself[1] "the Mathematical Adam" because of the many mathematical terms he invented; for example, mathematic—to denote the science itself in the same way as we speak of logic, rhetoric or music, while the ordinary form is reserved for the applications of the science. He referred to the cultures of mathematics and music "not merely as having arithmetic for their common parent but as similar in their habits and affections."[2] "May not Music be described," he wrote, "as the Mathematic of Sense, Mathematic as the Music of reason?[3] the soul of each the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music,—Music the dream, Mathematic the working life,—each to receive its consummation from the other when the human intelligence, elevated to the perfect type, shall shine forth glorified in some future Mozart-Dirichlet, or Beethoven-Gauss—a union already not indistinctly foreshadowed in the genius and labors of a Helmholtz"![4]

But such intimacies in these cultures are not discoveries and imaginings of a later day. For two thousand years music was regarded as a mathematical science. Even in more recent times the mathematical dictionaries of Ozanam,[5] Savérien,[6] and Hutton,[7] contain long articles on music and considerable space is devoted to the subject in Montucla's revised history,[8]—which brings us to

"Quibus musicis proportionibus homines delectantur?" and "Quando rationes surdæ in musica commode locum inueniunt?"

T. W. Preyer corrupted Leibniz's sentence into "Arithmetica est exercitium musicum occultum nescientis se sonos comparare animi" (compare M. Lecat, *Pensées sur la Science, la Guerre, et sur des Sujets très variés*, Brussels, 1919, p. 438). Preyer's thought in this connection will be apparent on turning to his monograph, "Ueber den Ursprung des Zahlbegriffes aus dem Tonsinn und über das Wesen der Primzahlen," pages 1–36 of *Beiträge zur Psychologie und Physiologie der Sinnesorgane, Hermann von Helmholtz als Festgruss zu seinem siebzigsten Geburtstag.* Gesammelt und herausgegeben von A. König, Hamburg and Leipsic, 1891.

[1] J. Sylvester, in a footnote to "Note on a Proposed Addition to the Vocabulary of Ordinary Arithmetic," *Nature*, vol. 37, p. 152, 1887.

[2] J. Sylvester, British Assoc. for the Adv. of Science, *Report*, 1869, page 7 of Notices and Abstracts.

[3] Compare "Die Mathematik ist die Musik des Verstandes, die Musik die Mathematik des Gefühls" as employed by Josef Petzval, *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 12, 1903, p. 327.

[4] This passage occurs in a footnote in the midst of Sylvester's memoir "Algebraical researches containing a disquisition on Newton's rule for the discovery of imaginary roots . . .," *Philosophical Transactions* for 1864, vol. 154, 1865, p. 613.

[5] J. Ozanam, *Dictionaire Mathematique*, Amsterdam, or Paris, 1691.

[6] A. Savérien, *Dictionnaire Universel de Mathematique et de Physique*, Paris, 1753, vol. 2.

[7] C. Hutton, *A Philosophical and Mathematical Dictionary*, London, 1795, vol. 2; new edition, 1815.

[8] J. F. Montucla and J. de La Lande, *Histoire des Mathématiques*, Paris, vols. 1 and 4, 1799, and 1802. Compare D. E. Smith, "The threatened loss of the second edition of Montucla's History of Mathematics" in this MONTHLY, *1921*, 207–208.

the threshold of the nineteenth century. It is, therefore, not surprising that many mathematicians wrote on musical matters. I shall presently consider these at some length. But certain other facts may first be reviewed.

The manner in which music, as an art, has played a part in the lives of some mathematicians is recorded in widely scattered sources. A few instances are as follows.

Maupertuis was a player on the flageolet and German guitar and won applause in the concert room for performance on the former.[1] At different times William Herschel served as violinist, hautboyist, organist, conductor, and composer (one of his symphonies was published) before he gave himself up wholly to astronomy.[2] Jacobi had a thorough appreciation of music.[3] Grassmann was a piano player and composer, some of his three-part arrangements of Pomeranian folk-songs having been published; he was also a good singer and conducted a men's chorus for many years.[4] János Bolyai's gifts as a violinist were exceptional and he is known to have been victorious in 13 consecutive duels where, in accordance with his stipulation, he had been allowed to play a violin solo after every two duels.[5] As a flute player De Morgan excelled.[6] The late G. B. Mathews knew music as thoroughly as most professional musicians; his copies of Gauss and Bach were placed together on the same shelf.[7] It was with good music that Poincaré best liked to occupy his periods of leisure. The famous concerts of chamber music held at the home of Emile Lemoine during half a century exerted a great influence on the musical life of Paris.[8] And in

[1] *Basler Jahrbuch*, 1910, p. 46 in "Maupertuis" (pp. 29–53) by F. Burckhardt; see also "Maupertuis' Lebensende" by the same author in *Basler Jahrbuch*, 1886, pp. 153–159. These very interesting articles contain new material concerning the closing days of Maupertuis at the home of his friend Johann Bernoulli the second. Compare D. E. Smith, "Maupertuis and Frederick the Great" in this MONTHLY, *1921*, 430–432. The first memoir presented to the French Academy by Maupertuis was "Sur la forme des instruments de musique," *Mémoires de l'Academie Royale des Sciences, 1724*, Paris, 1726, pp. 215–226 + 1 plate; *Histoire*, pp. 90–92. This memoir is not contained in *Oeuvres de Mr. Maupertuis* published at Dresden in 1752 and at Lyons in 1756.

[2] Concerning the musical activities of Herschel, see especially *The Scientific Papers of Sir William Herschel*, vol. 1, London, 1912, pp. xiv–xxii; F. J. Fétis, *Biographie*, etc., vol. 4, *loc. cit.*, and A. Noyes, *The Torch-Bearers, Watchers of the Sky*, New York, 1922, p. 231 f. Noyes perpetuates the old error about Herschel "deserting from the army." Herschel was born in 1738 and died in 1822.

[3] S. Hensel, *Die Familie Mendelssohn, 1729–1847*, second ed., Berlin, 1880, vol. 2, pp. 364–365; English edition, London, 1881, vol. 2, p. 324.

[4] Compare F. Engel, *Grassmann's Leben* in *Hermann Grassmann's Gesammelte Mathematische und Physikalische Werke*, vol. 3, part 2, Leipsic, 1911, pp. 250–253, 371–372. Grassmann was born in 1809 and died in 1877.

[5] F. Schmidt, "Lebensgeschichte des ungarischen Mathematikers Johann Bolyai," *Abhandlungen zur Geschichte der Mathematik*, Heft 8, 1898, p. 141. Bolyai was born in 1802 and died in 1860; sketches by G. B. Halsted appeared in this MONTHLY, *1896*, 1–5, and *1898*, 35–38.

[6] Sophia E. De Morgan, *Memoir of Augustus De Morgan*, London, 1882, p. 16. See also A. M. Stirling, *William De Morgan and his Wife*, London, 1922, p. 61. De Morgan was born in 1806 and died in 1871.

[7] *Proceedings of the Royal Society*, London, vol. 101 A, 1922, p. xiv; also in *Proceedings of the London Mathematical Society*, 2 series, vol. 21, 1923, p. l. Mathews was born in 1861 and died in 1922.

[8] L. Augé de Lassus, *La Trompette. Un demi-siècle de Musique de Chambre*, Paris, 1911. See also D. E. Smith, "Emile Michel Hyacinthe Lemoine," in this MONTHLY, *1896*, 29–33. Lemoine was born in 1840 and died in 1912.

America we have only to recall colleagues in the mathematics departments of the Universities of California, Chicago and Iowa, and of Cornell University, who are, to use Shakespeare's phrase, "cunning in music and mathematics."

While Friedrich T. Schubert, the Russian astronomer and mathematician, played the piano, flute, and violin in an equally masterly fashion,[1] his great-grand-daughter Sophie Kovalevsky was devoid of musical talent; but she is said to have expressed her willingness to part with her talent for mathematics could she thereby become able to sing.[2] Abel had no interest in music as such, but only for the mathematical problems it suggested. His close attention to a performer at a piano was once explained by the fact that he sought to find a relation between the number of times that each key was struck by each finger of the player.[3] Lagrange welcomed music at a reception because he could by the fourth measure become oblivious to his surroundings and thus work out mathematical problems; for him the most beautiful musical work was that to which he owed the happiest mathematical inspirations.[4] Dirichlet seemed to be sensible to the charms of music in a similar manner.[5]

Such are a few instances, which could be considerably multiplied, of the relation of mathematicians to the art of music

> "that gentlier on the spirit lies
> Than tir'd eyelids upon tir'd eyes."

They suggest the accuracy of at least a part of the following observations of Möbius in his book on mathematical abilities:[6] "Musical mathematicians are frequent . . . but there are wholly unmusical mathematicians and many more musicians without any mathematical capability." That there are musicians with some mathematical ability will be granted when we recall, not only that Henderson, the prominent New York music critic and the author of many works on musical topics, has written a little book on navigation,[7] but also that the late Sergei Tanaïeff, pupil of Rubenstein and Tchaikovsky, successor of the latter as professor of composition and instrumentation at the Moscow Conservatory of Music, and one of the most prominent of modern Russian composers, found algebraic symbolism and formulæ of fundamental importance in his lectures and work on counterpoint.[8]

---

[1] *Allgemeine deutsche Biographie*. Schubert was born in 1758 and died in 1825.

[2] Sophie von Adelung, *Deutsche Rundschau*, vol. 89, 1896, p. 405. Sophie Kovalevsky was born in 1850 and died in 1891.

[3] *Niels Henrik Abel Memorial publié à l'occasion du centenaire de sa naissance*, Christiania, 1902, pp. 57–58. Abel was born in 1802 and died in 1829.

[4] *Œuvres de Lagrange*, Paris, vol. 1, 1867, p. xlviii. Lagrange was born in 1736 and died in 1813.

[5] *G. Lejeune Dirichlets Werke*, vol. 2, Berlin, 1897, p. 343. Dirichlet was born in 1805 and died in 1859.

[6] P. J. Möbius, *Ueber die Anlage zur Mathematik*, zweite vermehrte und veränderte Auflage, Leipsic, 1907, p. 124.

[7] W. J. Henderson, *The Elements of Navigation*, New York, 1895; new and enlarged edition, 1917. He was formerly lieutenant in the first battalion, naval militia of New York.

[8] A. E. Hull, "Music and Mathematics," *The Monthly Musical Record*, London, May 1, 1916, vol. 46, p. 133 f. Tanaïeff was born in 1856 and died in 1915. See also "Russian Music, A Taneyef Souvenir," *The Monthly Musical Record*, vol. 46, pp. 313–314; and *Life and Letters*

A question which has interested more than one group of inquirers is: Can one establish any relationship between mathematical and musical abilities? Within the past year two Jena professors, Haecker and Ziehen, published the results of an elaborate inquiry as to the inheritance of musical abilities in *musical* families.[1] As a by-product of the inquiry they arrived at the result that in only about 2 per cent. of the cases considered was there any appreciable correlation between talent for music and talent for mathematics; they found also that the percentage of males lacking in talent for music but showing a talent for mathematics was comparatively high, about 13 per cent. At the Eugenics Record Office of Cold Spring Harbor, Long Island, there has been collected a considerable body of data upon which a study of the correlation of mathematical and musical abilities could be based. It will be interesting to see if the conclusions of Haecker and Ziehen are here checked, and also if some results are found as to the extent to which musical abilities are present in a group of mathematicians.[2]

II

Turning now to the *theory* of music, it is natural to inquire: What are the relations of mathematics to music? What have mathematicians written about music or its theory? Even on the part of one fully informed and competent, to answer these questions with any degree of completeness would require not one hour only, but many hours. I shall therefore limit myself to brief statements, with references to only a score or so of the better known mathematicians.

In any consideration of the history of music and its relation to mathematics it is important to have in mind the general character of music of different periods. With Helmholtz [3] these may be stated as follows:

(*a*) The Homophonic or Unison Music of the ancients, including the music of the Christian era up to the eleventh century, to which also belongs the existing music of Oriental and Asiatic nations.

(*b*) Polyphonic Music of the middle ages, with several parts, but without regard to independent musical significance of the harmonies, extending from the tenth to the seventeenth century, when it passes into

(*c*) Harmonic or Modern Music, characterized by the independent significance attributed to the harmonies as such.

of Peter Ilich Tchaikovsky, by M. Tchaikovsky, edited from the Russian by R. Newmarsh, London, 1916, with many references.

[1] "Ueber die Erblichkeit der musikalischen Begabung, nebst allgemeinen methodologischen Bemerkungen uber die psychische Vererbung," *Zeitschrift für Psychologie*, 1922, vol. 88, pp. 265–307; vol. 89, pp. 273–312; vol. 90, pp. 204–306; see particularly pp. 290–298. This was reprinted in book form with the title, *Zur Vererbung und Entwicklung der musikalischen Begabung*, Leipsic, 1922, 3 + 186 pages. Haecker is also the author of *Der Gesang der Vögel, seine anatomischen und biologischen Grundlagen*, Jena, 1900, 6 + 102 pp.

[2] The correlations between arithmetic and singing, space intuition and singing and other subjects studied by a group of 42 boys in the sixth year of a grammar school at Kiel, Germany have been set forth in M. Lobsien, "Korrelationem zwischen den unterrichtlichen Leistungen einer Schülergruppe," *Zeitschrift für experimentelle Pädagogik*, Leipsic, vol. 11, 1910, pp. 146–164

[3] *Loc. cit.*, p. 236.

Our first consideration is therefore to be given to the homophonic music of the *Greeks:* for in music as in mathematics the period of real development began in the sixth century B.C. with Pythagoras. Before his time tones an octave or a fifth apart, above and below, were regarded as consonant and as the basis of ordinary needs in declamation. If the *c* be taken as a point of departure, its fifth is *g*, and its fifth below is *f*. If this last note *f* be raised an octave so as to bring it nearer to the other notes, and if the octave of *c* be also added, the following four notes are obtained: *c, f, g, c*. Tradition affirms that these four notes constituted the range of the lyre of Orpheus. As Blaserna remarks,[1] "Musically speaking it is certainly poor, but the observation is interesting that it contains the most important musical intervals of declamation. In fact, when an interrogation is made, the voice rises a fourth. To emphasize a word, it rises another tone and goes to the fifth. In ending a story, it falls a fifth, etc. Thus it may be understood that Orpheus' lyre, notwithstanding its poverty, was well suited to a sort of musical declamation."

The notable contribution of Pythagoras was his enunciation of the law governing such sounds which are found in all the musical scales known. "He proclaimed the remarkable fact, of which the proof existed in his famous experiments with stretched strings of different lengths, that the ratios of the intervals perceived as consonant could all be expressed by the numbers 1, 2, 3, 4. His method of demonstration was afterward improved and rendered more exact by the invention of the monochord, and his law may now be stated as follows:[2]

"If a string be divided into two parts by a bridge, in such a manner as to give two consonant sounds when struck, the lengths of those parts will be in the ratio of two of the first four positive integers. If the bridge be so placed that two thirds of the string lie to the right and one third to the left, so that the two lengths are in the ratio of $1:2$, they produce the interval of the octave, the greater length being given to the deeper note. If the bridge be so placed that three fifths of the string lie to the right and two fifths to the left, the ratio of the two lengths is $2:3$ and the interval produced is the fifth. If the bridge be again shifted to a position which gives four sevenths on the right and three sevenths on the left, the ratio is $3:4$ and the interval is the fourth." Thus corresponding to the successively higher notes *c, f, g,* and *c* we have the numbers 1, 3/4, 2/3, and 1/2 for the relative lengths of the strings corresponding to the different notes.

The fourth and fifth gave the means of fixing a much smaller interval, called a tone, corresponding to which is the number $8/9 = 2/3 \div 3/4$. Starting with a fundamental *c* and inserting two tones between it and its fourth, two more between its fifth and its octave, the corresponding numbers for the succession *c d e f g a b c* would be 1, 8/9, 64/81, 3/4, 2/3, 16/27, 128/243, 1/2. The numbers corresponding to successive *pairs* of notes would be 8/9, 8/9, 243/256, 8/9, 8/9,

[1] P. Blaserna, *The Theory of Sound in its relation to Music*, London, 1875, p. 117. Compare Helmholtz, *loc. cit.*, p. 255, and D. B. Monro, *The Modes of Ancient Greek Music*, Oxford, 1894, p. 113 f.

[2] H. E. Wooldridge, *The Oxford History of Music*, vol. 1, Oxford, 1901, p. 11.

8/9, 243/256, the 243/256 being that number by which it is necessary to multiply into $8/9 \times 8/9$ in order to give 3/4.

Pythagoras looked upon the diatonic scale to which we have just referred in quite a different manner, namely, as derived from a succession of fifths. Thus starting from a prime *c* we have

$$c\ g\ d\ a\ e\ b.$$

Reducing *d* an octave, *a* an octave, *e* two octaves, and *b* two octaves, we have the series

$$c\ d\ e\quad g\ a\ b.$$

To obtain the *f* missing in this series and to fill up the wide interval between *e* and *g* it appears that *c* as a fifth below the prime was raised an octave. It may be readily verified that we are thus led to the same results as before; for example, *d*, the second fifth above the prime, is given by $2/3 \times 2/3$; to the *d* an octave lower corresponds $2 \times 2/3 \times 2/3 = 8/9$.

Pythagoras proposed to find in the order of the universe, where whole numbers and simple ratios prevail, an answer to the question: Why is consonance (the beautiful in sound) determined by the ratio of small whole numbers? The correct numerical ratios existing between the seven tones of the diatonic scale corresponded, according to Pythagoras, to the sun, moon and five planets, and the distances of the celestial bodies from the central fire, etc.

"It was the elaboration of these figments of philosophy, and because the fifth as the central tone of the octave corresponded to the astronomical order in which the Samian sage ranged the sun and planets, that he laid such a deep stress upon the *c* scale obtained from fifths only." [1]

Pythagoras limited himself to the insertion of seven notes within the octave. But from the primal scale he evolved six others. This was done not by setting up a new succession of fifths on the several notes of the primal scale but by making the second note of his first scale the prime of his second and so for each of five remaining notes. In this way, for example, we get the scale *d, e, f, g, a, b, c, d* with the corresponding numbers 1, 8/9, 27/32, 3/4, 2/3, 16/27, 9/16, 1/2. To the succession *b, c, d, e, f, g, a, b* corresponds 1, 243/256, 27/32, 3/4, 729/1024, 81/128, 9/16, 1/2.

It is not apparent in this latter scale that the method of Pythagoras can be said to illustrate the principle that the beautiful in sound must depend upon a succession of notes related to each other and a prime, by the simplest possible ratios.

The most noted of all the musical theorists of antiquity was Aristoxenus of Tarantum, a contemporary and pupil of Aristotle. To him as author have been assigned no less than 453 works but of these none now remain except the Harmonics,[2] portions of a treatise on rhythm, and some fragments recently found in

[1] H. Wylde, *The Evolution of the Beautiful in Sound*, London, 1888. Compare "The music of the spheres," *Harper's Magazine*, vol. 63, 1881, pp. 286–288. But best of all in this connection see chapter 12 of T. L. Heath, *Aristarchus of Samos, the Ancient Copernicus*, Oxford, 1913, with references to still more elaborate discussions.

[2] ΑΡΙΣΤΟΞΕΝΟΥ ΑΡΜΟΝΙΚΑ ΣΤΟΙΧΕΙΑ. *The Harmonics of Aristoxenus, Edited with*

Egypt. According to Macran (page 87), his great service was rendered "firstly, in the accurate determination of the scope of musical science, lest on the one hand it should degenerate into empiricism, or on the other hand lose itself in mathematical physics; and secondly, in the application to all questions and problems of music of a deeper and truer conception of the ultimate nature of music itself."

Of two treatises on music attributed to Euclid,[1] only the Theory of Intervals or Section of the Canon, as it is sometimes called, may be regarded as genuine.[2] It is based on the Pythagorean theory of music, "is mathematical, and clearly and well written, the style and form of the propositions agreeing well with what we find in the Elements."

The way in which the work starts out seems somewhat remarkable when we remember that it was written about three hundred years before Christ. It commences as follows:[3] "If all things were at rest, and nothing moved, there must be perfect silence in the world; in such a state of absolute quiescence nothing could be heard. For motion and percussion must precede sound; so that as the immediate cause of sound is some percussion, the immediate cause of all percussion must be motion. And whereas of vibratory impulses or motions causing a percussion on the ear, some there be returning with a greater quickness which consequently have a greater number of vibrations in a given time, whilst others are repeated slowly and of consequence are fewer in an assigned time, the quick returns and greater number of such impulses produce the higher sounds, whilst the slower which have fewer courses and returns, produce the lower. Hence it follows, that if sounds are too high they may be rendered lower by a diminution of the number of such impulses in a given time, and that sounds which are too low, by adding to the number of their impulses in a given time, may be made as high as we choose. The notes of music may be said then to

translation, notes, introduction, and index of words by H. S. Macran, Oxford, 1902. See also L. Laloy, *Aristoxène de Tarente et la musique de l'Antiquité*, Paris, 1904; C. F. A. Williams, *The Aristoxenian Theory of Musical Rhythm*, Cambridge, 1911; and F. A. Wright, *The Arts in Greece*, London, 1923, pp. 52–55.

[1] The best text with Latin translation, for the musical works attributed to Euclid, is that edited by H. Menge in *Euclidis Opera Omnia* edited by Heiberg and Menge, Leipsic, vol. 8, 1916. There is a critical introduction, "De scriptis musicis," pages xxxvii–liv. These same works, in Greek and Latin, are to be found in David Gregory's edition of Euclid's works (Oxford, 1703, pp. 531–536). A Latin-French edition by P. Herigone appears in his *Cursus Mathematicus*, vol. 5, Paris (1637), 1644, pp. 802–856. There is a French translation by P. Forcadel, *Le livre de la musique d'Euclide*, Paris, 1566; this is also in L. Lucas, *Une révolution dans la musique . . .*, Paris, 1849, and in L. Lucas, *L'Acoustique nouvelle . . .*, Paris, 1854. Another French translation is by C. E. Ruelle, *L'introduction harmonique de Cléonide, La division du canon d'Euclide . . .*, Paris, 1884. An English translation appears in C. Davey, *Letters addressed chiefly to a young gentleman upon subjects of Literature: including a translation of Euclid's section of the Canon; and his treatise on Harmonic; with an explanation of the Greek musical modes according to the Doctrine of Ptolemy*, Bury St. Edmunds, 1787, vol. 2, pp. 264–410. The Theory of Intervals, in Greek, is also given in K. v. Jan, *Musici Scriptores Græci*, Leipsic, 1895–1898, vol. 1, pp. 148–166; there is a "prolegomena" (in Latin), by Jan, pp. 115–147.

[2] Compare Heath, *The Thirteen Books of Euclid's Elements*, Cambridge, 1908, vol. 1, p. 17, and vol. 2, p. 295.

[3] C. Davey, *loc. cit.*, pp. 264–265; the punctuation and slight changes in the wording have been made in the quotation.

consist of parts, inasmuch as they are capable of being rendered precisely and exactly tunable, either by increasing or diminishing the number of the vibratory motions which excite them. But all things which consist of numerical parts when compared together, are subject to the ratios of numbers, so that musical sounds or notes compared together, must consequently be in some numerical ratio to each other."

Nearly two thousand years passed before Galileo went one step further,[1] and proved that the lengths of strings of the same size and tension were in the inverse ratios of the numbers of the vibrations of the tones they produced.[2] It was not for another seventy years that the actual number of vibrations corresponding to a given tone was determined;[3] but we shall return to this a little later.

Euclid's work contains 19 theorems. They are mostly concerned with results which may be obtained by the division of a monochord, or string to be experimented upon, which Euclid calls Proslambanomenos. Let this be named A.[4]

This string A was first divided into four parts; three parts were taken and the perfect fourth established with the ratio 3 : 4; two parts were taken and the sound of the octave established; one part was taken and the sound of the double octave A was given.

The next experiment was to divide the length which produced the fourth of the prime into two equal parts, when the sound, the octave of the fourth, was established.

Proslambanomenos was then divided into two equal parts, and one of these being again divided into three parts, two parts were taken and the octave of the fifth was established.

And so till all the tones in two octaves were determined. By beginning with different letters in the series thus determined, Euclid got the seven Pythagorean scales covering two octaves instead of one. Euclid arrived at these sounds by the division of the monochord instead of by successions of fifths employed by Pythagoras.

[1] Galileo Galilei, *Discorsi e Dimostrazioni Matematiche*, Leyden, 1638, at the end of the "first day." A new English translation by H. Crew and A. de Salvio appeared at New York in 1914 with the title: *Dialogues concerning Two New Sciences*. The manuscript of the original work was sent to the printer in 1636 and the printing was completed in 1637; but many of the results were given by Galileo in lectures long before.

[2] Credit for this result is given to Galileo with full knowledge of the claims of Mersenne. Compare F. Rosenberger, *Die Geschichte der Physik*, part 1, Braunschweig, 1882, pp. 35–36.

In J. C. Poggendorff, *Histoire de la Physique*, translated by E. Bibart and G. de la Quesnerie, Paris, 1883, the following sentence occurs on page 487: "Deux siècles après *Pythagore*, *Aristote* écrivit sur les sons, et fit preuve d'une connaissance exact des faits, donts il est peut-être redevable aux Pythagoriciens. Il savait, par example, que dans les cordes de tension égale et dans les tuyaux, le nombres des vibrations est en raison inverse des longueurs, et que les sons sont produits par des vibrations qui passent des corps sonores à l'air qui les transmet à notre oreille." I can find no verification of this statement as to Aristotle's knowledge.

[3] Compare Newton, *Philosophiæ Naturalis Principia Mathematica*, London, 1687, book 2, section 8, prop. 50, pp. 369–372; second ed., Cambridge, 1713, pp. 342–344.

[4] Compare "Euclid's mathematical divisions of a string, and resulting series of sounds" in H. Wylde, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, pp. 84–93. See also T. P. Thompson, *Theory and Practice of Just Intonation*, London, 1850, pp. 79–80; fourth ed., 1860, pp. 104–108. The fourth edition has the title *On the Principles and Practice of Just Intonation*.

Two of Euclid's theorems prove that an octave is less than six tones, the ratio of the interval being $(8/9)^6 \div (1/2) = 524288/531441$, or nearly $80 : 81.0915$. This same ratio is got from $(2/3)^{12} \div (1/2)^7$. In other words it is the ratio determined by the difference of tones derived by counting 12 fifths and 7 octaves from a fundamental. This interval, between notes theoretically the same, was noted by Pythagoras and is called a Pythagorean comma.[1]

The scales of Pythagoras and Euclid differ in two important respects from our major scales, namely, in the ratios for the intervals of a third and a sixth. In the scale of c, the interval of a major third from the tonic is now $4/5 = 64/80$ instead of the Pythagorean $64/81 = (8/9)(8/9)$. This substitution of $4/5$, even though not mentioned by Euclid, is not modern, but was already suggested in the late Pythagorean school.[2] The second substitution of $3/5$ for the major sixth interval from the tonic naturally followed from this, since it is the octave of the fifth below the third. In this way the ratios of the intervals of the major scale became[3] 1, 8/9, 4/5, 3/4, 2/3, 3/5, 8/15, 1/2, while the intervals between successive pairs of notes became 8/9, 9/10, 15/16, 8/9, 9/10, 8/9, 15/16.

In such a scale if we tune up four perfect fifths on the one hand and two octaves and a major third on the other, we ought to arrive at the same note. The resulting comma here is 80/81 instead of the 80/81.0915 already referred to. It is the distribution of this comma which is ordinarily carried through in our equal-tempered scale. This temperament is said to have been proposed by Aristoxenus.

And last among the Greeks to whom we shall refer is the celebrated mathematician, astronomer and geographer, Claudius Ptolemy, who flourished in the second century of the Christian era. Apart from the Almagest, works on optics and mechanics, a book on stereographic projection, a book in which he tried to show that the possible number of dimensions is limited to three, and other works, Ptolemy wrote a remarkable treatise on music.[4] In it he discusses critically the earlier Pythagorean and Aristoxenean modes and tonalities and presents new developments. But the restrictions made in connection with the music seem to indicate the beginning of a decline.

Some interesting suggestions have been made by Paul Tannery as to the

---

[1] Compare Helmholtz, *On Sensations* . . ., *loc. cit.*, p. 432.

[2] D. B. Monro, *The Modes of Ancient Greek Music*, Oxford, 1894, p. 123.

[3] If in the series of ratios for the major scale we substitute 5/6 for a minor third, instead of 4/5 as for the major, and 9/16 for the 8/15, we have the succession at which Newton arrived, in an experiment with the prismatic colors of pure light published in his *Optiks*, London, 1704, p. 92. Measuring from an origin to the left to determine the points 1, 8/9, 5/6, 3/4, 2/3, 3/5, 9/16, 1/2, and erecting cross lines he found, as he states, "the said cross lines divided after the manner of musical chord . . . to represent the Chords of that Key, and of a Tone, a third Minor, a fourth, a fifth, a sixth Major, a seventh, and an eighth above the Key: And the intervals . . . will be spaces which the several Colours (red, orange, yellow, green, blue, indiao, violet) take up."

[4] A somewhat defective text of this work, together with a Latin translation by John Wallis, was published at Oxford in 1680. Compare Fétis, *Biographie* . . ., vol. 7. See also *The Oxford History of Music*, vol. 1, by H. E. Wooldridge, Oxford, 1901, pp. 15–22; H. Wylde, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, chapter XI, etc.; and D. B. Monro, *The Modes of Ancient Greek Music*, Oxford, 1894, pp. 108–112.

possible rôle of Greek music in the development of pure mathematics.[1] One of these is to the effect that the idea of logarithms may have been suggested by such mathematical relations as the following going back to Pythagoras:

$$\frac{1}{2}=\frac{2}{3}\times\frac{3}{4} \qquad \frac{1}{2}=\left(\frac{3}{4}\right)^2\times\frac{8}{9}$$

being immediately interpreted in music by: The octave is composed of a fifth and a fourth; the octave is composed of two fourths and of a major tone. Thus mathematical multiplication is changed into musical addition.

Another of Tannery's suggestions involves finding solutions of a Diophantine equation in three variables. In the first four notes of the major scale we had the relation

$$\frac{8}{9}\times\frac{9}{10}\times\frac{15}{16}=\frac{3}{4}.$$

Ptolemy derived many scales[2] in which the relations were similar; for example,

$$\frac{7}{8}\times\frac{9}{10}\times\frac{20}{21}=\frac{8}{9}\times\frac{7}{8}\times\frac{27}{28}=\frac{9}{10}\times\frac{10}{11}\times\frac{11}{12}=\frac{3}{4}.$$

In other words the question of the composition of the tetrachord reduces to the following mathematical problem: "Determine all possible ways of decomposing the ratio 3/4 into a product of three ratios of the form $n/(n+1)$." From these results, those were finally selected which seemed practicable after trial with the monochord.

In my brief sketch of the work done by the Greeks, I have not intended to give you any idea of their music, but merely to select a few illustrations of the manner in which their music is connected with mathematics. On the varieties of their scales and their coloring through chromatics (as the name implies) and quarter tones, I have not touched. Nor have I commented on the great beauties of the music even though it was homophonic. Authorities agree with the following summing up of Helmholtz:[3] "Of course where delicacy in any artistic observations made with the senses come into consideration, moderns must look upon the Greeks in general as unsurpassed masters. And in this particular case they had very good reason and abundance of opportunity for cultivating their ears better than ours. From youth upwards we are accustomed to accommodate our ears to the inaccuracies of equal temperament, and the whole of the former variety of tonal modes, with their different expression, has reduced itself to such an easily apprehended difference as that between major and minor. But the varied gradations of expression, which moderns attain by harmony and modulation, had to be effected by the Greeks and other nations that used homo-

[1] P. Tannery, "Du rôle de la musique grecque dans le developpement de la mathématique pure," *Bibliotheca Mathematica*, series 3, vol. 3, 1902, pp. 161–175; also in P. Tannery, *Mémoires Scientifiques*, vol. 3, 1915, pp. 68–89.

[2] Compare "Examen des series d'Archytas" in L. Laloy, *Aristoxène de Tarente et la musique de l'antiquité*, Paris, 1904, pp. 364–365.

[3] *On Sensations . . .*, *loc. cit.*, p. 266.

phonic music by a more delicate and varied gradation of tonal modes. Can we be surprised, then, if their ear became much more finely cultivated for differences of this kind than it is possible for ours to be?"

The next outstanding figure in our survey is Boetius who flourished in the early part of the sixth century of our Christian era. He was a Roman senator and a philosopher,—"the last of the Romans whom Cato or Tully could have acknowledged for their countryman," as Gibbon expresses it. Not only did Boetius exert great influence in his own time through his summaries of logical and scientific works of the ancients, but for six centuries after his death they were the leading authorities. He wrote works on arithmetic, geometry and music. While the first printed edition of the arithmetic appeared at Venice in 1488, all three united seem to have first been published in 1492. Details of the mathematical works have been given by Cantor.[1] His extensive treatise on music[2] is a valuable repertory of the knowledge of the ancients in this art. It was long used as a text at the Universities of Oxford and Cambridge.[3] Boetius sets forth the details of the accomplishments of the Pythagoreans and the teachings of such writers as Aristoxenus which were opposed to those of Pythagoras. He also surveys the Ptolemaic musical scheme in connection with those of Pythagoras and Aristoxenus. Since the doctrine of Boetius was mainly Pythagorean, this was the system which prevailed for centuries later.

As the Roman absorbed the Greek, so the Christians accepted the Roman organization of learning. In the medieval curriculum the scope of this learning on the secular side was comprised within the seven liberal arts[4] and philosophy. The seven liberal arts, divided into the *Trivium* (grammar, dialectic, rhetoric), and the more advanced *Quadrivium* (geometry, arithmetic, music and astronomy), were an inheritance from a period at least as early as the second century before Christ; indeed the Quadrivium division of mathematical studies is Pythagorean.[5] Some explanation of the nature of the subjects of the Trivium is necessary in order to make their scope clear; but we are only concerned with the mathematical sciences of the Quadrivium in which, early in the middle ages, the course in geometry was more a course in geography and surveying than in the subject matter of Euclid's *Elements* which later became a text. The study of music consisted mainly in becoming acquainted with the mathematics of the subject, and with the mystic properties of its numbers,—much as taught by the Pythagoreans. As a liberal art it concerned itself neither with singing (apart from its rules), nor with playing on an instrument. Astronomy with its practical applications to the calendar and sun dial was the most popular of the Quadrivium subjects but there was probably more of astrology in it than astronomy as we now understand the term.

[1] *Vorlesungen über Geschichte der Mathematik*, vol. 1, third ed., Leipsic, 1907. See also D. E. Smith, *Rara Arithmetica*, Boston, 1908.

[2] See Fétis, *Biographie* . . ., vol. 1, *loc. cit.*, and Eitner, *Quellen-Lexikon*, vol. 2, *loc. cit. Arithmetica et Musica* of Boetius, ed. by Friedlein, Leipsic, 1867, is the standard edition.

[3] See article "Boetius" in *Encyclopœdia Britannica*, eleventh edition.

[4] Compare P. Abelson, *The Seven Liberal Arts*, New York, 1906.

[5] T. L. Heath, *A History of Greek Mathematics*, Oxford, 1921, vol. 1, pp. 11–12.

Some four hundred years after the time of Boetius, the polyphonic period in the development of music had its inception in the composition of certain two-part song-forms. During the six hundred years which followed, that is, till towards the close of the sixteenth century, polyphonic music adorned with canon, fugue, and counterpoint was developed to a notable degree.

Of mathematicians who flourished in this period I shall refer to only one, Girolamo Cardano.[1] Among those of the sixteenth century achieving a reputation in mathematics and medicine none was better known than he, whose greatest mathematical work, *Ars Magna* (1545), contains the first solution of the general cubic equation in print.

Cardano was an ardent lover of music and while living in Milan his house was constantly filled with men and boys of somewhat sinister reputation but capable of joining with him in part-singing so popular in the polyphonic period. During the last twenty-five years of his life he spent considerable time in writing a work on music,[2] which was in many respects original and must have been welcomed by all musical students as a valuable contribution to the literature of the subject. This work begins by laying down at length the general rules and principles of the art, and then goes on to treat of ancient music in all its forms; of music as Cardano knew and enjoyed it; of the system of counterpoint and composition, and of the construction of musical instruments.[3]

An interesting glimpse of Cardano's personality may be gleaned in another place from his listing of the joys of home and children. Incidentally he suggests: "Let the young child . . . be shut out from the sight or hearing of all ill. When he is about seven years old let him be taught elements of geometry to cultivate his memory and imagination. With syllogisms cultivate his reason. Let him be taught music, and especially to play upon stringed instruments; let him be instructed in arithmetic and painting, so that he may acquire taste for them, but not be led to immerse himself in such pursuits. He should be taught also a good hand-writing, astrology, and when he is older, Greek and Latin."

In the early part of the *third* period in the development of music, namely, the period of Harmonic or Modern Music, we have the first opera and the first oratorio, and, as I have already said, the discovery by Galileo that the simple ratios of the lengths of strings existed also for the pitch numbers of the tones they produced, an observation later generalized by Newton. By the time of Rameau, the most eminent French composer and writer on the theory of music in the eighteenth century, the harmonics or upper partial tones of the human voice had been recognized and made the basis for more satisfying harmonic development. A string, for example, vibrates not only as a whole but also, at

---

[1] 1501–1576.

[2] G. Cardano, *Opera Omnia*, Leyden, 1663, vol. 4. See also fragment no. 6 in vol. 10. Volume 4, no. 10 is *Opus novum de proportionibus numerorum, motuum, ponderum, sonorum aliarumque rerum mensurandarum* . . . first published at Basle in 1570.

[3] W. G. Waters, *Jerome Cardan, a biographical Study*, London, 1898, p. 256; see also pp. 163, 235. In the more extensive biography by H. Morley, *The Life of Girolamo Cardano of Milan, Physician*, 2 vols., London, 1854, there are references to music on the following pages: vol. 1, pp. 41, 45, 202, 295; vol. 2, pp. 19, 43, 53.

the same time, in each of its aliquot parts 1/2, 1/3, 1/4, 1/5, 1/6, and so on. Thus the first upper partial tone is the upper octave of the prime tone, the second is the fifth of this octave, the third upper partial is the second higher octave, the fourth is the major third of this second higher octave, the fifth is the fifth of the second higher octave, making six times as many vibrations as the prime in the same time; and so on, each successive upper partial tone being fainter than the preceding. It may be shown that beginning with the twenty-fourth upper partial all the notes of a major scale may be obtained from the dominant, that is, the fifth.[1] The dominant and not the tonic is thus the root, of the whole scale. In the bugle, trumpet, French horn, and other instruments only the fundamental tone of the instrument and some of its harmonics can be sounded. On a horn about four feet long the notes are c, c', g', c'', e'', and g'', —the primes denoting tones in higher octaves.

Not all upper partials need exist in connection with a fundamental musical tone. Certain tuning forks have no upper partials.[2] In 1800, the noted physicist, Thomas Young, who first furnished the key to decipher Egyptian hieroglyphics, was also the first to show that "when a string is plucked or struck, or, as we may add 'bowed' at any point in its length which is the node of any of its so-called harmonics, those simple vibrational forms of the string which have a node in that point are not contained in the compound vibrational form. Hence if we attack at its middle point, all the simple vibrations due to the even numbered partials, each of which has a node at that point, will be absent. This gives the sound of the string a peculiarly hollow or nasal twang."[3] Because of this law piano makers eliminate[4] certain undesirable upper partials by striking the middle strings of their instruments at a point 1/7 to 1/9 of their lengths from their extremities. So too in making other instruments it is possible to eliminate, or reinforce, certain partials.

But we have got ahead of our story. Returning to the beginning of the Harmonic period let us consider the musical writings which were issued in the seventeenth century by such mathematicians as Kepler, Wallis, Mersenne, Desargues, Descartes and Christian Huygens.

Pythagorean ideas on the ratios of numbers and of proportions applied to the constitution of the universe seem to have been the point of departure of Kepler in his famous work *Harmonices Mundi* published in 1619.[5] It is in the fifth book of this work that one first finds the third fundamental law of modern astronomy, "The squares of the periodic times of the several planets are proportional to the cubes of their mean distances from the sun," demonstration of which furnished Newton with the basis for his theory of gravitation. The third book of the work is especially devoted to music and it may be characterized

[1] The scale is made from the following partials: 24, 27, 30, 32, 36, 40, 45, 48.

[2] Helmholtz, *On Sensations* . . ., *loc. cit.*, pp. 54, 528.

[3] Helmholtz, *On Sensations* . . ., *loc. cit.*, p. 52.

[4] Helmholtz, *On Sensations* . . ., *loc. cit.*, p. 77; but compare pp. 545–546.

[5] *Joannis Kepleri astronomi opera omnia*, ed. C. Frisch, vol. 5, Frankfurt, 1864. Compare Fétis, *Biographie* . . ., *loc. cit.*, vol. 5. Kepler was born in 1571 and died in 1630.

as mainly a work on the philosophy of music. The fifth book to which I have referred is somewhat allied to the third, since in it the author endeavored to establish curious analogies between the harmonic proportions of music and astronomy.

Markedly contrasted to Kepler in abilities and habits of thought was John Wallis, the notably able Savilian professor at Oxford University, where a brilliant mathematical school was developed under his direction. He is well known as mathematician and cryptographer,[1] but few have observed his extensive writings on musical matters [2] filling more than 500 folio pages in the third volume of his collected works. The first of these is a Greek and Latin edition of Ptolemy's Harmony, and Porphyry's third century commentary [3] on the same, with an extensive appendix by Wallis on ancient and modern music. Then comes the only published text, with Latin translation, of a musical work by Manuel Bryenne, a fourteenth century Greek, four manuscripts of whose work are to be found at the Bodleian. Among other writings of Wallis on acoustics and music may be mentioned four memoirs published in the *Philosophical Transactions*,[4] and bearing the following titles: "On the trembling of consonant strings," "On the division of the monochord, or section of the musical canon," "On the imperfections of an organ," and "On the strange effects of music in former times."

The Franciscan friar Marin Mersenne, Wallis's senior by nearly 30 years, is known to the general run of mathematicians through the numbers with which his name is associated and which arise in discussion of perfect numbers. He was widely acquainted with French and foreign contemporary mathematicians and actively corresponded with them. His work in physics [5] dealt chiefly with questions in acoustics. He determined ratios of the vibration numbers of strings varying in thickness and tension, results included in those of Brook Taylor derived mathematically about 70 years later. I have not been able to verify the statement [6] that Mersenne noticed, but attached no importance to the observation, that a vibrating string gave forth not only the fundamental tone but also higher sounds. We have already remarked that Rameau made much of the fact in the following century. Mersenne wrote half a dozen works on harmony and musical instruments [7] but his most notable one is *L'Harmonie Universelle*, a great work of 1500 pages with an immense quantity of engraved plates and musical examples. This was published in 1636–7. It is really a combination of several treatises, for example, On the Nature of Sounds and

[1] Compare D. E. Smith, "John Wallis as a Cryptographer," *Bulletin of the American Mathematical Society*, vol. 24, pp. 82–96, 1917. Wallis was born in 1616 and died in 1703.

[2] Compare Fétis, *Biographie . . .*, vol. 8, *loc. cit.* Also H. Mendel, *Musikalisches Conversations-Lexikon*, vol. 11, 1878.

[3] Jan believes that this was probably mostly compiled by Pappus or some other competent mathematician; see K. v. Jan, *Musici Græci Scriptores Græci*, Leipsic, 1895, p. 116.

[4] 1677–1698.

[5] F. Rosenberger, *Die Geschichte der Physik*, part 1, Braunschweig, 1882, pp. 93–95, etc.; also J. C. Poggendorff, *Histoire de la Physique*, Paris, 1883, pp. 488–489.

[6] J. K. Fischer, *Geschichte der Physik*, vol. 1, Göttingen, 1801, pp. 468, 470.

[7] Compare Fétis, *Biographie . . .*, vol. 6, *loc. cit.*; compare Eitner, *Quellen-Lexikon . . .*, vol. 6, *loc. cit.*

Movements of All Sorts of Bodies, On Voice and Songs, and On Instruments. There is also a treatise on mechanics, by Roberval, which no one but a Mersenne could regard as appropriately placed in his work on harmony. While no sections of the work are of transcendent merit, one finds a great amount of information, especially regarding Frenchmen, which is no longer to be found elsewhere. It is only here, for example, that we learn that the geometer Desargues was the author of a method of singing.

Among Mersenne's friends was one, some eight years his junior, René Descartes. That he was interested in music [1] is attested by the fact that a score of his published letters treat of motions of vibrating strings and various musical topics. Moreover in 1618, when 22 years of age, he wrote a *Compendium Musicæ*, but this was first published as a little tract of 58 pages [2] in 1650, the year of his death. The material is arranged under about a dozen headings such as: the object of music is the sound; number and time that one should observe in the sounds; concerning the diversity of sounds; consonances; the octave; the fifth; the fourth; the second, minor third, and sixth; the degrees or tones of music; dissonances; and the manner of composing—in connection with which five principles are laid down in an interesting manner.

A copy of the manuscript of the *Compendium* found its way to one afterwards to become a particular friend of Descartes.[3] This was Constantin Huygens, a many-sided genius possibly best known as a poet and a musician; he was a competent performer on several instruments and author of several musical works. His second son was Christian the great Dutch mathematician, mechanician, astronomer and physicist. Two publications dealing with musical matters were written by Christian Huygens. The first of these is a brief sketch of 1691, entitled "Novus Cyclus Harmonicus," and occupying only 8 quarto pages.[4] In them he suggests another solution of the problem of how suitably to arrive at a tempered scale. If we divide the octave into twelve equal parts or degrees, we have a cycle in which a fifth of 7 and a major third of 4 degrees approximates to Pythagorean intonation. A cycle of 53, with a fifth of 31 and a major third of 18, had also been proposed, and led to similar results. The new harmonic cycle of Huygens contained 31 degrees, with a fifth of 23 and a major third of 10, and closely imitates mean tone temperament.[5] He refers to the writings of

[1] Compare Fétis, *Biographie* . . ., vol. 3, *loc. cit.*; and Eitner, *Quellen-Lexikon* . . ., vol. 3, *loc. cit.* Descartes was born in 1596 and died in 1650.

[2] See also *Œuvres de Descartes* publiées par Charles Adam et Paul Tannery, Paris, vol. 10, 1908, pp. 79–150. An English Translation was published at London in 1653. There were four other French editions.

[3] See many references in *Œuvres de Descartes*, vol. 12, 1910, *Vie & Œuvres de Descartes* by C. Adam. See also in this MONTHLY, *1921*, 167, where in the course of an article by D. E. Smith on "Descartes's appreciation of Huygens the elder" a letter from Descartes dated May 23, 1632, contains the following clause: "I do not know how to respond to the courtesy of Monsieur Huygens, except that I cherish the honor of his acquaintance as one of the greatest pieces of fortune that has come to me."

[4] In *L'Histoire des Ouvrages des Sçavans*, Rotterdam, October, 1691, p. 78; also C. Huygens, *Opera Varia*, Amsterdam, 1724, vol. 3, pp. 747–754. Christian Huygens was born in 1629 and died in 1695.

[5] Compare Helmholtz, *On Sensations* . . ., *loc. cit.*, p. 436. See also *Proc. Roy. Soc. of London*, vol. 13, 1864, p. 412.

Mersenne, and of Zarlino, "one of the most learned and enlightened music theorists of the sixteenth century." I have already drawn attention to the natural way in which logarithms enter into the discussion of musical intervals. So far as I have been able to determine, this little publication of Huygens is the first to illustrate this fact.

The second work of Huygens containing musical material was finished for the press just before his death. Three years later it appeared simultaneously in Latin and English and is an exceedingly entertaining work. It is entitled *The Celestial Worlds discover'd: or Conjectures concerning the Inhabitants Plants and Productions of the Worlds in the Planets.*[1] In order adequately to present an idea of a section on mathematics and music I shall quote somewhat extensively.

The author surmises that if the surfaces of Jupiter and Saturn are divided like ours into sea and land it is reasonable to suppose that the inhabitants must know of the art of navigation. He then infers that they must have the "Mechanical Arts and Astronomy, without which Navigation can no more subsist, than they can without Geometry." Huygens then continues (page 84): "But Geometry stands in no need of being prov'd after this manner. Nor doth it want assistance from other Arts which depend upon it, but we may have a nearer and shorter assurance of their not being without it in those Earths. For that Science is of such singular Worth and Dignity, so peculiarly imploys the Understanding, and gives it such a full Comprehension, and infallible certainty of Truth, as no other Knowledge can pretend to: it is moreover of such a Nature, that its Principles and Foundations must be so immutably the same in all Times and Places, that we cannot without Injustice pretend to monopolize it and rob the rest of the Universe of such an incomparable Study. Nay Nature itself invites us to be Geometricians: it presents us with Geometrical Figures, with Circles and Squares, with Triangles, Polygons, and Spheres, and proposes them as it were to our Consideration and Study which abstracting from its usefulness is most delightful and ravishing. Who can read *Euclid* or *Apollonius*, about the Circle, without Admiration? Or *Archimedes* of the Surface of the Sphere, and Quadrature of the Parabola without Amazement? Or consider the late ingenious Discoveries of the Moderns, with Boldness and Unconcernedness? And all these Truths are as naked and open, and depend upon the same plain Principles and Axioms in *Jupiter* and *Saturn* as here, which makes it not improbable that there are in the Planets some who partake with us in these delightful and pleasant studies." Then a little later the author continues (page 86): "It's the same with Musick as with Geometry, it's everywhere immutably the same, and always will be so. For all Harmony consists in Concord, and Concord is all the World over fix'd according to the same invariable Measure and Proportion so that in all Nations the Difference and Distance of Notes is the same, whether they be in a continued gradual Progression, or the Voice skips over one to the next. Nay, very credible Authors report, that there's a sort of Bird in *America*, that can plainly sing in

[1] London, 1698; "second edition corrected and enlarged" in 1722; "new edition corrected," Glasgow, 1762. The Latin edition published at The Hague is entitled Κοσμοθεωρός *sive de terres cœlestibus, earumque ornatu, conjecturae*; second edition, 1699.

order six musical Notes: Whence it follows, that the Laws of Musick are unchangeably fix'd by Nature, and therefore the same Reason holds for their Musick, as we e'en now shewed for their Geometry."

Discussing the probability of other planets' being inhabited and of the inhabitants' possible interest in music and invention of musical instruments, he continues (page 88): "What if they should excell us in the Theory and practick part of Musick, and outdo us in consorts of vocal and instrumental Musick, so artificially compos'd, that they shew their skill by the Mixtures of Discords and Concords and of this last sort 'tis very likely the 5th and 3d are in use with them.

"This is a very bold Assertion, but it may be true for aught we know, and the Inhabitants of the Planets may possibly have a greater insight into the Theory of Musick than has yet been discover'd among us. For if you ask any of our Musicians why two or more perfect Fifths cannot be used regularly in Composition; some say 'tis to avoid that Sweetness and Lushiousness which arises from the repetition of this pleasing Chord. Others say, this must be avoided for the sake of that Variety of Chords that are requisite to make a good Composition; and these Reasons are brought by Descartes [1] and others. But an Inhabitant of *Jupiter* or *Venus* will perhaps give you a better Reason for this, *viz.* because when you pass from one perfect Fifth to another, there is such a Change made as immediately alters your Key, you are got into a new key before the Ear is prepared for it, and the more perfect Chords you use of the same kind in Consecution, by so much the more you offend the Ear by these abrupt Changes."

It may interest harmony students of our day to learn that the prohibition of consecutive fifths [2] was not something recently invented for their undoing, but was a matter of fundamental importance adequately explained over two hundred years ago. And this is not the only passage of interest for such students in the last work of Christian Huygens.

I have already referred to Thomas Young's memoir of 1800 and his explanations of varied qualities of tone through agitation of a string at different points. The mere fact of such differences of quality had been already noted by Huygens [3] in connection with harpsichords—those precursors of pianos in the sixteenth, seventeenth, and eighteenth centuries.

III.

In the eighteenth century when calculus had become a tool, there was a notable series of theoretical discussions of vibrating strings. But before considering these I wish to draw special attention to the first English scientific treatment of harmony, a work of high order, by Robert Smith.[4] It was entitled *Harmonics or the Philosophy of Musical Sounds* and was first published [5] in 1749.

[1] This was simply "*Cartes*" in the original.

[2] Compare "Modern music and 'fifths,'" *Monthly Musical Record*, vol. 46, 1916, pp. 43–44.

[3] Helmholtz, *On Sensations* . . ., *loc. cit.*, p. 77.

[4] Born 1689, died 1768. See *Dictionary of National Biography*, Oxford. Eitner, *Quellen-Lexikon* . . ., *loc. cit.*, vol. 9, confuses this Robert Smith with an earlier musician and composer; see *Musical Antiquary*, Oxford, vol. 2, pp. 171–173, 1911.

[5] "Second edition much improved and augmented," London, 1759, 20 + 293 pp. + 28 plates.

The theory of intervals and various systems of temperament are discussed in a manner very attractive even for a reader in the present day. Smith held the Plumian chair at Cambridge, the one of which A. S. Eddington is the present incumbent, and his work on harmonics contained the substance of lectures he had delivered for many years. It was he who was the author of the notable work on *Optics* which has been translated into several languages. He was also the founder of the well-known Smith's Prizes "annually awarded to those candidates who present the essays of greatest merit on any subject in mathematics or natural philosophy." [1]

First in the series of theoretical discussions to which I have referred are those of Brook Taylor, who, according to his biographer,[2] "possessed considerable ability as a musician and an artist." His discussions appeared in the *Philosophical Transactions* [3] for 1713 and 1715 and in his book *Methodus Incrementa Directa et Inversa*,[4] the first treatise dealing with finite differences, and the one which contains the celebrated theorem regarding expansions, now connected with Taylor's name. He solved the following problem which he believed to be entirely new: "To find the number of vibrations that a string will make in a certain time having given its length, its weight, and the weight that stretches it." In discussing the *form* of the vibrating string, his suppositions regarding initial conditions, including that it vibrated only as a whole, led to a differential equation whose integral gave a sine curve. Thus started a discussion which was to culminate a century later in the work of Fourier.

I have already referred to the discovery of upper partial tones by Rameau and how he made this the basis of a system of harmony; his first work on this subject was published in 1726, but the first mathematician who seemed to take account of the fact was Daniel Bernoulli in a memoir of 1741–43 though not published [5] till 1751. About this time D'Alembert's thorough acquaintance with Rameau's theories was shown by his publication in 1752 of a volume entitled "Elements of theoretical and practical music according to the principles of Monsieur Rameau, clarified, developed, and simplified." Of this work six French editions [6] and one in German were published.[7] Helmholtz remarks [8] that D'Alembert's book "is an extremely clear and masterly performance, such

---

A *Postscript* (12 pp. + 1 plate) was published in 1762. A German edition was published at Berlin in 1771.

[1] *Cambridge University Calendar*, 1921–22.

[2] E. I. Carlyle in *Dictionary of National Biography*, Oxford. Taylor was born in 1685 and died in 1731.

[3] Vol. 28, London, 1714, "De motu tensi," pp. 26–32; vol. 29, 1715, "An account of a book entitled Methodus Incrementorum by the author," pp. 339–350.

[4] London, 1715; other title pages are dated 1717.

[5] D. Bernoulli, *Comment. acad. sc. Petrop.*, vol. 13 (1741/3), 1751, p. 173, § 8. Bernoulli was born in 1700 and died in 1782.

[6] *Eléments de musique théorique et pratique, suivant les principes de M. Rameau, éclaircis, développés et simplifiés*, Paris, 1752; second edition, 1759; third and fourth editions, Lyons, 1762, 1766, 1772, and 1779. D'Alembert was born in 1717 and died in 1783. Compare Eitner, *Quellen-Lexikon* . . ., vol. 1, *loc. cit.*

[7] The German edition by F. W. Marpurg was published at Leipsic in 1757.

[8] *On Sensations* . . ., *loc. cit.*, p. 232.

as was to be expected from a sharp and exact thinker, who was at the same time one of the greatest physicists and mathematicians of his time. Rameau and D'Alembert lay down two facts as the foundation of their system. The first is that every resonant body audibly produces at the same time as the prime its twelfth and next higher third as upper partials. The second is that the resemblance between any tone and its octave is generally apparent. The first fact is used to show that the major chord is the most natural of all chords, and the second to establish the possibility of lowering the fifth and the third by one or two octaves without altering the nature of the chord, and hence to obtain the major triad in all its different inversions and positions."

D'Alembert wrote also a long essay on the liberty of music [1] and articles of musical interest in the great *Encyclopédie Methodique*. But from a mathematical point of view, his memoir of 1747 dealing [2] with Taylor's problem of the vibrating string, and taking account of matters previously overlooked, is very notable. He was led to the differential equation (with a, a constant, equal to unity)

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2},$$

where the origin of coördinates was at the end of the chord whose length is l, the axis of x in the direction of the chord, and y the displacement at any time t. Of this equation he found the solution

$$y = f(at + x) - f(at - x),$$

where f represents any function such that $f(z) = f(z + 2l)$. He then found certain equations for determining the functions satisfying this relation of periodicity.

Euler immediately raised the question of the generality of the solution and set forth *his* interpretation. D'Alembert had supposed the initial form of the string to be given by a single analytical expression, while Euler regarded it as lying along any arbitrary continuous curve, different parts of which might be given by different analytical expressions. Lagrange [3] joined in the discussion, to which Daniel Bernoulli contributed chiefly from physical rather than mathematical considerations. He started with Taylor's particular solution and found, in effect, that the function for determining the position of the string after starting from rest could naturally be expressed in a form later called a Fourier series.[4] Thus

[1] "*De la liberté de la musique*" in his *Mélanges de Littérature et de Philosophie*, Amsterdam, 1767–1773, and reprinted in d'Alembert, *Œuvres Philosophiques, historiques et littéraires*, Paris, 1805, vol. 3, pp. 335–409. The first sentence gives a clue as to the meaning of the title. It is: "Il y a chez toutes les nations, deux choses qu'on doit respecter, la *religion* et le *gouvernement;* en France on y en ajoute une troisième, la musique du pays."

[2] "Recherches sur la courbe que forme une corde tendue mis en vibration," *Memoires de l'Acad. Royale des Sciences et belles Lettres, 1747*, 1749, pp. 214–219; "Suite des Recherches sur . . .," pp. 220–249.

[3] Euler (1707–1783); Lagrange (1736–1813).

[4] The series is not stated explicitly by Bernoulli in the memoir, "Sur le mêlange de plusieurs espaces de vibrations simples isochrones, qui peuvent coexister dans un même système de corps," *Histoire de l'Acad. Royale des Science, 1753*, Berlin, 1755; but it can be put together from different places, no. 17, p. 160, and no. 23, p. 165.

were such series first introduced into mathematical physics. Bernoulli remarked that since his solution was perfectly general it should include those of Euler and D'Alembert. In this way mathematicians were led to consideration of the famous problem of expanding an arbitrary function as a trigonometric series. No mathematician would admit even the possibility of its solution till this was thoroughly demonstrated, in connection with certain problems in the flow of heat, by Fourier who gives due credit to the suggestiveness of the work of those in the previous century to whom I have referred. Fourier's results were contained in a memoir crowned by the French Academy in 1812 but not printed till more than a decade later. It is sometimes asserted that the first mathematical proof of Fourier's results, with the limits of arbitrariness of the function carefully stated, was given by Dirichlet in his classic memoirs of 1829 and 1837. So far as the limits of arbitrariness are concerned this is correct; but that Fourier rigorously established his expansion of an arbitrary function seems to admit of no denial or qualification.[1]

One of Euler's most notable papers connected with the history of Fourier's series did not appear in print till 1793, ten years after his death. Thus for eighty years, from Taylor to Euler and Lagrange, mathematicians were occupied with the problem of the vibrating string[2] and allied problems including the vibration of a column of air and of an elastic rod. Then thirty years of silence and the great advance by Fourier.

I have indicated only a few bald facts since details in this regard are readily available elsewhere.[3]

Although more than twenty years Fourier's senior, Gaspard Monge,[4] so well known as an expounder of the applications of analysis to geometry,[5] and of descriptive geometry, was associated with him in more than one undertaking. They were professors at the École Polytechnique in Paris, which Monge was largely instrumental in founding. They both accompanied Napoleon to Egypt where Monge was the first president of the Institute of Egypt and Fourier its secretary. Monge was a passionate devotee of music and made a journey to Italy in order to procure copies of all of the musical works in the chapel of St. Mark's, Venice. He was also an ardent republican and, according to Arago,[6] an enthusiast for the "Marseillaise" which he sang every day at the top of his

[1] Compare Darboux, in *Œuvres de Fourier*, vol. 1, Paris, 1888, p. 512.

[2] D'Alembert's own account of this is interesting. See his article "Cordes, (*vibrations des*)" in his *Encyclopédie Methodique, Mathématiques*, vol. 1, Paris, 1784.

[3] I have found Burkhardt's monumental report to be the most complete and most reliable source of information. It is entitled "Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik," *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 10, Heft 2, 1901–1908.

[4] Compare Fétis, *Biographie* . . ., vol. 6, *loc. cit.* See also D. E. Smith, "Monge and the American colonies," in this MONTHLY, *1921*, 166. Monge was borne in 1746 and died in 1818.

[5] When 25 years of age Monge published his "Recherche des équations des surfaces d'apres leurs mode de génération" in the memoirs of the academy of Turin which had been founded through Lagrange's influence. On reading this paper Lagrange exclaimed, "Avec son application de l'analyse à la représentation des surfaces, ce diable d'homme sera immortel." (D. F. J. Arago, *Œuvres Complètes*, vol. 2, Paris, 1854, pp. 447–448.)

[6] Arago, *Œuvres*, vol. 2, p. 516.

voice before seating himself at the table. He, too, occupied himself with the problem of the vibrating string [1] and constructed a model of a surface, certain parallel sections of which give the form of the curve of the vibrating string at any time under conditions which Monge states. This model which was made in 1794 is still preserved in the École Polytechnique.

And finally in connection with great mathematicians of the eighteenth century, the extent of Euler's contributions to the theory of vibrating bodies, acoustics, and music, may be indicated somewhat further.[2] About 30 of his published memoirs, and a treatise, *Tentamen novæ theoriæ musicæ*,[3] not to speak of letters in his Letters to a German Princess, deal with such subjects. They appeared during about 60 years [4] from the first, a dissertation on sound, published in 1727, when he was 20 years old. Among the topics of memoirs not already referred to are: On the sound of bells, Conjectures as to the reason of some dissonances generally accepted in music, The true character of modern music, and On the vibratory motion of drums. It is in this last mentioned memoir of 1766 that the general so-called Bessel's functions of integral order first occur.

Euler's treatise on music was first published in 1739, but we learn from a letter Euler wrote to Daniel Bernoulli in May, 1731, that he had already almost completed the manuscript of the work. This letter describing the ideals of the work in some detail, as well as Bernoulli's reply in the following August, are readily accessible.[5] I shall therefore make but brief extracts from the early parts of the letters. Euler explains, "My main purpose was that I should study music as a part of mathematics and deduce in an orderly manner, from correct principles, everything which can make a fitting together and mingling of tones pleasing. In the whole discussion I have necessarily had a metaphysical basis, wherein the cause is contained why a piece of music can give one pleasure and the basis for it is to be located, and why a thing to us pleasing is to another displeasing." To this Bernoulli replied, "I cannot readily divine wherein that principle should exist, however metaphysical it may be, whereby the reason could be given why one could take pleasure in a piece of music, and why a thing pleasant for us, may for another be unpleasant. One has indeed a general idea of harmony that it is charming if it is well arranged and the consonances are well managed; but, as it is well known, dissonances in music also have their use since by means of them the charm of the immediately following consonances is brought out the better, according to the common saying *opposita juxta se posita magis elucescunt* [opposites placed together shine brighter]; also in the art of painting, shadows must be relieved by light."

[1] Monge, "Construction de l'équation des cordes vibrantes," *Journal de l'Ecole Polytechnique*, cahier 15, tome 8, Paris, 1809, pp. 118–145.

[2] Compare Fétis, *Biographie* . . ., vol. 3, *loc. cit.*

[3] First published at St. Petersburg in 1739; there were French editions in 1839 and 1865.

[4] These are all listed on pages 319–321, 332–333, of G. Eneström's remarkable *Verzeichnis der Schriften Leonard Eulers*, Leipsic, 1910, 1913.

[5] *Bibliotheca Mathematica*, third series, vol. 4, 1903, pp. 383–388. Bernoulli's letter was published earlier in *Journal für Mathematik*, vol. 23, 1842, pp. 199–200, and in *Correspondance mathématique et physique* . . ., ed. by Fuss, St. Petersburg, 1843, vol. 2, pp. 8–11.

Euler's treatise does not seem to have met with unqualified favor. Brewster reports Fuss to have said[1] "it had no great success as it contained too much geometry for musicians, and too much music for geometers." Helmholtz gives a good deal of space[2] to setting forth the psychological considerations which Euler explains had influenced him to found his relations of consonances to whole numbers.

But here we must leave this "myriad-minded" eighteenth century genius.

And now there is time for but the briefest references to mathematics and music during the past one hundred years—the century in which niceties of mathematical calculation were surely contributory to the improvement of such instruments as the flute and organ, to the wonders of phonograph-record manufacture, of broadcasted concerts, and of sound-wave photography—the century in which Helmholtz and Rayleigh lived and worked.

Helmholtz's epoch-making work, *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik,* first appeared in 1862. The great feature of this work is the formulation and proof of the laws by which the ear hears musical sounds from one or more distinct sources; how the theory of combined musical sounds is reduced to the theory of combined simple sounds. The starting point of these discoveries was the fact, recognized by Rameau just two hundred years ago, that upper partials were associated with fundamental tones. From these laws we learn the *nature* of consonance and dissonance, knowledge so necessary for building up a system of harmony; we learn the principles which determined those degrees of musical sound selected by various nations at various times; we understand the reasons for the simple ratios of the lengths of strings producing consonant tones and the limitation of the numbers of these ratios; and we appreciate the value of temperaments for different instruments.[3]

In his *Tonempfindung* Helmholtz relegated to appendices the purely mathematical discussions. For example, the third appendix is "On the motion of plucked strings;" the fifth is "On the vibrational forms of pianoforte strings"; the sixth is an "Analysis of the motion of violin strings"; and the seventh is "On the theory of pipes."[4] He goes into such matters more extensively in the volume of his lectures on the mathematical principles of acoustics.[5]

Such subjects are also treated in masterly fashion by Rayleigh in his *Theory of Sound*[6] and in his papers. Among other works will be mentioned only the mathematical elements of music, as presented some twenty-five years earlier by Airy,[7] senior wrangler and astronomer.

[1] *Letters of Euler on Different Subjects in Natural Philosophy to a German Princess,* ed. by D. Brewster, third ed., vol. 1, Edinburgh, 1823, p. xxv.

[2] Helmholtz, *On Sensations . . ., loc. cit.*, pp. 229–231. Compare H. Wylde, *The Evolution of the Beautiful in Sound,* Manchester and London, 1888, pp. 171–172.

[3] Compare Helmholtz, *On Sensations . . .*, first English edition, 1875, p. vi.

[4] Helmholtz, *On Sensations . . ., loc. cit.*, III, pp. 374–477; V, pp. 380–384; VI, 384–387; VII, pp. 388–396.

[5] Helmholtz, *Vorlesungen . . ., loc. cit.;* see, for example, discussion regarding the violin on pp. 121–139.

[6] *Loc. cit.*

[7] G. B. Airy, *On Sound and Atmospheric Vibrations with the Mathematical Elements of Music,* London and Cambridge, 1868; second edition, 1871.

In such works, in the comparatively recent notable paper in this country by Harvey Davis,[1] on vibrations of a rubbed string, and, of course, in other mathematical treatments of similar material, Fourier series must enter in a fundamental manner. With specified conditions the series and its coefficients for a given tone or combination of tones may be determined. Or, if we have a graph of the vibrations corresponding to such tones, the series may also be calculated, various terms in the series corresponding to simple elements compounded in the tone or tones.

During the past twenty years photography has contributed in a remarkable manner to the analysis of musical sounds. In England, from 1905 to 1912, E. H. Barton and his associates published [2] a series of papers illustrated by photographs of vibration curves particularly as issuing from the violin strings, bridge, and belly.

In India, five years ago, R. C. V. Raman published an extensive bulletin "On the mechanical theory of the vibration of bowed strings and of musical instruments of the violin family, with experimental verification of the results." [3] It is illustrated by reproductions of many photographs, those of the wolf-notes, so well known to stringed-instrument players, having especial interest. The more recent publications of S. Garten and F. Kleinknecht contain a discussion of tones produced by the voice.[4] And with us the work that D. C. Miller, of the Case School of Applied Science, has done in this connection is known to many, not only through his volume on *The Science of Musical Sounds*,[5] but also through his remarkably interesting public lectures where his extraordinary instrument called the phonodeik, which photographically records sound waves, may also be used for projecting traces of the waves, as generated, on the screen of a lecture platform.

For the mathematician a great advantage of a photograph is that he can, after much labor, from it calculate the corresponding Fourier series. But in the laboratory, work of this kind is often saved by the employment of a machine called the harmonic analyzer.[6] The first instrument of this kind was made by Lord Kelvin in 1878; two were put forth by Henrici in 1894, and among others is that of Michelson and Stratton, constructed in 1898. By means of a Henrici machine, when the stylus of the instrument is moved along the curve of the

[1] H. N. Davis, "The Longitudinal Vibrations of a Rubbed String," *Proceedings of the American Academy of Arts and Sciences*, vol. 41, 1906, pp. 691–727 + 3 plates.

[2] *Philosophical Magazine*, 1905–1907, 1909, 1910, 1912.

[3] *Indian Association for the Cultivation of Science, Bulletin no. 15*, Calcutta, 1918, part 1, 158 pp.

[4] "Beiträge zur Vokallehre," I–III, *Abhandlungen der mathematischen-physikalischen Klasse der Sächsischen Akademie der Wissenschaften*, vol. 38, 1921–22; the sub-title for part III is: "Die automatische Analyse der gesungenen Vokale."

[5] New York, 1916.

[6] Compare G. A. Carse and J. Urquhart, "Harmonic Analysis" in *Modern Instruments and Methods of Calculation*, edited by E. M. Horsburgh, London, 1914, pp. 220–247. H. de Morin, *Les Appareils d'Intégration*, Paris, 1913, pp. 147–190. W. Dyck, *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente*, Munich, 1892, pp. 212–222, and 1893, pp. 34–36.

photograph the numerical values of the coefficients in the corresponding Fourier series may be read off. In 1910 Miller reconstructed a Henrici analyzer so as to care for thirty components with precision.[1] That is, a tone made up of 30 simple tones can be analyzed and the coefficients of the corresponding number of terms in the Fourier series written down. Regarding this kind of work I must not pause to do more than suggest that it has applications of high importance for tone generation and for perfecting musical instruments.

In concluding references to activities of the past one hundred years, I should, however, take time to recall that when, in these latest days, there arose a question as to the manner in which our present musical notation for equal temperament scales could best be simplified, it was a former president of this Association who brought forward a scheme[2] so beautifully simple that further advance in this regard cannot be imagined.

Speculation as to music of the future furnishes tempting themes for discussion. I shall merely mention some of these in conclusion.

The possibilities of melody and harmony in the trinity of musical fundamentals have, within the limits of our hampering scale systems, been largely explored. But what is to be the future of the almost untried vast *rhythmic* possibilities so intimately bound up with mathematical relations? Practically all of our music is modulo 2, 3, 4, 6, 8, 9, 12; but why not have modulo 5, 7, 10, 11, 13, for example, or combinations of these moduli in the same measures?

Again, is it not within the realms of possibility that some day the inadequacies of the present vehicle of musical expression may lead us to revive some of the ideals of Greek music during the golden period of Aristoxenus?

And yet again, when we recall the many results in connection with musical tones found empirically by makers of musical instruments but for which no satisfactory explanations have been furnished by the mathematician or physicist, may we not conclude that when such explanations are forthcoming, a new era shall have dawned in the evolution of musical instruments?

---

[1] D. C. Miller, *loc. cit.*, p. 100.

[2] E. V. Huntington, "A simplified musical notation," *The Scientific Monthly*, vol. 11, 1920, pp. 276–283. The following footnote occurs on p. 282: "By the addition of an occasional single letter (Ellis's 'duodenal'), the new notation can even be made to indicate the note required for 'just intonation' with complete accuracy. A discussion of this phase of the subject is reserved for another occasion." In *Science*, 1921, T. P. Hall endorses the scheme (January 28, pp. 91–92) and R. P. Baker raises several pertinent considerations (March 11, pp. 235–236).

THE GEOMETRY OF RIEMANN AND EINSTEIN.

By JAMES PIERPONT, Yale University.

PART II

7. Some Metrical Relations. The equations (49), as we have seen, are thus the parametric equations of an R-straight in terms of the parameter s. To the parameter value s' will correspond a point z'. For z to coincide with z' it is necessary that $z_i' = z_i$ or

$$a_i \cos\frac{s}{R} + b_i \sin\frac{s}{R} = a_i \cos\frac{s'}{R} + b_i \sin\frac{s'}{R}; \qquad i = 1, \cdots, 4,$$

or

$$a_i \left\{ \cos\frac{s}{R} - \cos\frac{s'}{R} \right\} = b_i \left\{ \sin\frac{s'}{R} - \sin\frac{s}{R} \right\},$$

or

$$- a_i \sin\frac{s + s'}{2R} \sin\frac{s - s'}{2R} = b_i \cos\frac{s' + s}{2R} \sin\frac{s' - s}{2R} \cdot \tag{54}$$

Now,

$$\sum a_i b_i = 0.$$

Therefore,

$$\sum a_i b_i \cos\frac{s + s'}{2R} \sin\frac{s - s'}{2R} = 0,$$

or, using (54),

$$\sum a_i^2 \sin\frac{s + s'}{2R} \sin\frac{s - s'}{2R} = 0.$$

Therefore,

$$\sin\frac{s + s'}{2R} \sin\frac{s - s'}{2R} = 0.$$

Thus either $s + s' = 2R\pi n$, or $s - s' = 2R\pi n$, n an integer.

The first of these in (54) leads to $\sin s/R = 0$ which is obviously impossible; we must therefore adopt the second. Hence as s increases from $s = 0$, z' never coincides with z until $s = 2\pi R$. Hence the

THEOREM: *The length of all R-straights is $2\pi R$ in R-measure.*

As in 2-way space we may define a *restricted R-geometry* as follows:

1°. Diametrically opposite points on the fundamental sphere F are to be regarded as one and the same point.

2°. Points outside the fundamental sphere are to be regarded as non-existent or imaginary.

We may denote this geometry by R^*. In this R^*-geometry
Two points determine an R^*-straight.
Two R^*-straights cut once and only once.
The R^*-straight is closed and its length is πR.

The truth of the last statement is seen thus: Let the R^*-straight meet F in A, A'. If A has the coördinates $(a_1, a_2, a_3, 0)$, A' has the coördinates $(-a_1, -a_2, -a_3, 0)$. In (49) let us begin to measure s at A. The point z reaches A' when $s = \pi R$ and not before. As $A = A'$ in R^* this is the length of a straight in R^*.

Let us multiply the equations (49) by a_i and add; we get

$$\sum a_i z_i = \cos\frac{s}{R}\sum a_i^2 + \sin\frac{s}{R}\sum a_i b_i,$$

or using, (50), (51),

$$\cos\frac{s}{R} = \frac{a_1 z_1 + \cdots + a_4 z_4}{R^2}; \qquad s = \text{dist }(a, z). \tag{55}$$

This formula which gives the distance between two arbitrary points a, z is fundamental; it is of course measured along the R-straight joining a and z.

We have seen that

$$u_1 z_1 + \cdots + u_4 z_4 = 0 \tag{56}$$

defines an R-plane. We may call $u_1, \cdots, u_4$ the *plane coördinates* of this plane since when these are given, this plane is determined. As the plane whose coördinates are ku_1, ku_2, ku_3, ku_4 is the same as (56), we may suppose that the u's satisfy the relation:

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = R^2, \tag{57}$$

i.e., the z-coördinates of a point and the coördinates of a plane satisfy the same type of relation (24), (56). This fact is important.

Let us find the angle in R-measure between the plane (56) and

$$v_1 z_1 + \cdots + v_4 z_4 = 0, \tag{58}$$

where the plane coördinates v_i satisfy the relation

$$v_1^2 + \cdots + v_4^2 = R^2. \tag{59}$$

If we pass to the x-coördinates, the two equations (56), (58) become as we have already seen

$$u_4(x_1^2 + x_2^2 + x_3^2) - 4R(u_1 x_1 + u_2 x_2 + u_3 x_3) - 4u_4 R^2 = 0,$$
$$v_4(x_1^2 + x_2^2 + x_3^2) - 4R(v_1 x_1 + v_2 x_2 + v_3 x_3) - 4v_4 R^2 = 0.$$

The E-angle θ between these spheres is given [1] by

$$\cos\theta = \frac{u_1 v_1 + \cdots + u_4 v_4}{R^2}, \tag{60}$$

[1] *Cf.* Snyder and Sisam, *Analytic Geometry of Space*, p. 56.

using (57), (59). As angles in R-measure have the same value as in E-measure, (60) gives the angle in R-measure between the R-planes u, v. The reader should note the similarity between the formula (55) for the distance between two points and the formula (60) for the angle between two planes. On the right side of (55) we have the coördinates of the points; on the right side of (60) the coördinates of the two planes.

From (60) we see that when $\theta = \pi/2$,

$$u_1v_1 + \cdots + u_4v_4 = 0, \tag{61}$$

and conversely.

The distance s between the two points $a = (a_1, \cdots, a_4)$ and $-a = (-a_1, \cdots, -a_4)$ is given by (55), *viz.*,

$$\cos\frac{s}{R} = -\frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{R^2} = -1.$$

Therefore, $s = \pi R$. Thus one of the points a or $-a$ falls within F, the fundamental sphere. For this reason we may regard $(a_1, \cdots, a_4)$ or $(-a_1, \cdots, -a_4)$ as defining the same point in R^*.

8. Poles and Polars. Two points a, z whose distance apart is $s = \pi R/2$ are said to be *conjugate*. Setting this value of s in (55), we get

$$a_1z_1 + \cdots + a_4z_4 = 0. \tag{62}$$

Thus the locus of all points z conjugate to a given point a is an R-plane (62). We say (62) is the polar plane of a and call a the pole of (62). The coördinates $a_1, \cdots, a_4$ of the point a satisfy the relation (24), that is, $a_1^2 + \cdots + a_4^2 = R^2$. The point $-a$ whose coördinates are $-a_1, \cdots, -a_4$ also has (62) as polar plane. Thus $-a$ is also a pole of (62), and a given plane has two poles in R and one pole in R^*. To avoid confusion, we will suppose the following to refer to R^*; the extension to R is obvious. From the above we have the

THEOREM 1. *The polar plane of the point (a_1, a_2, a_3, a_4) has the plane coördinates $a_1, \cdots, a_4$ and the pole of the plane whose coördinates are $a_1, \cdots, a_4$ is the point $a = (a_1, \cdots, a_4)$.*

Let δ be the distance between the two points a, b. Then by (55)

$$\cos\frac{\delta}{R} = \frac{a_1b_1 + \cdots + a_4b_4}{R^2}. \tag{63}$$

The polar planes of a, b are

$$a_1z_1 + \cdots + a_4z_4 = 0,$$
$$b_1z_1 + \cdots + b_4z_4 = 0,$$

and by (60), if θ is the angle between these planes,

$$\cos\theta = \frac{a_1b_1 + \cdots + a_4b_4}{R^2}.$$

Comparing this with (63) gives

$$\cos\frac{\delta}{R} = \cos\theta, \qquad \text{or} \qquad \theta = \frac{\delta}{R}. \tag{64}$$

Hence the

THEOREM 2. *The angle between two planes is $1/R$ the distance between their poles.*

Let us find the distance δ from the point $P = (a_1, \cdots, a_4)$ to the plane

$$u_1z_1 + \cdots + u_4z_4 = 0.$$

The coördinates of the pole Q of this plane are $(u_1, \cdots, u_4)$. Let the R-straight joining Q and P cut this plane in the point U. Then $\overline{QU} = \pi R/2$, since Q, U are conjugate. From $\overline{QP} + \overline{PU} = \overline{QU}$ we have, if we call $\overline{QP} = d$, $d + \delta = \pi R/2$, since by hypothesis $\overline{PU} = \delta$. Then by (55)

$$\cos\frac{d}{R} = \frac{u_1a_1 + \cdots + u_4a_4}{R^2} = \cos\frac{\pi R/2 - \delta}{R} = \sin\frac{\delta}{R}\,.$$

Thus δ is given by

$$\sin\frac{\delta}{R} = \frac{a_1u_1 + \cdots + a_4u_4}{R^2}\,. \tag{65}$$

Let α be the polar plane of A; if B is a point of α, the polar plane β of B passes through A. For B lying in α is conjugate to A; as β contains all points conjugate to B it must contain A.

Let the above planes α, β cut [1] in the line l; let C be a point of l and let γ be the polar of C. Then γ passes through the join m of A, B. For C lying on α is conjugate to A; also since C lies on β, it is conjugate to B. But γ containing all points conjugate to C must contain A, B and hence their join m.

We call l, m *reciprocal polars*. The foregoing gives the

THEOREM 3. *When a point describes an R-straight l, its polar plane rotates about the reciprocal polar m of l.* Any point of l is conjugate to any point of m.

We have also the

THEOREM 4. *Any plane through l is perpendicular to any plane through m.*

For their poles L, M lie respectively on m and l and dist. $(L, M) = \delta = \pi R/2$, as L, M are conjugate. This value of δ in (64) gives $\theta = \pi/2$.

Let ABD be the polar plane of C cutting l in D; we show easily that the polar plane of D is the plane ABC. These four points A, B, C, D and their opposite faces form a tetrahedron called a *polar tetrahedron*. The polar plane of each vertex is the opposite face, opposite edges are reciprocal polars, and two faces meeting on an edge are perpendicular.

Let A_1, A_2, A_3, A_4 be a polar tetrahedron, T, whose vertex A_4 is the origin O and whose vertices A_1, A_2, A_3 lie respectively on the x_1, x_2, x_3-axes. The length of the edge OA_i, $i = 1, 2, 3$, in R-measure is $\rho = \pi R/2$; hence by (9) its length in E-measure is $r = 2R \tan \pi/4 = 2R$. Thus the polar plane of $O = A_4$ is in E-geometry the fundamental sphere F. Obviously the z-coördinates of the vertex A_k are all zero except the kth for which $z_k = R$.

[1] The reader is requested to make for himself the figure of a tetrahedron whose vertices are A, B, C, D.

Let now $u_1z_1 + \cdots + u_4z_4 = 0$ be an arbitrary plane; if η_k is the distance of the vertex A_k from this plane, we have by (65)

$$\sin\frac{\eta_k}{R} = \frac{u_kR}{R^2} = \frac{u_k}{R}; \qquad k = 1, 2, 3, 4,$$

since by the foregoing all the terms on the right of (65) except the kth are 0. Thus

$$u_k = R\sin\frac{\eta_k}{R}. \tag{66}$$

Thus the four coördinates of a plane are simple functions of the distance of the four vertices of the above tetrahedron from this plane. There is an analogous interpretation of the z-coördinates of a point. For the plane coördinates of the face α_k opposite the vertex A_k are all 0 by (66) except the kth for which $u_k = R$. Thus if ζ_k is the distance of an arbitrary point of space $z = (z_1, \cdots, z_4)$ to the face α_k, we have by (65)

$$\sin\frac{\zeta_k}{R} = \frac{z_kR}{R^2} \qquad \text{or} \qquad z_k = R\sin\frac{\zeta_k}{R}; \qquad k = 1, \cdots, 4. \tag{67}$$

Thus the four coördinates $z_1, \cdots, z_4$ of a point are simple functions of the distance from this point to the four faces of the above tetrahedron.

Let B_1, B_2, B_3, B_4 be the vertices of another polar tetrahedron T'; let the equation of the face β_k opposite B_k be

$$b_{k1}x_1 + b_{k2}z_2 + b_{k3}z_3 + b_{k4}z_4 = 0; \qquad k = 1, \cdots, 4,$$

where as usual the plane coördinates satisfy the relation

$$\sum_j b_{kj}^2 = R^2; \qquad j = 1, \cdots, 4.$$

Since the faces of T' are mutually perpendicular, we have also

$$\sum_j b_{kj}b_{ij} = 0 \qquad \text{if} \qquad i \neq k. \tag{68}$$

Let ζ_k' be the distance of the point $P(z_1, \cdots, z_4)$ from the face β_k. Then by (65)

$$\sin\frac{\zeta_k'}{R} = \frac{b_{k1}z_1 + \cdots + b_{k4}z_4}{R^2}.$$

Let us set analogous to (67)

$$y_k = R\sin\frac{\zeta_k'}{R}; \tag{69}$$

we may call $y_1, \cdots, y_4$ the coördinates of P relative to the tetrahedron T'. Comparing the last two equations gives

$$y_k = a_{k1}z_1 + \cdots + a_{k4}z_k; \qquad k = 1, \cdots, 4, \tag{70}$$

where we have set $b_{kj} = Ra_{kj}$. Thus when we pass from the original tetrahedron

T to the tetrahedron T', the new coördinates y of a point P are related to the old coördinates z by the homogeneous linear relation (70). This transformation may be represented by the accompanying table A which we read as in ordinary analytic geometry.

$$\begin{array}{c|cccc} & z_1 & z_2 & z_3 & z_4 \\ \hline y_1 & a_{11} & a_{12} & a_{13} & a_{14} \\ y_2 & a_{21} & a_{22} & a_{23} & a_{24} \\ y_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ y_4 & a_{41} & a_{42} & a_{43} & a_{44} \end{array} \qquad (A)$$

The orthogonal relations (67), (68) give

$$\begin{aligned} a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} + a_{i4}a_{j4} &= 0, \qquad i \neq j, \\ &= 1, \qquad i = j. \end{aligned} \tag{71}$$

Such bilinear expressions of the elements of two rows of the table A are called *scalar products*. We have thus: The scalar product of any two rows of A is 1 or 0 according as the two rows are the same or different. It is easily shown[1] that the same holds for two columns of A. The determinant a of the coefficients of (A) is ± 1. We shall take the $a = +1$.

9. Motion, Displacements. One of the most important notions in E-geometry is that of rigid bodies and their displacement. Rigid bodies may be moved or displaced from one position to another without altering any of their dimensions. Thus if $d\sigma$ is the distance between two adjacent points before displacement and $d\sigma'$ their distance afterwards, $d\sigma' = d\sigma$ for all such pairs. The question at once arises can bodies be displaced in R-geometry so that $ds' = ds$. The answer is "yes." For let us look at the transformation defined by (70) or by the adjoined table (A) in another way. We may in fact regard (70) as a transformation which converts every point of space z into the point $y = (y_1, \cdots, y_4)$ relative to the original tetrahedron T which we regard as fixed. Since a linear relation

$$u_1z_1 + \cdots + u_4z_4 = 0$$

goes over into a linear relation, say

$$v_1y_1 + \cdots + v_4y_4 = 0,$$

after this transformation, we see A transforms R-planes into R-planes and hence R-straights into R-straights. For this reason we call A a collineation.

To show that $ds' = ds$ after (A), we have, differentiating (70),

$$dy_k = \sum_j a_{kj}dz_j.$$

Therefore,

$$dy_k^2 = \sum_j a_{kj}dz_j \sum_i a_{ki}dz_i = \sum_{ij} a_{kj}a_{ki}dz_jdz_i; \qquad i, j = 1, \cdots, 4.$$

[1] See Snyder and Sisam, *loc. cit.*, p. 40.

Thus

$$ds'^2 = \sum_k dy_k^2 = \sum_k \sum_{ij} a_{kj}a_{ki}dz_j dz_i = \sum_{ij} dz_i dz_j \cdot \sum_k a_{ki}a_{kj}.$$

Now the last sum on the right is the scalar product of the ith and jth columns of A, hence this sum vanishes unless $i = j$, in which case it is 1. Thus

$$ds'^2 = \sum_i dz_i^2 = ds^2.$$

We have thus proved the fundamental

THEOREM: *A linear transformation* (70) *subject to the orthogonal conditions* (71) *defines a displacement, that is,* $ds' = ds$.

In particular, we see that it is possible to make a given polar tetrahedron coincide with another given polar tetrahedron.

An immediate application of the free mobility of our figures in R-space enables us to establish the

THEOREM: *The geometry on any R-plane is the same as that on an E-sphere of radius R.*

For the given R-plane α may be made to coincide with the x_1, x_2-plane without altering any dimensions in R-measure and the geometry on this plane we have considered in § 2.

Let us consider the special transformation U defined by the adjoined table. Its determinant $= 1$ and the scalar products of rows or columns are 0 or 1 as required by (71). Hence U is a displacement. Let us see how it moves the points of space.

	z_1	z_2	z_3	z_4	
y_1	$\cos\theta$	$\sin\theta$	0	0	
y_2	$-\sin\theta$	$\cos\theta$	0	0	(U)
y_3	0	0	1	0	
y_4	0	0	0	1	

First, we note that each point P on the x_3-axis is left unaltered by U. For by (30) the z-coördinates of P are:

$$z_1 = z_2 = 0, \qquad z_3 = R \sin \rho/R, \qquad z_4 = R \cos \rho/R. \tag{72}$$

After U, P has the same coördinates, hence P is unmoved.

Let us next consider the polar tetrahedron whose vertex A_4 is the origin O, while the other vertices A_1, A_2, A_3 lie on the x_1, x_2, x_3-axes, respectively. The z-coördinates of a point Q on the edge A_1A_2 are by (30)

$$z_1 = R \cos \alpha, \qquad z_2 = R \cos \alpha_2 = R \sin \alpha, \qquad z_3 = z_4 = 0, \tag{73}$$

where we have set $\alpha = \alpha_1$ to save the subscript. On applying U, the point Q has the coördinates:

$$y_1 = z_1 \cos\theta + z_2 \sin\theta = R(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = R\cos(\alpha - \theta),$$
$$y_2 = -z_1 \sin\theta + z_2\cos\theta = R\sin(\alpha - \theta); \qquad y_3 = z_3, \qquad y_4 = z_4.$$

Thus the point Q has been moved through the angle $-\theta$ along the edge A_1A_2, and the plane A_3OQ has rotated about the x_3-axis through this angle.

Let L be an arbitrary point in space; let the R-plane ω through L perpendicular to the x_3-axis cut this axis in M. U leaves ω unchanged as a whole while the R-straight ML is turned through the angle $-\theta$, leaving the length $\overline{LM}$ unaltered. We see U is a rotation about the x_3-axis in the Euclidean sense.

We wish now to consider the analogous displacement V defined by the adjoined table.

	z_1	z_2	z_3	z_4
y_1	1	0	0	0
y_2	0	1	0	0
y_3	0	0	$\cos\theta$	$\sin\theta$
y_4	0	0	$-\sin\theta$	$\cos\theta$

(V)

On applying V to the point Q lying on the edge A_1A_2, whose coördinates are given by (73), we see Q is unmoved. On the other hand, V moves the point P defined by (72) lying on the x_3-axis to the point P' whose coördinates are

$$y_1 = 0, \qquad y_2 = 0,$$
$$y_3 = R\sin\rho/R\cos\theta + R\cos\rho/R\sin\theta,$$
$$y_4 = -R\sin\rho/R\sin\theta + R\cos\rho/R\sin\theta.$$

Thus V leaves the x_3-axis unmoved as a whole. To find the distance δ that P has moved along this axis, we have by (55)

$$\cos\frac{\delta}{R} = \frac{y_1z_1 + \cdots + y_4z_4}{R^2} = \cos\theta.$$

Therefore, $\delta = R\theta$. As this is independent of the position of P on the x_3-axis, we have the

THEOREM: *The displacement V rotates all R-planes through the edge A_1A_2 through the angle θ; the poles of these planes P move through the distance $\delta = R\theta$ along the x_3-axis.*

We see the displacement V is a rotation about the edge A_1A_2 analogous to the displacement U relative to the edge A_3A_4.

Let us now see the effect of applying first U and then V to the points of space. U converts $z_1, \cdots, z_4$ into

$$y_1 = z_1\cos\theta + z_2\sin\theta, \qquad y_2 = -z_1\sin\theta + z_2\cos\theta, \qquad y_3 = z_3, \qquad y_4 = z_4,$$

while V converts the point $y_1, \cdots, y_4$ into the point z' say, whose coördinates are:

$$\begin{aligned} z_1' &= z_1\cos\theta + z_2\sin\theta, & z_2' &= -z_1\sin\theta + z_2\cos\theta, \\ z_3' &= z_3\cos\theta + z_4\sin\theta, & z_4' &= -z_3\sin\theta + z_4\cos\theta, \end{aligned} \tag{74}$$

which we can represent by the subjoined table.

	z_1	z_2	z_3	z_4
z_1'	$\cos\theta$	$\sin\theta$	0	0
z_2'	$-\sin\theta$	$\cos\theta$	0	0
z_3'	0	0	$\cos\theta$	$\sin\theta$
z_4'	0	0	$-\sin\theta$	$\cos\theta$

(W)

The determinant of W is 1 and its rows satisfy the condition of orthogonality (71); W is thus a displacement.

Since W is the result of the application of U and then V, we see that now all points on both the edges A_1A_2 and A_3A_4 are moved along these edges through the distance $\delta = R\theta$. This is true not only for these points, but for all points of space. For let $a = (a_1, \cdots, a_4)$ be an arbitrary point. On applying W it goes over into the point z whose coördinates are:

$$\begin{aligned} z_1 &= a_1\cos\theta + a_2\sin\theta, & z_2 &= -a_1\sin\theta + a_2\cos\theta, \\ z_3 &= a_3\cos\theta + a_4\sin\theta, & z_4 &= -a_3\sin\theta + a_4\cos\theta. \end{aligned} \tag{75}$$

If we regard θ as variable, these equations are the parameter equations of the path that the point a describes. The distance δ between a and z is of course measured along the R-straight joining a and z; and as yet we do not know that this straight is the same as the path curve (75). However by (55) we have

$$\cos\frac{\delta}{R} = \frac{a_1z_1 + \cdots + a_4z_4}{R^2}.$$

On substituting the values of $z_1, \cdots, z_4$ given in (75) this gives

$$\cos\frac{\delta}{R} = \cos\theta.$$

Therefore, $\delta = R\theta$, which was to be proved.

Let ds be an element of arc on the path curve (75), we wish to show that

$$ds = Rd\theta \qquad \text{or} \qquad \frac{ds}{d\theta} = R. \tag{76}$$

In fact let z, z' be the two points on (75) corresponding to θ and $\theta + d\theta$. Then

$$\cos\frac{ds}{R} = \frac{z_1z_1' + \cdots + z_4z_4'}{R^2} = \cos d\theta.$$

Therefore, $ds/R = d\theta$, which is (76).

To show that the path curves (75) are R-straights we employ the equations (48) which define straights. Changing the variable from s to θ in (48) by using (76) we have

$$\frac{d^2z_i}{d\theta^2} + z_i = 0, \qquad i = 1, \cdots, 4. \tag{77}$$

The curve (75) will be a R-straight if $z_1, \cdots, z_4$ as there given satisfy (77). That this is so is seen at once on differentiating twice the equation (75). We have thus the

THEOREM: *The displacement W causes all the points of space to describe R-straights of length $R\theta$. All R-planes through the edge A_3A_4 are rotated through an angle $-\theta$ while planes through the edge A_1A_2 rotate through the angle θ. It is thus a double rotation.*

10. Clifford Parallels. We have seen that all straights in an R-plane meet, there are no parallels in this Euclidean sense. From another standpoint there are; in fact, one property of E-parallels is that their distance apart is constant. Such straights exist in R-geometry and are called *Clifford parallels* after their discoverer.

Let P be an arbitrary point of space; from this point drop[1] an R-perpendicular on the x_3-axis meeting this in the point L. We now apply the transformation W. In allowing θ to vary from $\theta = 0$ to $\theta = \theta$, the point P describes an R-straight, moving to P' say, such that $\overline{PP'} = R\theta$. Meanwhile L moves along the x_3-axis

[1] The reader is requested to draw the figure.

to L' such that $\overline{LL'} = R\theta$ also. The R-straight PL goes over into the R-straight $P'L'$ also perpendicular to the x_3-axis and $\overline{P'L'} = \overline{PL}$ in R-measure. Thus the path curve l which P describes under the displacement W is an R-straight such that each point of l is at a constant R-distance from the x_3-axis which is also an R-straight. These two lines are thus parallel in the sense of Clifford.

There is another Clifford parallel through the given point P which we obtain as follows. In the rotation V let us replace θ by $-\theta$, we get a rotation V_1 about the edge A_1A_2 in the opposite direction to V. If we combine this with U we get the double rotation W_1 defined in the adjoined table. This is entirely analogous to W. The equations of the path curve through the point a are analogous to (75):

	z_1	z_2	z_3	z_4
z_1'	$\cos\theta$	$\sin\theta$	0	0
z_2'	$-\sin\theta$	$\cos\theta$	0	0
z_3'	0	0	$\cos\theta$	$-\sin\theta$
z_4'	0	0	$\sin\theta$	$\cos\theta$

(W_1)

$$\begin{aligned} z_1 &= a_1\cos\theta + a_2\sin\theta, & z_2 &= -a_1\sin\theta + a_2\cos\theta, \\ z_3 &= a_3\cos\theta - a_4\sin\theta, & z_4 &= a_3\sin\theta + a_4\cos\theta. \end{aligned} \tag{78}$$

The same reasoning as above shows that these lines are parallel to the x_3-axis or the edge A_3A_4. Thus through a given point a there pass two Clifford parallels relative to the x_3-axis.

We wish to show now that these Clifford parallels lie on a certain *torus* whose axis is the x_3-axis. Let P be an arbitrary point of space. Let the meridian plane μ through the x_3-axis and P cut the edge A_1A_2 of the polar tretrahedron $A_1, \cdots, A_4$ we have used so often in the point Q. The coördinates $y_1, \cdots, y_4$ of Q are as we saw in (73):

$$y_1 = R\cos\alpha, \qquad y_2 = R\sin\alpha, \qquad y_3 = y_4 = 0. \tag{79}$$

Thus the equation of the meridian plane μ is

$$z_2 - z_1\tan\alpha = 0 \qquad \text{or} \qquad z_2/z_1 = \tan\alpha; \tag{80}$$

and the coördinates $z_1, \cdots, z_4$ of P lying in this plane must satisfy (80). The R-straight through P, Q cuts the x_3-axis in L, say, and as the edge A_1, A_2 is the reciprocal polar of the x_3-axis, *i.e.*, of the edge A_3A_4, we have $\overline{QL} = \overline{QP} + \overline{PL} = \pi R/2$. If we set $\overline{PL} = \delta$, $\overline{PQ} = d$, this last relation gives $d + \delta = \pi R/2$.

To find the surface on which P lies we have only to remember that the double rotation W rotates the meridian plane μ about the x_3-axis and P moves in this plane so that $\overline{LP} = \delta$ is constant, since all lengths are unchanged by a displacement. We have now, by (55) and (79),

$$\begin{aligned} \cos\frac{d}{R} &= \frac{y_1z_1 + \cdots + y_4z_4}{R^2} = \frac{R(z_1\cos\alpha + z_2\sin\alpha)}{R^2} \\ &= \frac{Rz_1(\cos\alpha + z_2/z_1\sin\alpha)}{R^2} = \frac{z_1(\cos\alpha + \tan\alpha\sin\alpha)}{R} \end{aligned}$$

by (80). Hence

$$\cos\frac{d}{R} = \frac{z_1\sec\alpha}{R} = \sin\frac{\delta}{R} \qquad \text{as} \qquad d + \delta = \frac{\pi R}{2}\,.$$

Thus

$$z_1^2 \sec^2\alpha = R^2 \sin^2 \delta/R \qquad \text{or} \qquad R^2 \sin^2 \delta/R = z_1^2 (1 + \tan^2 \alpha)$$

or, using (80),

$$R^2 \sin^2 \delta/R = z_1^2 + z_2^2. \tag{81}$$

Hence replacing z_1, z_2 by their x values as given by (22),

$$(x_1^2 + x_2^2) \frac{16R^4}{(r^2 + 4R^2)^2} = R^2 \sin^2 \frac{\delta}{R}; \qquad r^2 = x_1^2 + x_2^2 + x_3^2.$$

Thus finally the point P moves on the surface

$$(x_1^2 + x_2^2 + x_3^2 + 4R^2)^2 = 16R^2(x_1^2 + x_2^2) \operatorname{cosec}^2 \delta/R. \tag{82}$$

This is a torus in E-geometry, generated by rotating the E-circle

$$(x_1 - 2R \operatorname{cosec} \delta/R)^2 + x_3^2 = 4R^2 \cot^2 \delta/R \tag{83}$$

about the x_3-axis. The coördinates of the center C of this circle are:

$$x_1 = 2R \operatorname{cosec} \delta/R, \qquad x_2 = 0, \qquad x_3 = 0; \tag{84}$$

its radius

$$t = 2R \cot \delta/R. \tag{85}$$

Since every R-straight lies in an E-plane passing through the origin, the Clifford parallels to the x_3-axis are plane sections of this torus. We show that these sections are made by the planes ω which are tangent internally to the torus.

For let P be one of the two points where a Clifford parallel γ meets the fundamental sphere. The meridian plane μ passing through the x_3-axis and P cuts[1] the fundamental sphere in a circle of radius $2R$ in E-measure. It cuts out two circles K, K' from the torus whose centers C, C' lie on a line OM perpendicular to the x_3-axis such that $\overline{OC} = \overline{OC'} = 2R \operatorname{cosec} \delta/R$ in E-measure, by (84), and the radius of each is $t = 2R \cot \delta/R$ by (85). Since $\overline{OC}^2 = \overline{OP}^2 + \overline{PC}^2$, the triangle OPC is a right triangle and OP is tangent to the circle K. Since curves meeting at P and drawn on the torus have their tangents at P lying in the tangent plane to the torus at P, the plane ω passes through OP and is tangent to the torus at P.

Let l, l' be the two Clifford parallels through the point $a = (a_1, \cdots, a_4)$. Giving θ the value $d\theta$ in (75), (78), we get

$$\begin{array}{lllll} dz_1 = a_2 d\theta, & dz_2 = -a_1 d\theta, & dz_3 = a_4 d\theta, & dz_4 = -a_3 d\theta, & \text{on } l, \\ \delta z_1 = a_2 d\theta, & \delta z_2 = -a_1 d\theta, & \delta z_3 = -a_4 d\theta, & \delta z_4 = a_3 d\theta, & \text{on } l'. \end{array}$$

The angle φ between these two arcs of length $ds = \delta s = R d\theta$ is given by

$$\begin{aligned} \cos \varphi &= \frac{dz_1}{ds} \cdot \frac{\delta z_1}{\delta s} + \cdots + \frac{dz_4}{ds} \cdot \frac{\delta z_4}{\delta s} \\ &= \frac{(a_2^2 + a_1^2 - a_4^2 - a_3^2) d\theta^2}{R^2 d\theta^2} = \frac{a_2^2 + a_1^2 - a_4^2 - a_3^2}{R^2}. \end{aligned} \tag{86}$$

[1] It will aid the reader to draw the figure.

Now the distance δ of a from the x_3-axis is given by (81), *viz.*,

$$\sin^2\frac{\delta}{R} = \frac{a_1^2 + a_2^2}{R^2} = \cos^2\left(\frac{\pi}{2} - \frac{\delta}{R}\right). \tag{87}$$

As

$$\cos^2\frac{1}{2}\varphi = \frac{1+\cos\varphi}{2} = \frac{a_1^2 + a_2^2}{R^2}$$

by (86) we have, comparing this with (87),

$$\cos\frac{1}{2}\varphi = \cos\left(\frac{\pi}{2} - \frac{\delta}{R}\right).$$

Therefore,

$$\frac{1}{2}\varphi = \frac{\pi}{2} - \frac{\delta}{R}.$$

If, instead of φ, we use the supplementary angle $\psi = \pi - \varphi$, we get

$$\psi = 2\delta/R.$$

Since a displacement enables us to make any R-straight coincide with the x_3-axis leaving all distances and angles unaltered, we have the

THEOREM: *Through any point of space P there pass two Clifford parallels l, l' to a given line m. If ψ is the angle between l, l' and $\delta = PQ =$ the R-distance of P to m, $\psi = 2\delta/R$. The R-plane in which l, l' lie is perpendicular to the R-straight PQ, and ψ is bisected by the R-plane through m and P.* The surface on which these parallels lie is called a Clifford surface.

11. Areas and Volumes. These notions we define briefly. Suppose in E-geometry we pass from rectangular coördinates x_1, x_2, x_3 to curvilinear coördinates u_1, u_2, u_3. In the new coördinates

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

becomes

$$ds^2 = \sum a_{ij}du_idu_j; \qquad a_{ij} = a_{ji}, \qquad i, j = 1, 2, 3. \tag{88}$$

We call

$$a = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{89}$$

the determinant of the quadratic differential form (88). It is shown in the calculus [1] that the element of volume

$$dv = dx_1dx_2dx_3$$

becomes, in terms of the new variables,

$$dv = \sqrt{|a|}\,du_1du_2du_3. \tag{90}$$

[1] See, for example, Goursat-Hedrick, *Mathematical Analysis*, vol. 1, Boston, 1904, p. 305.

We shall adopt (90) as our definition of volume in R-geometry where ds^2 is replaced by (5) or one of its equivalent forms, as (10) for example.

Similarly it is shown in the calculus that if

$$\begin{aligned} ds^2 &= a_{11}du_1^2 + 2a_{12}du_1du_2 + a_{22}du_2^2 \\ &= \sum a_{ij}du_idu_j, \qquad a_{ij} = a_{ji}, \qquad i, j = 1, 2, \end{aligned} \tag{91}$$

defines the element of arc on a surface S, then

$$dA = \sqrt{|a|}\,du_1du_2 \tag{92}$$

defines the element of area on S. Here a is the determinant of (91) or

$$a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \tag{93}$$

We adopt (92) as our definition of area in R-geometry, the form (91) being now expressed in R-measure.

An important and very necessary property of these definitions of area and volume is that they are unaltered after a displacement; that is to say, if for example v is the volume of a body B, let B be moved into the position B'. If v' is the volume of B', then $v = v'$.

To illustrate these definitions let us find the area and volume of an R-sphere. This we define as the locus of all points P, whose distance in R-measure from a fixed point C is constant. This distance $\overline{PC} = \rho$, say, is of course measured along the R-straight PC; we call PC a radius.

Without loss of generality, we may suppose the center C is the origin and ds is given by (10). Then

$$a = \begin{vmatrix} 1 & 0 & 0 \\ 0 & R^2 \sin^2 \rho/R & 0 \\ 0 & 0 & R^2 \sin^2 \rho/R \sin^2 \theta \end{vmatrix}.$$

Therefore

$$\sqrt{a} = R^2 \sin^2 \frac{\rho}{R} \sin \theta.$$

Thus

$$\begin{aligned} v &= R^2 \int_0^\rho \sin^2 \frac{\rho}{R}\, d\rho \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi \\ &= 4\pi R^3 \int_0^\rho \sin^2 \frac{\rho}{R} \cdot d\left(\frac{\rho}{R}\right) = 2\pi R^2 \rho - \pi R^3 \sin \frac{2\rho}{R}. \end{aligned} \tag{94}$$

The radius of the fundamental sphere F in R-measure is $\rho = \pi R/2$. This in (94) gives as the volume of all R^* space the following:

$$V_F = \pi^2 R^3. \tag{95}$$

For $\rho = \pi R$, (94) becomes

$$V_\infty = 2\pi^2 R^3, \tag{96}$$

the volume of all R-space.

To find the area of an R-sphere of radius ρ, we may suppose as before that its center is at O. As $\rho =$ constant on this sphere S, the element of arc ds on S is obtained by setting $d\rho = 0$ in (10), which gives

$$ds^2 = R^2 \sin^2 \frac{\rho}{R} (d\theta^2 + \sin^2 \theta d\varphi^2).$$

This takes the place of (91); hence

$$a = \begin{vmatrix} R^2 \sin^2 \rho/R & 0 \\ 0 & R^2 \sin^2 \rho/R \sin^2 \theta \end{vmatrix}$$

and

$$A = R^2 \sin^2 \frac{\rho}{R} \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi = 4\pi R^2 \sin^2 \frac{\rho}{R} \cdot \qquad (97)$$

Let S be an R-sphere of radius ρ in R-measure and center C. We show that S is also an E-sphere whose center of course is not C in general. For let $z_1, \cdots z_4$ be the coördinates of a point P of S and let $A_1, \cdots A_4$ be the coördinates of its center C. Then

$$\cos \frac{\rho}{R} = \frac{A_1 z_1 + \cdots + A_4 z_4}{R^2} = k,$$

where k is a constant since ρ is constant. Hence, setting $- kR^2 = B$,

$$A_1 z_1 + A_2 z_2 + A_3 z_3 + A_4 z_4 + B = 0.$$

Replacing the z's by the x's, using (22) and (23), we get

$$4R^2(A_1 x_1 + A_2 x_2 + A_3 x_3) + A_4(4R^2 - r^2) + B(r^2 + 4R^2) = 0$$

or

$$(x_1^2 + x_2^2 + x_3^2)(B - A_4) + 4R^2(A_1 x_1 + A_2 x_2 + A_3 x_3) + 4R^2(B + A_4) = 0$$

which is an E-sphere.

Let us define a circle in R-geometry as the locus of all points in an R-plane at a constant distance from a fixed point in this plane. The foregoing shows that R-circles are also E-circles with different center in general. The intersection of two R-spheres or the intersection of an R-plane with an R-sphere is an R-circle.

The length in R-measure of an R-circle of radius ρ is

$$s = 2\pi R \sin \rho/R$$

and its area is by (92)

$$A = 2\pi R^2(1 - \cos \rho/R).$$

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. THE CONSEQUENCES OF ROLLE'S THEOREM.

By A. A. BENNETT, University of Texas.

Rolle's Theorem may be stated as follows:

If $F(x)$ *be defined as a one-valued function on the interval*[1] $a \leq x \leq b$, *and* (1) *vanishes at* $x = a$ and $x = b$, (2) *is continuous from within the interval at* a *and at* b, (3) *has a derivative at every point of the segment* a, b; *then there exists a point* ξ *of the segment such that* $F'(\xi) = 0$. We will assume that this theorem has been established in the usual manner.

Among the numerous important consequences of this theorem that have been enumerated there are two involving only first derivatives, that are given in most texts on the calculus. One due to Lagrange and given in all standard modern texts states that under suitable conditions

$$\frac{f(b) - f(a)}{b - a} = f'(\xi), \ \xi \textit{ being some point between } a \textit{ and } b.$$

The other, due to Cauchy, and somewhat less extensively quoted, states that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}, \ \xi \textit{ as before.}$$

The former, on account of its extensive utility, is often called the "law of the mean," or "mean value theorem," the latter being styled, in contrast, the "second law of the mean," or "extended mean value theorem." These names have the advantage of avoiding reference to the discoverers or reputed discoverers, since the ascribing to historical personages is usually controversial, often misleading, of little educational advantage, and of no logical import. On the other hand, these particular terms are unsatisfactory, since many writers call the second of these extensions the "theorem of the mean" and prove it first.

Even the clearest texts do not hesitate in this connection to introduce apparently artificial functions which can be identified with the F of Rolle's theorem. These functions are justified by the fact that they serve to establish the desired theorem as a consequence of a known theorem, but are not psychologically motivated. The result is that the student is mystified, and the particular form of the function is difficult to keep in mind. There is no need of this artificiality, and in the case of Lagrange's theorem, there seems to be no excuse for it. When

[1] The term *interval* will be used to include the end-points, assumed distinct, while *segment* refers to the same set exclusive of end-points.

Lagrange's theorem of the mean is proved independently of Cauchy's, the function used for F is usually although not universally that given by the following obvious geometrical consideration. The ordinate difference between the curve and the chord (which becomes the ordinate under a simple transformation of axes) obviously satisfies the conditions required for F, and is therefore used. Osgood mentions this fact but there is scarcely a single foreign treatise that hints at the meaning of the function used. In the case of Cauchy's theorem, silence seems unanimous. This is the more surprising since this may be given the same geometrical interpretation as before, by using the independent variable as a parameter. If $f(t)$ be called y, and $g(t)$ be called x, Cauchy's theorem like Lagrange's (although now applicable to a wider class of curves) states that there exists a point on the arc for which the tangent is parallel to the chord.

The proof of Cauchy's theorem is not materially simplified by first proving Lagrange's, although Lagrange's is an immediate corollary of Cauchy's if $g(x)$ be identified with x. If both theorems are to be demonstrated, the total space devoted to proof will be reduced if Cauchy's be proved from Rolle's theorem, and Lagrange's be inferred as a corollary. Since the hypothesis of Rolle's theorem is much more restrictive than that of Cauchy's, some new functions must be introduced. We may either state the theorem to be proved, and then seek to relate it to Rolle's theorem, or we may set up a simple function which obviously satisfies Rolle's theorem and as obviously leads to Cauchy's. We shall do the former, but by reversing the steps we would follow the latter course.

We may write the conclusion of Cauchy's theorem in the following symmetrical algebraic form: There exists at least one value, ξ, between a and b such that

$$\begin{vmatrix} x'(\xi) & y'(\xi) & 0 \\ x(a) & y(a) & 1 \\ x(b) & y(b) & 1 \end{vmatrix} = 0. \tag{1}$$

This[1] may be compared with the more usual form and identified with it, or may be directly verified from the first principles of the analytic geometry of plane vectors as the condition that the infinitesimal vector from the origin to $[x'(\xi), y'(\xi)]$ shall be parallel to the line joining $[x(a), y(a)]$ and $[x(b), y(b)]$. It is at once obvious that (1) is of the form $F'(\xi) = 0$, where $F(t)$ satisfies the condition

$$F(t) \equiv \begin{vmatrix} x(t) & y(t) & 1 \\ x(a) & y(a) & 1 \\ x(b) & y(b) & 1 \end{vmatrix}. \tag{2}$$

This in turn is on the face of things one of the simplest possible functions involving the two arbitrary functions $x(t)$ and $y(t)$, which will vanish for $t = a$ and $t = b$, as desired for Rolle's theorem.

It is clear from the form of the proof that in place of (2) we might have

[1] Vivanti, *Analisi Infinitesimale*, page 100, uses this expression.

taken[1]

$$F(t) \equiv \begin{vmatrix} x(t) & y(t) & z(t) \\ x(a) & y(a) & z(a) \\ x(b) & y(b) & z(b) \end{vmatrix}, \tag{3}$$

or even

$$F(t) \equiv \begin{vmatrix} x(t) & y(t) & z(t) & 1 \\ x(a) & y(a) & z(a) & 1 \\ x(b) & y(b) & z(b) & 1 \\ A & B & C & 1 \end{vmatrix}. \tag{4}$$

From (4) we obtain the geometric theorem that there is at least one point on the general space arc considered at which the tangent line is parallel to the plane through (A, B, C) and the chord, the point (A, B, C) being any given point not on the line of the chord. From (3) we have the special case in which (A, B, C) is the origin.

II. Note on the Comparison of Aggregates.

By H. L. Slobin, University of New Hampshire.

Professor R. S. Underwood in his article in the Monthly (*1922*, 346) supplements a theorem with the words "The above theorem is interesting in that it brings out in a striking way for pedagogical purposes the great preponderance of irrational over rational numbers."

It is obvious that in any continuous interval there is a correspondence between the aggregate of the values of the argument of the trigonometric functions and the aggregate of the values of the functions themselves. The values of the functions are of two classes, the rational and the irrational. The irrational are of two classes, the algebraic irrational and the transcendental irrational. The irrationals determined by Professor Underwood are of the algebraic irrational class only. The method of comparing two aggregates is by their cardinal number; *i.e.*, the power or potency. The algebraic numbers are enumerable, and in particular the algebraic irrationals are enumerable, and the rational numbers are enumerable, and hence the aggregate of rational numbers and the aggregate of algebraic irrational numbers are equivalent. The cardinal number of the irrational or of the transcendental numbers in any interval is C, while that of an enumerable set is $\aleph_0$; where $C > \aleph_0$. It may indeed be permissible to speak in general of the "preponderance of irrational numbers over rational numbers," but I do not think it permissible or advisable thus to speak in comparing two enumerable aggregates such as those irrationals considered by Professor Underwood.

[1] Epstein in Pascal's *Repertorium*, 2d ed., page 469, gives (3).

RECENT PUBLICATIONS.

EDITED BY D. C. GILLESPIE, Cornell University, Ithaca, N. Y., to whom communications should be sent.

REVIEWS.

Vector Analysis and the Theory of Relativity. By F. D. MURNAGHAN. Baltimore, The Johns Hopkins Press, 1922. 8vo. x + 125 pp. Price $2.75.

The statement has commonly been attributed to Einstein at the time that he first published the generalized theory of relativity that there were only twelve people in the world who were capable of understanding his paper. If such a statement was made, it was of course true only in the sense that at that particular time very few scholars were familiar with all the physical and mathematical theories employed in the paper, and not at all in the sense that the relativity theory is so abstruse that it must remain beyond the reach of the majority of physicists and mathematicians. In particular, Einstein made use of the "absolute differential calculus" which had been developed by Ricci and Levi-Civita about fifteen years earlier, but which was little known outside of a narrow circle of mathematical specialists.

As soon as the importance of the generalized theory of relativity was realized, and especially after the prestige it acquired from the verification in 1919 of its prediction of the amount of deviation of a light-ray passing near the sun, a demand arose for an exposition which would make the details of the theory accessible to physicists of average training. The present book, which is based on a series of lectures delivered by the author to graduate students at Johns Hopkins, is an attempt to meet this need. An introductory note by Professor Joseph S. Ames testifies to its success in doing so.

In the first chapter the concept of a tensor is introduced by the aid of line and surface integrals; it is shown that for an integral along a curve to have a value independent of the particular system of coördinates used, the functions in the integrand must be the components of a covariant vector or tensor of the first rank, and similarly in a surface integral the functions must form a covariant tensor of the second rank, etc. The general definitions of covariant, contravariant, and mixed tensors are then given by means of their transformation equations.

The second chapter is devoted to the algebra of tensors. The importance of tensors is due to the fact that the vanishing of a tensor (or the equality of two tensors) is a property which is independent of the coördinate system used. The generalized theory of relativity regards all systems of coördinates as on the same footing, and aims to express the laws of physics in a form which will be the same for all observers, regardless of their relative motions. This object is attained if they can be expressed by the vanishing of tensors.

In chapters III and IV the further theory of tensors is developed, and its character as a generalization of the ordinary vector analysis adapted to all types of space and to all coördinate systems is brought out. The latter part of chapter IV and most of chapter V are devoted to applications of the tensor analysis to certain problems in electromagnetic theory.

In chapter VI the absolute differential calculus is taken up, leading to the introduction of the tensor of the second rank whose vanishing gives Einstein's law of gravitation. The chapter ends with a discussion of the curvature of space. In the seventh and last chapter the preceding analysis is applied to the classical problems of relativity, including the metrical character of the space near a heavy body, the motion of a particle under Einstein's law of gravitation, with special reference to Mercury, and the deviation of a light-ray in a gravitational field.

Misprints and minor discrepancies of notation are rather numerous, but most of them are not such as to cause any difficulty to the reader. Apparently no serious attempt was made to secure consistency in such matters as enclosing superscripts in parentheses or using commas between the components of multiple subscripts, and the squares of numbers with superscripts are printed in three or four different ways.

The subject-matter of the book is rather abstract, and it is by no means easy reading. This is, however, largely inherent in the nature of the topics treated; as Eddington says, "I doubt if there is any royal road to relativity, and it is scarcely possible to make serious progress except by analytic methods."

It is an interesting feature of the history of physics that the mathematical framework of a theory often outlives the theory itself. Thus much of the mathematics of the old theory that light consists of mechanical vibrations in an elastic medium called the ether was taken over by the electromagnetic theory; and if the latter theory is to be displaced in the near future by a new one in which the concept of the ether is entirely dispensed with, as seems quite possible, we may feel sure that Maxwell's equations will continue to play a prominent part. Similarly the mathematics which has been developed in connection with Einstein's theory will probably prove to be a permanent contribution to mathematical physics, whatever may be the fate of the physical hypotheses on which that theory is now based.

P. M. BATCHELDER.

James Stirling: A Sketch of his Life and Works, along with his Scientific Correspondence. By CHARLES TWEEDIE. Oxford, Clarendon Press, 1922. 10 + 213 pp. Price $5.35 (in England, 16 s.).

We have here, set in its background of politics, travel, commerce and science, an entertaining account of the life and achievements of James Stirling, a mathematician of no mean ability, but overshadowed by such contemporaries as his friends Newton, Euler, Bernoulli, De Moivre, Cramer, and others. In addition to a spirited biography enriched with numerous extracts from private letters, this book gives a judicious summary of his now rare published works: (A) "Enumeration of Cubics," (B) "Methodus Differentialis," (C) "Contributions to the *Philosophical Transactions*." These summaries are handled in an intelligent and efficient manner, showing a mathematical comprehension of the material that is delightful as it is unusual in a biographer. The last third of the book is

devoted to the publication of a collection of Stirling's scientific correspondence, enlightened by not a few explanatory notes by the author.

Stirling is best known for his asymptotic formula in connection with the Gamma Function, but this special result is merely characteristic of his bold and ingenious treatment of many rather intractable analytical expansions, and his recourse to difference equations. While the modern mathematician will not be as indulgent as he in questions of rigor, Stirling's originality during his brief productive period sets a high standard for modern progress. It is impossible here to summarize his wide range of activities but mention might be made in passing of his geometrical work on cubic curves, his study of series, and his investigation of the shape of the earth, all questions suggestive of and perhaps suggested by his friend and counsellor, Sir Isaac Newton.

As a strictly mathematical reference book this work must remain among those of secondary importance. It has not the comprehensiveness of a complete edition of collected works, and on the other hand most of the mathematical problems here published for the first time are no longer in the forefront of mathematical investigation. Many minor points of considerable interest and some importance in the history of mathematics and in the biography of mathematicians receive fresh light from the careful antiquarian research of the author. Despite a few misprints and other trivial objections, this volume is thoroughly to be commended in its purpose, its use of material and its delightful spirit. Its appeal is indeed a wide one, and its glimpse of the heroic days of the early eighteenth century will be suggestive and entertaining to many who will not have Stirling's ambition in mathematical research.

A. A. BENNETT.

Vorlesungen über die Theorie der algebraischen Zahlen. By E. HECKE. Leipzig, Akademische Verlagsgesellschaft, 1923. Paper, 8vo. viii + 266 pages. Price (bound) 14 Swiss francs.

In view of Hecke's important recent researches in the analytic theory of numbers, his book is certain to attract special attention from all workers in this field. Its subject matter has been so well selected, many of the proofs have been so materially simplified, and the presentation is so clear and attractive that the book is recommended without reservation to those seeking an adequate introduction to the theory of algebraic numbers as developed to date. It was possible to greatly shorten and simplify numerous proofs and to refer them to a common source by employing properties of finite and infinite abelian groups established in 28 pages in Chapter II, and applied in Chapter III to develop needed theorems in ordinary elementary theory of numbers.

Chapters IV and V devote 94 pages to the classical theory of the algebra and arithmetic of algebraic fields. Chapter VI presents in 18 pages the determination by the transcendental methods of Dirichlet and Dedekind of the number of classes of ideals and theorems on the distribution of prime ideals, including a proof of the existence of an infinitude of rational primes in any arithmetical

progression. Chapter VII devotes 45 pages to quadratic fields, including determinations of the number of classes of ideals with and without the use of zeta functions. The final Chapter, VIII, presents a new proof of the general quadratic reciprocity law in an arbitrary algebraic field. This proof, which is much shorter than all previous proofs, employs Gauss sums extended to any field and theta functions with n variables.

In 1918 Landau published his brief introduction to the elementary and analytic theories of algebraic numbers. This was followed by a larger book along similar lines by Bianchi. One of these books and Hecke's new book should certainly be in the hands of any one desiring to acquaint himself with the theory of algebraic numbers to date. That theory is so fundamental that no mathematician can afford to remain ignorant of it. The book by Hecke presupposes no acquaintance whatever with the theory of numbers and leads the reader by the shortest routes to the chief results to date concerning algebraic numbers. So valuable a book should be accessible to every mathematician.

L. E. DICKSON.

Geometry of Greek Vases. By L. D. CASKEY. Boston, Museum of Fine Arts, 1922. 4to. 11 + 235 pages. Price $6.00.

This book is to be regarded as a supplement to Jay Hambidge's *Dynamic Symmetry: the Greek Vase,* which was reviewed by Professor A. A. Bennett in this MONTHLY (*1922,* 169–171). Mr. Caskey has made careful measurements of practically all the examples of Attic pottery in the collection of the Boston Museum, and the book contains these together with an analysis of the way in which they fit into the "dynamic symmetry" plan of construction. The Introduction (pp. 1–34) summarizes the results, and contains all that is of interest from a mathematical viewpoint. It is found that in 195 measurements the departure from the dimensions computed according to a "dynamic" analysis amounts to less than 2 mm., while in 68 instances the discrepancy is over 2 mm. While this result is not so striking as to afford a substantial ground for accepting Hambidge's theory of "dynamic symmetry," it does indicate that in a large number of cases such a geometrical construction is a very plausible explanation of the design employed by the Greek artists. Indeed, when we consider the fondness that the Greeks constantly showed for geometric constructions in general, and, since the days of the early Pythagoreans, for the regular pentagon in particular, it does not seem at all idle fancy, still less mere "ingenious magic" as one writer has put it,[1] to imagine that geometric constructions of the simple sort indicated (ratios of $\sqrt{2}$, $\sqrt{5}$, and $(\sqrt{5}-1)/2$ for the most part) were actually employed as the basis for the proportions of the parts of many Attic vases.

The book is well printed, and the numerous drawings carefully and attractively reproduced.

R. B. MCCLENON.

[1] R. Carpenter in *American Journal of Archaeology,* 25 (1921), 18–36.

ARTICLES IN CURRENT PERIODICALS

American Journal of Mathematics, volume 45, April, 1923: "A class of numbers connected with partitions" by E. T. Bell, 73–82; "Note on a new type of summability" by N. Wiener, 83–86; "On mediate cardinals" by Dorothy Wrinch, 87–92; "Periodic oscillations of three finite masses about the Lagrangian circular solutions" by H. E. Buchanan, 93–121; "On certain chains of theorems in reflexive geometry" by Flora D. Sutton, 122–144; "A poristic system of equations" by L. B. Robinson, 145–153.

Annales de l'École Normale Supérieure, volume 58, nos. 7–9, July, August, September, 1923: "La théorie des marées et les équations intégrales" (continuation) by G. Bertrand, 193–258; "Analogie entre les séries trigonométriques et les séries sphériques au point de vue de leur sommabilité par les moyennes arithmétiques" by E. Kogbetliantz, 259–288.

Bulletin des Sciences Mathématiques, second series, volume 47, August, 1923: "Pascal mathématicien et physicien" by E. Picard, 257–267; "Mouvement d'un solide pesant fixé par un point voisin de son centre de gravité" (continuation) by H. Vergne, 268–281; "Le problème de Dirichlet et le potentiel de simple couche" (continued) by G. Bertrand, 282–288.

Bulletin of the American Mathematical Society, volume 29, July, 1923: "Report on continuous curves from the viewpoint of analysis situs" by R. L. Moore, 289–302; "A generalization of a property of an acnodal cubic curve" by H. Hilton, 303–308; "On uniform limitedness of sets of functional operations" by T. H. Hildebrandt, 309–315; "Beck on coördinate geometry" [review of H. Beck, *Koordinaten-Geometrie, I. Die Ebene* (Berlin, 1919)] by J. L. Coolidge, 316–318; "Keynes on probability" [review of J. M. Keynes, *A Treatise on Probability* (London, 1921)] by E. B. Wilson, 319–322; "Blaschke on differential geometry" [review of W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie* (Vol. I, Berlin, 1921)] by G. A. Bliss, 322–325; Reviews: by C. N. Moore of H. S. Carslaw, *Introduction to the Mathematical Theory of the Conduction of Heat in Solids* (2nd ed., London, 1921), 326–327; by E. Swift of C. Runge, *Praxis der Gleichungen* (2nd ed., Berlin, 1921), 327–328; by P. J. Daniell of A. E. Kennelly, *Les Applications élémentaires des Fonctions Hyperboliques à la Science de l'Ingénieur Electricien* (Paris, 1922), 238; and by J. B. Shaw of J. Boussinesq, *Cours de Physique Mathématique, Compléments au Tome III* (Paris, 1922), 329–330; Notes, 331–333; New publications, 334–336.—October: "On the Riemann Zeta function" by C. F. Craig, 337–340; "Note on a certain type of ruled surface" by W. C. Graustein, 341–344; "The differentiation of a function of a function" by H. S. Carslaw, 344; "On the relative curvature of two curves in V_n" by J. Lipka, 345–348; "Square partition congruences" by E. T. Bell, 349–355; "Determination of all systems of ∞^4 curves in space in which the sum of the angles of every triangle is two right angles" by J. Douglas, 356–366; "Volume III of Lie's Memoirs" [review] by R. D. Carmichael, 367–369; "Bolzano on paradoxes" [review of B. Bolzano, *Paradoxien des Unendlichen* (Leipzig, 1921)] by L. L. Silverman, 370–371; Reviews: by G. A. Miller of A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung, mit Anwendungen auf algebraische Zahlen und Gleichungen sowie auf Kristallographie* (Berlin, 1923), 372; by C. H. Sisam of E. Pascal, *Repertorium der höheren Mathematik* (2nd ed., Vol. II, part 2, Geometry of Space, Leipzig, 1922), 373; by C. H. Forsyth of H. Andoyer, *Tables logarithmiques à treize décimales* (Paris, 1922), 373, and of J. W. Glover, *Tables of Applied Mathematics in Finance, Insurance, Statistics* (Ann Arbor, 1923), 376–379; by D. N. Lehmer of A. Cunningham, *Fundamental Congruence Solutions* (London, 1923) and *Haupt-exponents, Residue-indices, Primitive Roots, and Standard Congruences* (London, 1922), 374–375; by H. B. Phillips of J. B. Shaw, *Vector Calculus* (New York, 1922), 375; by F. Cajori of Jamblichus, *Theologoumena Arithmetica* (Leipzig, 1922), 377; and by J. B. Shaw of S. Valentiner, *Vektoranalysis* (3rd ed., Berlin, 1923), 377; Notes, 378–380; New publications, 381–384.

Journal de Mathématiques Pures et Appliquées, series 9, volume 2, no. 3, 1923: "Sur les principes fondamentaux de la théorie des contacts dans l'hypergéométrie réelle ou imaginaire et sur les familles complètes de figures intégrales d'un système d'équations aux dérivées partielles du premier ordre" by Ch. Riquier, 215–280; "A new simple theory of hypercomplex integers" by L. E. Dickson, 281–325.

Journal für die reine und angewandte Mathematik (Crelle), volume 153, nos. 1–2, August, 1923: "Ein- und Zweischaligkeit von Flächen 3. Ordnung" by R. Sturm, 1–7; "Arithmetische Eigenschaften der Polynomialkoeffizienten" by K. Hensel, 8–11; "Symmetrische Matrizen im Körper der rationalen Zahlen" by H. Hasse, 12–43; "Beiträge zur Theorie der

Laguerreschen Polynome und zum Summationsproblem von Orthogonalsystemen" by G. Wiarda, 44–60; "Bemerkung zu Herrn A. Fraenkel's Aufsatz 'Die Berechnung des Osterfestes'" by K. Boecklen, 61–65; "Ueber die dreifach ausgedehnte Mannigfaltigkeit der ebenen Dreiecke mit reellen Seiten" by H. Wolff, 66–75; "Zur Theorie des quadratischen Hilbertschen Normenrestsymbols in algebraischen Körpern" by H. Hasse, 76–93; "Zur Bewertungstheorie der algebraischen Körper" by K. Rychlik, 94–107; "Bemerkungen zu Herrn Hensels Arbeit 'Die Zerlegung der Primteiler eines beliebigen Zahlkörpers in einem auflösbaren Oberkörper'" by T. Rella, 108–110; "Zur Newtonschen Approximationsmethode in der Theorie der p-adischen Gleichungswurzeln" by T. Rella, 111–112; "Darstellbarkeit von Zahlen durch quadratische Formen in einem beliebigen algebraischen Zahlkörper" by H. Hasse, 113–130; "Ueber die Zerlegung der endlichen Gruppen in direkte unzerlegbare Faktoren" by R. Remak, 131–140.

Mathematische Zeitschrift, volume 17, nos. 3–4, August, 1923: "Ueber die Einordnung der Affingeometrie in die Theorie der höheren Uebertragungen. I" by J. A. Schouten, 161–182; "Ueber die Einordnung der Affingeometrie in die Theorie der höheren Uebertragungen. II" by J. A. Schouten, 183–188; "Die Airysche Funktion für den Ellipsenring" by A. Timpe, 189–205; "Ueber die Picardschen Ausnahmewerte sukzessiver Derivierten" by W. Saxer, 206–227; "Ueber die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten" by M. Fekete, 228–249; "Zahlentheoretische Abschätzungen mit Anwendung auf Gitterpunktprobleme" by J. G. van der Corput, 250–259; "Ueber Bildschranken bei Potenzreihen und ihren Abschnitten" by W. Rogosinski, 260–276; "Ueber einen zahlentheoretischen Satz von Hurwitz" by W. Jänichen, 277–292; "Zur Charakterisierung der Drehungsgruppe" by H. Weyl, 293–320.

Messenger of Mathematics, volume 52, nos. 11–12, March and April, 1923: "On Pellian chains" by A. Cunningham, 161–167; "On symmetrical plane algebraic curves" by H. Hilton, 168–175; "A suggestion for a new symbolic treatment of probability" by I. M. Horobin, 176–181; "On a spherical configuration of eight points" by W. Burnside, 181–184; "On the zeros of an integral function represented by Fourier's integral" by G. Pólya, 185–188; "Three n-dimensionals" by F. C. Pitt-Bazett, 189–192.

Proceedings of the National Academy of Sciences of the U. S. A., volume 9, July, 1923: "Form of the numbers of the subgroups of a prime power group" by G. A. Miller, 237–238.—August: "The Einstein equations of the gravitational field for an arbitrary distribution of matter" by T. Y. Thomas, 275–278.—September: "Groups of order 2^m in which the number of the subgroups of at least one order is of the form $1 + 4k$" by G. A. Miller, 326–328.

Scientific Monthly, volume 17, September, 1923: "Inaccuracies in the mathematical literature" by G. A. Miller, 216–228; "A survey of mathematics and astronomy" by E. D. Roe, Jr., 245–254.—November: "Mathematics as a career" by C. J. Keyser, 489–497.

Transactions of the American Mathematical Society, volume 25, January, 1923: "The (1, 1) correspondence associated with the cubic space involution of order two" by F. R. Sharpe and V. Snyder, 1–12; "Sur certaines équations aux différences finies" by N. E. Norlund, 13–98; "Differential geometry of an m-dimensional manifold in a Euclidean space of n dimensions" by C. E. Wilder, 99–122; "Expansions in terms of solutions of partial differential equations. First paper: Multiple Fourier series expansions" by C. C. Camp, 123–134; "Euler algebra" by E. T. Bell, 135–154.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 54, no. 3, published July 20, 1923: "Der mathematische Unterricht und der Bergbau" by F. Requard, 129–132; "Ueber die Sichtung des mathematischen Unterrichtsstoffes in den höheren Lehranstalten" by R. Fuchs, 132–138; "Experimentelle Geometrie der Richtung ausgeführt für die Ebene" by K. Bochow, 139–148; "Die Zahl e im Unterricht" by A. Harnack, 148–152; "Beiträge zur Behandlung der Kombinatorik" by W. König, 152–154; "Die Schnittpunkte zweier Kreise" by A. Witting, 155–156; "Die Behandlung einiger Grundbegriffe der Mechanik im Schulunterricht" by F. Hund, 156–162; "Die Verwendung der Glimmlampe im Unterricht" by A. Lindemann, 162–168; "Die Entstehung der Brennpunkte im Kugelschatten" by A. Larmer, 168–169; "Die Lehre von der Chordale in lagengeometrischer Behandlung" by R. Roth, 169–170; "Leonhard Eulers Stellung zur Relativität" by E. Hoppe, 181–184.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL AND H. L. OLSON.

Send communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3035 [1923, 337]. Proposed by R. M. MATHEWS, Wesleyan University.

This problem was incorrectly stated. It should read: Generalize projectively and prove that the envelope of the bisectors of the angles between corresponding lines of two perspective pencils is a curve of the *third* class.

3050. Proposed by C. N. MILLS, State Normal, Aberdeen, South Dakota.

Eliminate x, y, z from the equations

$$x^4/a^{5/4}b^{3/4} + y^4/a^{3/4}b^{5/4} = mz^2,$$
$$x^{3/2}/(az)^{1/4} = x + y = y^{3/2}/(bz)^{1/4}$$

and show that, if $ab > 0$, m cannot be less than 2^9.

3051. Proposed by NORMAN ANNING, University of Michigan.

Given the sequence: $u_1 = 2$, $u_2 = 8$, $u_n = 4u_{n-1} - u_{n-2}$ $(n = 3, 4, 5, \cdots)$, show that

$$\frac{\pi}{12} = \sum_{n=1}^{n=\infty} \text{arc cot } u_n^2.$$

3052. Proposed by DR. JOSEF LEWAMDONSKI, Pfaffsatten, Austria.

The ellipse whose parametric equations are:

$$x = a \cos \varphi, \qquad y = b \sin \varphi \tag{1}$$

intersects the conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{2}$$

If φ_1, φ_2, φ_3, φ_4 are the eccentric angles of the four points of intersection, prove that

$$\tan \tfrac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = abB/(a^2A - b^2C).$$

3053. Proposed by DR. JOSEF LENSE, University of Vienna, Austria.

Let $a_1 = a$, $a_2 = a^{a_1}$, $a_3 = a^{a_2}$, $\cdots$, $a_{n+1} = a^{a_n}$, $\cdots$. Discuss the $\lim_{n=\infty} a_n$ as a function of a. (The limit exists for some values of $a > 1$.)

3054. Proposed by S. A. COREY, Des Moines, Iowa.

Construct three triangles with vector sides a, b, c; d, e, f; g, h, k, such that $d = a$, $g = 2a$, $e = 2b$, $h = b$. Prove that $a^2 + b^2 + 4c^2 = f^2 + k^2$.

SOLUTIONS.

2936 [1921, 467; 1923, 339–341]. Proposed by J. P. BALLANTINE, Columbia University.

A person in drinking from a conical drinking glass tips it at a constant angular rate. At what angle will the delivery be the maximum and at what angle will the surface of the water be a maximum.

CORRECTION BY OTTO DUNKEL, Washington University.

In the solution of problem **2936** (*1923*, 340, 341) the reasoning on page 341 should be replaced by the following:

The greatest delivery is given by the acute angle φ determined from the equation above, provided $1 > \cos 2\alpha > 1/3$. For the derivative of $\cos(\varphi + \alpha)\sin^2(\varphi + \alpha)/\cos^3(\varphi - \alpha)$, multiplied by a factor which is positive in the range $0 \leqq \varphi \leqq \pi/2 - \alpha$, is equal to

$$f(\varphi) = 3\cos(\varphi + \alpha)\cos 2\alpha - \cos(\varphi - \alpha)$$

and, in this case, $f(\varphi)$ is positive from $\varphi = 0$ until it vanishes for the above-mentioned acute angle φ, and then it becomes negative.

In every other case the greatest delivery is given by $\varphi = 0$; for if $\cos 2\alpha \leqq 1/3$, then $f(\varphi) \leqq -2\sin\varphi\sin\alpha$ and hence it is never positive in the range for φ. Therefore the delivery decreases throughout this range. (*End of correction.*)

It is worth noting that the long algebraic inequality on page 340 can be reduced to

$$(1 - x)^2(1 + x)^3(4x^3 + 2x - 7)[(2x + 1)^2 - 8] < 0$$

which is satisfied, in the given range, by $\cos 2\alpha > (2\sqrt{2} - 1)/2 = 0.91421$.

2986 [1922, 356]. Proposed by C. F. GUMMER, Queen's University.

A triangle is inscribed in a circle. The arcs into which it divides the circumference are bisected at points forming the vertices of a second triangle. A third triangle is derived in the same way from the second, and so on. Prove that each set of alternate triangles approaches a limiting position.

SOLUTION BY B. F. YANNEY, College of Wooster.

We shall first prove that, regardless of position, the limiting triangles of the two sets are equilateral. Let $\theta_1^{(n)}$, $\theta_2^{(n)}$, $\theta_3^{(n)}$ be the arcs into which the vertices of the nth triangle divide the circumference and suppose that they are arranged in descending order of magnitude $\theta_1^{(n)} \geqq \theta_2^{(n)} \geqq \theta_3^{(n)}$; then the next set is $\theta_1^{(n+1)} = (\theta_1^{(n)} + \theta_2^{(n)})/2$, $\theta_2^{(n+1)} = (\theta_1^{(n)} + \theta_3^{(n)})/2$, $\theta_3^{(n+1)} = (\theta_2^{(n)} + \theta_3^{(n)})/2$ and we have $\theta_1^{(n)} \geqq \theta_1^{(n+1)} \geqq \theta_2^{(n+1)} \geqq \theta_3^{(n+1)} \geqq \theta_3^{(n)}$. Also $\theta_1^{(n+1)} - \theta_3^{(n+1)} = (\theta_1^{(n)} - \theta_3^{(n)})/2$. This shows that $\theta_1^{(n)}$ and $\theta_3^{(n)}$ approach limits and that these limits are equal. It then follows that $\theta_2^{(n)}$ approaches the same limit which must be $2\pi/3$, if we suppose that the radius of the circle is unity.

Let δ be a small number less than $\pi/12$ and choose N so large that $|\theta_i^{(N)} - 2\pi/3| < \delta/2$, for $i = 1, 2, 3$. Let A be a vertex of the Nth triangle and mark off on the circle the points A, C', B, A', C, B' at intervals of $\pi/3$. Take these points as mid-points of intervals of length 2δ on the circumference. Then the vertices of the Nth triangle will lie respectively in the A, B, C intervals. Since the vertices of the next triangle are obtained by bisecting the arcs for the Nth triangle, the vertices of the $(N + 1)$th triangle will lie in the A', B', C' intervals. The vertices of all following triangles will fall alternately in these two sets of intervals. By choosing smaller values of δ we obtain in the case of each interval a nest of intervals each lying within the previous interval. Hence in each of the intervals A, B, C there is a limit point, α, β, γ respectively; similarly in A', B', C' there are limit points α', β', γ', respectively. By choosing N large enough we obtain a triangle whose vertices lie as near α, β, γ as we please and at the same time such that the subtended arcs are as near $2\pi/3$ as we please. Hence α, β, γ are the vertices of an equilateral triangle. The vertices of the triangles obtained by bisection from the first set must approach the mid-points of the arcs $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ and, since they also approach α', β', γ', these latter points must be the mid-points.

Also solved by T. BENNETT, A. PELLETIER and J. ROSENBAUM.

2990 [1922, 356]. Proposed by R. M. MATHEWS, Wesleyan University.

If a circle be bitangent to a conic, its center is on one of the axes of the curve.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Take axes with the origin at the point of tangency of the two curves and the X-axis tangent to both. Let the radius of the circle be the unit. Then the equation of the circle will be

$$x^2 + (y - 1)^2 = 1 \tag{1}$$

and that of the conic may be written as

$$Ax^2 + 2Bxy + Cy^2 + 2Ey = 0, \tag{2}$$

the term in x and the constant term dropping out on account of the choice of axes.

Two curves are said to have contact of the nth order at a point, if the values of y and the derivatives of y up to and including the nth are equal, respectively, at the point. In this case the contact is of the third order, so that at the origin the values of y, y' ($= dy/dx$), y'' ($= d^2y/dx^2$) and y''' ($= d^3y/dx^3$) must be the same for the two curves.

For points on the circle near the origin we write

$$y = 1 - \sqrt{1 - x^2} = x^2/2 + x^4/8 + \cdots;$$

and this expansion gives, for $x = 0$, $y = 0$, $y' = 0$, $y'' = 1$, $y''' = 0$.

For the conic, take equation (2) and differentiate it three times. The values above (for $x = 0$) must satisfy the resulting equations, which are

$$Ax^2 + 2Bxy + Cy^2 + 2Ey = 0,$$
$$2Ax + 2B(xy' + y) + 2Cyy' + 2Ey' = 0,$$
$$2A + 2B(xy'' + 2y') + 2C(yy'' + y'^2) + 2Ey'' = 0,$$

and

$$2B(xy''' + 3y'') + 2C(yy''' + 3y'y'') + 2Ey''' = 0.$$

Substituting, we find from the last equation that $B = 0$, so that equation (2) becomes $Ax^2 + Cy^2 + 2Ey = 0$, a conic symmetrical with respect to the Y-axis, which is consequently an axis of the curve. But the center of the circle lies on the Y-axis and hence on an axis of the curve, which was to be proved.

NOTE BY THE EDITORS.—The third equation gives $A = -E$, so that the equation of the conic becomes $Ax^2 + Cy^2 - 2Ay = 0$. But this apparent restriction is due to the choice of the radius r of the circle as unity. To return to the general case we have merely to make the transformation $X = rx$, $Y = ry$, which gives

$$AX^2 + CY^2 - 2E'Y = 0, \qquad r = E'/A,$$

where r is the radius of curvature at the extremity of an axis of the conic.

This problem may be handled without the use of derivatives by applying the theorem: If a conic and a circle intersect in four points, the angle between each of the three pairs of common chords is bisected by the perpendicular from their intersection to an axis of the conic. This may be proved by writing the equation of the conic without the xy term as $S = 0$, and the equation of the circle in the general form as $C = 0$. Then $S - kC = 0$ gives the equation of a pair of common chords by a suitable choice of k. Hence, for this value of k,

$$S - kC = l(y - mx + n)(y + mx + n'),$$

which proves the theorem.

A number of special theorems follow from this. We shall consider two which are related to this problem. By considering the limiting case in which the chords reduce to the tangent to the conic and the chord through the point of contact, we have the known theorem that the angle between the common tangent and the common chord of the conic and the osculating circle is bisected by the perpendicular from the point of osculation to an axis of the conic. If now we let the chord and tangent fall into coincidence, we have contact of the third order, and the tangent must now be perpendicular to an axis. But this can only happen at a vertex of the conic and where the center of the circle falls on an axis.

If, on the other hand, a pair of common chords coincide, this chord of double contact must be perpendicular to an axis, and here again the center of the circle lies on an axis.

Also solved by MAURICE BAUDIN, WILLIAM HOOVER, A. PELLETIER, A. V. RICHARDSON, J. K. WHITTEMORE and the PROPOSER.

2991 [1922, 356]. Proposed by E. J. OGLESBY, New York University.

Sum the infinite series

$$S_2(x) = 1 + \frac{3x^2}{2!} + \frac{4x^4}{4!} + \frac{6x^6}{6!} + \cdots,$$

where the numerators of the coefficients form a series of numbers whose third differences are all equal to 2.

SOLUTION BY MAURICE HOME, University of British Columbia.

The problem will be considered as a special case of the following summation:

$$S_p(x) = a_0 + a_p \frac{x^p}{p!} + a_{2p} \frac{x^{2p}}{(2p)!} + \cdots \qquad (p = \text{a positive integer}).$$

Let

$$f(x) = \sum_{r=0}^{r=\infty} \frac{a_r x^r}{r!}.$$

With the usual notation of finite differences,

$$f(x) = e^{xE} \cdot a_0 = e^{x(1+\Delta)} \cdot a_0 = e^x \cdot e^{x\Delta} \cdot a_0 = e^x \sum_{r=0}^{r=\infty} \Delta^r a_0 \frac{x^r}{r!}.$$

If now a_r is a rational integral function of r of degree n,

$$\Delta^j a_0 = 0 \qquad \text{for} \qquad j > n$$

and

$$f(x) = e^x \sum_{r=0}^{r=n} \Delta^r a_0 \frac{x^r}{r!}.$$

If p is prime, let θ be a special root of $x^p - 1 = 0$. Then it is readily found that

$$S_p(x) = \frac{1}{p} \sum_{j=0}^{j=p-1} f(\theta^j x).$$

The case of p not prime may be made to depend on the above solution. In the particular case given,

$$\cdot a_r = \frac{1}{24}(r^3 - 9r^2 + 38r + 24),$$

$$a_0 = 1, \qquad \cdot\Delta a_0 = \frac{5}{4}, \qquad \Delta^2 a_0 = -\frac{1}{2}, \qquad \Delta^3 a_0 = \frac{1}{4},$$

$$\Delta^j a_0 = 0, \quad j > 3; \qquad \theta = -1, \qquad p = 2.$$

$$\begin{aligned} S_2(x) &= \frac{1}{2}\{f(x) + f(-x)\} \\ &= \frac{e^x}{2}\left(1 + \frac{5}{4}x - \frac{1}{4}x^2 + \frac{1}{24}x^3\right) + \frac{e^{-x}}{2}\left(1 - \frac{5}{4}x - \frac{1}{4}x^2 - \frac{1}{24}x^3\right) \\ &= \left(1 - \frac{x^2}{4}\right)\cosh x + \left(\frac{5x}{4} + \frac{x^3}{24}\right)\sinh x. \end{aligned}$$

Also solved by E. H. CLARKE, A. PELLETIER, J. F. REILLY, ELIJAH SWIFT, PINCUS SCHUB, and the PROPOSER.

2992 [1922, 420]. Proposed by AUGUSTUS BOGARD, Teresian University, Winona, Wis.

A semicircle rotates at a uniform velocity about its diameter and slides along the line of that diameter at such a uniform rate as just to pass the full length of the diameter while making one revolution about it. Find the equation of the surface thus generated.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let the plane of the semi-circle in its initial position be the xz plane, with the diameter $OA = 2a$ along the z-axis. When the plane rotates through the angle θ toward the positive y-axis, the center C rises to the position (o, o, z_c) and we have from the conditions of the problem $(z_c - a)/a = \theta/\pi$. In the new position the equation of the semi-circle is $x' = \sqrt{a^2 - (z - z_c)^2}$ if we take as x'-axis the intersection of the xy-plane with the plane of the circle. Now if (x, y, z) are the coördinates of a point P on the semi-circle, we have at once

$$y = \sin\theta \sqrt{a^2 - \left(z - a - \frac{a\theta}{\pi}\right)^2},$$

$$x = \cos\theta \sqrt{a^2 - \left(z - a - \frac{a\theta}{\pi}\right)^2},$$

as the parametric equations of the surface generated by the *semi-circle*. The minus sign before the radical gives the surface generated by the other half of the circle. By eliminating θ we have for the equation of the surface generated by the entire circle

$$y = x \tan \frac{\pi}{a} (z - a \pm \sqrt{a^2 - x^2 - y^2}).$$

Also solved by MAURICE BAUDIN, THEODORE BENNETT, MALCOLM FOSTER, H. HALPERIN and A. PELLETIER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

A. D. PITCHER, Professor of Mathematics at Adelbert College, Western Reserve University, died on October 5, aged 43 years. While he had suffered from a weak heart for a number of years, his sudden death came as an unexpected blow to his family and colleagues. In fact he had met his classes as usual on October 4 and gave every indication of normal health.

Professor Pitcher was born and reared in Kansas and, having graduated from the University of Kansas in 1906 and received the degree of Doctor of Philosophy from the University of Chicago in 1910, he had held positions at the University of Kansas, at Dartmouth College, and during the past eight years at Adelbert College. Since receiving his doctorate, Professor Pitcher had carried on extensive researches in Moore's General Analysis, the Frechet Theory, and the Theory of Functions of a Real Variable, his contributions to these fields of mathematics giving him a high place in the esteem of American mathematicians. At all times he maintained a high standard of work, and because of his patient and sympathetic temperament he enjoyed an unusual degree of loyalty and affection from his students as well as the admiration of his associates.

Professor C. B. WILLIAMS, professor of mathematics and dean of Kalamazoo College, and his wife both lost their lives at Yokohama, Japan, in the recent severe earthquake in that country. They had left Kalamazoo during the summer expecting to spend the coming year in travel and study and with the intention of making a trip around the world.

Dr. J. N. VAN DER VRIES, charter member of the Association, has been appointed manager of the North Central Division of the Chamber of Commerce of the United States of America. Dr. Van der Vries was for seventeen years a member of the mathematical staff at the University of Kansas, and was chairman of the department from 1911 to 1917.

Professor ARCHIBALD HENDERSON, of the University of North Carolina, has a year's leave of absence on full pay from the Kenan Research Foundation. He sails for England on October 17 and will spend the year in research work on relativity in the Universities of Cambridge, Berlin, Rome and Paris.

Professor D. T. WILSON, of the department of astronomy of Case School of Applied Science, died on Friday, October 12, 1923, at Washington, D. C., after

a long illness. Dr. Wilson was born at Clinton, N. C., was graduated from the University of North Carolina in 1887, received his degree of A.M. from Vanderbilt University in 1896 and the degree of Ph.D. from the University of Chicago in 1905. He spent a number of years as a computer in the United States Observatory at Washington, then taught the two years 1901–03 at the University of Cincinnati, before coming to Case School in 1903 as an assistant professor. He was made associate professor in 1911. Illness compelled him to give up his work at the end of the college year in 1921. When the Warner and Swasey Observatory was being planned, Dr. Wilson assisted in the designing of the building and the equipment. He made a special study of ballistics and during the war taught classes in that subject, coöperating with the Naval War College and the Coast Artillery Corps. The special astronomical research to which he gave his attention was the computation of the perturbations of a group of asteroids, printed in Upsala in 1912.

Professor MALCOLM MCNEILL of Lake Forest University died suddenly on October 5, 1923. He was born at Galena, Illinois, 1855, graduated from Princeton University in 1877 where he became fellow in astronomy and received the Master's degree in 1880. He was instructor in astronomy at Princeton in 1881–2, then assistant professor until he came to Lake Forest in 1888 as professor and head of the department of astronomy. His work, however, was largely in the department of mathematics where he served the institution for 35 years.

Professor McNeill was greatly beloved by the students who, by a large majority vote, nominated him as the most popular professor in the institution and who were to have presented his portrait to the College as a feature of the home-coming day this fall. The portrait, which is a special gift of the class of 1923, will be hung in the Durrand Institute. Professor McNeill was a charter member of the Mathematical Association of America.

A letter just received by Professor D. E. SMITH from Mr. YOSHIO MIKAMI, the well-known Japanese writer upon the history of mathematics in his country, states that the Imperial Academy with which he is connected was not destroyed by the recent earthquake and fire. He also states that his own valuable collection of Japanese mathematical books and manuscripts is safe. For a time all this material was in danger of fire, but the wind changed, and it was thus saved. The University Library, however, and the Colleges of Letters, Justice, and Economics were entirely destroyed with a large quantity of valuable books, but most of the other important libraries were fortunately saved. It cannot fail to be a source of gratification to American scholars to know that at least the most valuable part of the historical mathematical libraries was saved.

On the two hundredth anniversary of the birth of Euler, a committee of the *Society of Swiss Naturalists* launched the project of international coöperation for the publication of his collected works. Academies, Societies, including the American Mathematical Society, and individuals subscribed for about 300 sets. Eighteen of the estimated seventy volumes have been published. By reason of the European War nearly one half of the subscribers have been unable to

meet their obligations in full. Under these circumstances, a considerable number of new subscribers must be secured if the completion of the undertaking is to be possible in the near future. Those libraries or individuals wishing information with a view to promoting this great international undertaking should communicate with Professor R. C. ARCHIBALD, Brown University, Providence, R. I.

Professor NIELS BOHR, of the University of Copenhagen, will deliver the Silliman lectures at Yale University, and also the Simpson lectures at Amherst College, during the present academic year.

Professor F. N. COLE, of Columbia University, has been granted leave of absence for the second half of the present academic year.

Dr. IRWIN ROMAN, of Northwestern University, has been appointed associate professor of mathematics at Vanderbilt University.

Mr. A. D. CAMPBELL, of Cornell University, has been appointed assistant professor of mathematics at the University of Arkansas.

Professor O. W. ALBERT, of Grinnell College, has been appointed head of the department of mathematics at the University of Redlands.

Mr. D. L. HOLL, of the University of Chicago, has been appointed assistant professor of mathematics at Ohio Wesleyan University.

Dr. G. M. ROBISON, of Cornell University, has been appointed assistant professor of mathematics at Trinity College, Durham.

Dr. G. E. RAYNOR, of Princeton University, has been appointed assistant professor of mathematics at Wesleyan University.

The following have been appointed to instructorships of mathematics: Dr. F. H. MURRAY, at Dalhousie University; Mr. T. ANDREW and Mr. J. N. NIXON, at the University of Kentucky; Mr. H. A. ROBINSON, at the Texas Agricultural and Mechanical College.

Professor C. N. LITTLE, dean of the college of engineering of the University of Idaho, died September 7, 1923. Professor Little was known for his contributions to the theory of knots.

Dr. H. W. WILSON, assistant professor of mathematics at the University of Iowa, has been granted a year's leave of absence on account of ill health; he is spending it at Albion, Michigan.

At the meeting in New York of the American Mathematical Society, officers were elected as follows: vice-president for one year, Professor E. V. HUNTINGTON, Harvard University; for two years, Professors T. H. HILDEBRANDT, University of Michigan and J. H. M. WEDDERBURN, Princeton University; secretary for two years, Professor R. G. D. RICHARDSON, Brown University; treasurer for two years, Professor W. B. FITE, Columbia University; librarian for three years, Professor R. C. ARCHIBALD, Brown University.

The Bôcher memorial prize for mathematical research was awarded to Professor G. D. BIRKHOFF, Harvard University, for his memoir on "Dynamical systems with two degrees of freedom."

The *Harvard Alumni Bulletin* for January, 1924, contains an extended article by Professor J. L. COOLIDGE entitled "The story of mathematics at Harvard."

Interest in the science at the university, it appears, may be dated from the year 1727 at which time the Hollis professorship in mathematics and natural philosophy was founded. The first incumbent was Isaac Greenwood, who was however afterwards dismissed for bad behavior. The famous Benjamin Peirce served from 1840 until 1880 but under the title of Perkins professor, the Hollis professorship having become ultimately identified with the department of physics. Aside from other details, the article contains photographic reproductions of paintings of Hollis and Peirce, also of the set of mathematical models to be seen in the Widener Library.

It is understood that the recent Act of Congress in reducing the army and navy will affect the instructional staff at the United States Naval Academy, several instructors in mathematics being automatically dropped at the end of this year. While the MONTHLY has never undertaken to conduct a bureau of information with respect to vacancies and candidates, it would seem that this situation at the Academy is sufficiently unique to warrant specific mention. Institutions which may be looking for instructors in mathematics may secure information concerning the men who will be losing their positions by addressing Professor H. E. Slaught, University of Chicago.

CONTENTS

EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF for 1923, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of

America the sum of .... Dollars,

to be known as the ....

.... Fund, and to be used

or[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witnesses: Signature ....

....

....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924
NUMBER 2, FEBRUARY

PUBLISHED BY THE ASSOCIATION,
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others

MATHEMATICAL METHODS IN ECONOMIC RESEARCH.[1]

By C. C. MORRIS, Ohio State University.

Within the last few years there has been a great expansion of the use of mathematical methods in the field of economics. This expansion has taken two well-defined directions: one, that of statistical analysis of time series, the other, that of laying a mathematical foundation for the science of economics. It is the purpose of this paper to give a brief summary of the methods used, and the results obtained, by those who are carrying forward these investigations. The expansion in the field of economic statistics will be first considered.

In 1900 the Bureau of Economic Research was organized for the investigation of practical subjects in economics, statistics and politics, from a non-partisan but progressive point of view, "to the end that thoughtful men, however diverse their views of public policy, might base their discussions on objective as distinguished from subjective opinion." In 1920, the National Bureau of Economic Research was chartered. The personnel of the new organization includes that of the old and in addition many new names. The scope of the bureau has been enlarged to include "quantitative investigations into all subjects that affect public welfare." Two publications, the first of which appeared in 1921, have been issued. They contain an exhaustive report on "Income in the United States; its amount and distribution." In 1917, the Committee on Economic Research in Harvard University was appointed. Of this Committee, Professor Persons is statistician and editor. The activities of this committee have resulted in the founding of two quarterly journals, the *Review of Economic Statistics*, the first number of which appeared in January, 1919, and the *Harvard Business Review*, the initial appearance of which was in October, 1922. In addition to the two quarterlies, the Harvard Economic Service was established in 1922. This consists of a weekly letter, the principal function of which is to forecast business conditions. In addition to the above, many papers have appeared in statistical literature on price-trend curves, seasonal variation, law of growth, business cycles and related topics. Especially valuable have been some recent papers in the *Journal of the American Statistical Society.*

It may be interesting to pause for a moment, as we stand at the threshold of what apparently is to be a still greater expansion of statistical methods in the field of business, and determine what considerations have made such an expansion possible. For many years mathematicians had been spinning pennies, and testing out the laws of probability by noting the number of heads and tails that appeared; had been throwing dice and observing whether particular combinations made their appearance as often as the theory of probability indicated they should appear; had been drawing balls of many colors from urns of fixed

[1] Read at the summer meeting of the Association at Vassar College, Poughkeepsie, New York, Sept. 6, 1923.

composition in order to develop a theory of random sampling. "Urn schemata" had become a well-worn phrase. Problems arising from such considerations were, however, wholly academic, since they were artificial in character and had no connection with practical affairs. But there came a time, and to me this seems to be the reason for the tremendous activity in the field of economic statistics indicated above, when the question arose as to whether the laws of probability which had been the outgrowth of experiments artificial in character did not have a practical application. The mathematician said "yes," and the researches of the last ten years have shown that the mathematician was right in his conviction that the actions of the mass man could be forecast with the same degree of accuracy as we forecast the different combinations of heads and tails when spinning pennies.

The problem of forecasting has become one of the new problems of business. Its importance is now universally recognized and business men who gauge not only their production but their purchases of raw material by these forecasts are finding themselves in a much stronger position than those who do not. As might be expected in the treatment of such a new problem, the methods used are rather sharply differentiated. For the purposes of this paper they may be classified as: (1) empirical, (2) graphical and (3) theoretical. Upon my own responsibility and without consulting those concerned, I would name the Harvard Committee as leaders in the first method of the analysis of data, the National Bureau of Economic Research and the private service companies, such as the Babson Service, in the second method, and Mr. Fisher, author of the *Mathematical Theory of Probabilities,* in the third method. It is realized that the classification here suggested is only approximately correct, that the various methods blend one into the other, yet such a classification will serve as a basis for some remarks regarding these methods.

In the initial number of the *Review of Economic Statistics,* Professor Persons gave in detail his method of analyzing a time series and the reasons upon which his choice was based. An abstract of this method, which will be included in the forthcoming "*Handbook of Mathematical Statistics,*" appeared in the June number of the *Journal of the American Statistical Society*. It may be summarized as follows:

(1) The determination of the curve of trend by means of the principle of least squares;
(2) The determination by months of the moving averages, or link relatives;
(3) The determination of the monthly medians of these values;
(4) The reduction of these medians to January as a base, in such a way as to secure a definite cycle, the error distributed by the compound interest formula.

Attention may be called to the three assumptions here involved:

(1) That the line of trend is given by a simple curve such as the linear, the parabolic or the exponential;

(2) That the median may be used to represent a series of values of the variate when the number of va ues is not large and the dispersion is considerable, and

(3) That the compound interest formula may be used to distribute errors even if they are of considerable size.

Each of these assumptions deserves consideration. The first one, that the law of growth can be closely approximated by a simple curve, seems to imply that growth itself is a simple affair. We will willingly grant this in the case of a principal at simple interest or simple discount, or in the case of a principal where the increase at any time is proportional to the principal at that time, but when we try to carry such simple laws of growth over to that of the monthly sales of automobiles or paint, we get into difficulty. The laws of growth in the demand for these two commodities cannot be so simply expressed. They are entirely different and rest upon different foundations. Paint is a necessity, the automobile is, to a great extent, a luxury. We must paint our houses in order to preserve them, and hence in order to discover the law of growth of the demand for paint we must discover the law of growth in the number of houses which must be painted. But the law of growth in the demand for automobiles is found in the accumulation of an excess of income over outgo and hence is a function of the prosperity of the country. In fact in most of the series which are worthy of study, growth is complex and it is an open question whether an attempt to express it in simple terms will in the end be satisfactory. These simple terms will not represent the law of growth with accuracy yet the method used may be as accurate as the data to which it is applied, and an attempt to use a general law of growth, or a different law for each series, may not be desirable.

Prescott [1] proceeded on the hypothesis that the law of growth passes through four distinct stages:

(1) Period of experimentation;
(2) Period of growth into the social fabric with a constantly increasing rate;
(3) Period of growth at a constantly diminishing rate, and
(4) Period of stability.

He found that a curve, due to the distinguished actuary, Benjamin Gompertz, and whose equation is

$$l_x = kg^{c^x},$$

closely approximated this cyclic growth. This is the formula which, modified by Makeham, has become a fundamental one in actuarial work. Prescott used this curve in his effort to forecast the number of automobiles that would be absorbed by the people of the United States in 1922. While the growth of any time series is usually a direct function of the population, which actuaries assume to be increasing according to a simple exponential law, yet in the periods of experimentation and stability such a law for a time series will evidently fail.

[1] Prescott, R. B., Law of Growth in Forecasting Demand. *Journal American Statistical Association*, Dec., 1922.

The second assumption, made by those who are using the empirical method in their analysis of a time series, that the median of the link relatives may be used to represent the series, raises the old question as to the value of this statistical average. In a recent note, Fisher [1] mentions the fact that its use was condemned by De Moivre, Bernoulli, and by many of the great mathematicians who have worked in the field of statistics. He also quotes Thiele and Charlier against this practice.[2] The objection is based upon the hypothesis that the errors of random sampling are so great that conclusions based upon less than a thousand values of the variate have little value. It is evident that if we are trying to find the seasonal variation of the sales of automobiles or paint, we do not have at our command statistics for anything like a thousand Januarys or Februarys. And even in those series where a complete record for many years is available, but where the unit of measurement is the dollar, we can not go back of 1918 or 1919 in any study that we may make. The question then of the choice of the method of averaging is indeed an important one. The determination of that value of the variate which has the greatest probability (the mode) is, theoretically, a comparatively simple matter. We may consider the fifteen or twenty values of the moving averages as a frequency distribution, pass through these a generalized frequency curve of the Gram-Charlier type, and determine the mode by solving the equation obtained by setting the first derivative equal to zero. The equation of this curve may be written

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\frac{\lambda_1\omega}{1!}i + \frac{\lambda_2\omega^2}{2!}i^2 + \frac{\lambda_3\omega^3}{3!}i^3 + \cdots} e^{-xi\omega}d\omega$$

where the semi-invariants λ_i $(i = 1, 2, 3, \cdots)$ are defined by the relation

$$e^{\frac{\lambda_1\omega}{1!} + \frac{\lambda_2\omega^2}{2!} + \frac{\lambda_3\omega^3}{3!} + \cdots} \int_{-\infty}^{+\infty} \varphi(x)dx = \int_{-\infty}^{+\infty} e^{x\omega}\varphi(x)dx.$$

The practical difficulty of this method is that the first derivative when set equal to zero can only be solved with great labor. I have Mr. Fisher's permission to quote from a private letter: "The methods I apply, based upon the work of Laplace, Poisson, and the extensions by Gram, Thiele, and myself, do not necessarily assure us that we thus obtain the most probable value (the mode) in the strict sense of the word. Neither Laplace, Poisson, Gram or Thiele made such an assertion; they speak of the method as the most advantageous method, as the method which ought to be preferred. What we obtain is not therefore necessarily the mode but the most advantageous value of the variate." Mr. Fisher did not say how this most advantageous value of the variate is obtained, whether or not it is an approximate solution of the first derivative when a sufficient number of terms of the frequency curve has been retained to secure a good fit. Such an elaborate procedure based upon the theory of probabilities as developed by Laplace appeals to the mathematician and theorist. Yet it is quite beyond

[1] *Journal American Statistical Association*, June, 1923.
[2] A. Fisher, *Mathematical Theory of Probabilities.*

the reach of the statistician who is not a mathematician and the question arises whether it is not too theoretical, and not sufficiently responsive to world conditions.

The third assumption that the errors of random sampling can best be distributed by the compound interest formula seems reasonable. Sometimes, however, the error is quite large and appreciable changes are made by it in the coefficients of seasonal variation. It might be well to try other methods and compare results.

I do not wish to be understood, in commenting upon the assumptions made by those using empirical methods, as criticizing those methods. They have already proved their worth. Great advances have been made by their use, and we look forward to still greater accomplishments. The business executive who studies the weekly letter of the Harvard Committee has definite grounds upon which to base his decisions.

The second method used in economic research is the graphical one. A typical study, that of turpentine, as made by the statistician of a large paper company, will make this method plain. The Babson line for general business is plotted for twelve or fifteen years. The area between this curve and the curve of trend shows the volume of business above or below the normal. As an additional guide the Babson line for the ten basic commodities is drawn. A curve is now drawn showing the monthly prices of turpentine, along with others showing exports, receipts, and stock on hand. Since eighty-five per cent. of the turpentine produced is used in the paint industry, and since this industry depends upon those financial conditions which govern building, a curve showing the amount of residence building permits is also drawn. These curves are now studied in the light of world conditions at any particular time, such as shortage of receipts, unusual weather conditions, business failures, price control, transportation difficulties, painter strikes, foreign buying, labor troubles, war conditions, advances in wages, speculation, etc. This analysis of a time series which in this case is the market price of a commodity is accomplished without the use of mathematical formulæ of any kind. The statistician has before him a picture of what has taken place and attempts to interpret the future with its aid. We may urge as a criticism of this method that too great emphasis is placed upon world conditions the effect of which it is easy at the moment to exaggerate, that there is no way to care for seasonal variation and that the long time movements due to conditions other than those of the present are ignored. Yet these line pictures drawn free-hand are wonderfully illuminating and the business executive who combines their information with keen judgment and reliable current information is in a far better position to act intelligently than one who buys when the spirit moves him.

The third method used in economic research, the theoretical, attempts to solve the same problems as those to which the other two methods are applied but insists upon rigorous theoretical treatment in every case. Attention has already been called to the two ways of drawing a frequency curve, the free-hand method, and the Gram-Charlier method, also to the two ways of averaging,

the median and the mode. It may be that experience will show that the method applied must depend upon the data under consideration, not only upon the accuracy of the values of the variate but also upon their number. The mathematician is naturally prejudiced in favor of rigorous methods of analysis but he must not forget that most of the data he deals with, for instance the monthly production of yellow pine or Douglas fir, are rough, and rough methods may be preferable.

In the study of a time series one is naturally led to the comparison in terms of their standard deviations of two series in order to find out if their time variations are the same or whether like changes occur with a constant time difference, whether there is a so-called lag. In the April, 1919, number of the *Index*, it is shown that for a six months' lag, the correlation between the production of pig iron and the rates on 60–90-day paper is .75 for the years 1903–1918. When variations in one series antedate those in a second series the first becomes a barometer for the second. If we grant the hypothesis that nothing just happens, that every variation has its cause, we must admit its corollary, that barometers exist for all kinds of businesses, and their discovery therefore becomes a challenge to the statistician. The business depression of 1920–21 did not come as a stroke out of a clear sky; it came because certain fundamental economic principles had been violated and its coming we now realize should have been foreseen. It differed from preceding periods of depression only in its severity. It was preceded by over-production, over-expansion of credit, waste in manufacture, inefficiency of labor and a mounting interest rate. The researches of the last two years have shown that these conditions have always preceded the low point in the business cycle. We did not appreciate fully the value of these symptoms as the means for a financial diagnosis in 1920. We do appreciate them in 1923. These sensitive barometers would have warned us of its approach and would have enabled us to discount its effects. Every coming event must cast its shadows before and it is the function of the business statistician to recognize these shadows so that the rigors of the storm may be abated. It is the unexpected that causes the greatest damage. The business men of the country are awakening to the value of economic research, are beginning to realize that the laws of probability are holding in the business world. A new profession is being born and if this new profession rises to its highest ideals, then those seasons of business depression which in the past have been such a curse to our country will, in a large measure at least, be levelled. The trough of the wave will not be so deep; the crest not so high.

Now we may ask what part the Mathematical Association of America and the Universities which are represented in its membership are to play in the production of men for the profession of the business statistician. This question demands serious consideration. It must not be left to those who are not qualified. Through the activities of the College of Agriculture, the Ohio State University has for years been serving the farmer. Now the business men of the state are demanding that some attention be given to them. A new type of investigator

must be equipped. May I suggest that he must first of all have a broad general education. He must know economics but not to the extent that he is an idealist; he must know calculus, the theory of least squares, the theory of errors, the theory of probabilities, the theory and practice of statistical procedure. He must be a skilled computer; know when to use a slide rule and when to use a six-place logarithm table. He must have a sense for accuracy and be able to tell at a glance whether data are reasonable or not. He must know how to fit the accuracy of his methods to the accuracy of his data. He must realize that precise methods do not make inaccurate data more accurate. He must associate with business men and learn something of their psychology. He must have common sense and must draw conclusions quickly and accurately. I think I am justified in the statement that the courses in statistics as given by a majority of the universities do not meet these demands. A course in statistics based upon freshman mathematics must be to a large extent a failure. The basis for the study of statistics must be as wide and deep as the subject itself.

So far in this paper our attention has been focussed upon the statistical methods that are used in analyzing time series and upon the application of these methods to the problem of forecasting. A much more fundamental problem of economic research however is that of determining whether or not the science of economics, or possibly I had better say theoretical economics, may be placed upon a firm mathematical basis, whether a theory of economics, expressed quantitatively in terms of formulæ and equations, may be built up sufficiently simple to be verified in actual societies. We were told at our December meeting that such would have to be the case before economics could be classed as a science. In the MONTHLY (*1922*, 371–380), Professor G. C. Evans laid the foundation for such a mathematical treatment. If

$$q(u) = Au^2 + Bu + C$$

represents, as a quadratic function, the total cost of producing (that is, putting on the market) u units of a commodity in unit time and

$$y = ap + b$$

represents as a linear function the amount of goods, y, which if the price is p, will be bought in the market in unit of time, then out of these equations may be deduced conclusions as to production and prices which correspond to the varying conditions of coöperation, competition and monopoly. We see in his treatment the beginnings of a general system of economics, and look forward to the further development. I wish also to refer to other papers dealing with such questions as "Rates of Exchange" by Bray; "Elasticity of Demand and Flexibility of Prices" by Moore and "The Law of Growth in Forecasting Demand" by Prescott. The attempt to analyze the business cycle by means of Fourier series is an interesting application of that series. Such papers as I have mentioned seem to foreshadow a new day, when the economist must be first of all a mathematician. It will then be seen that the great masses of economic data which have been

accumulating for the last twenty-five years are amenable to mathematical analysis, and can be written in terms of mathematical formulæ.

I wish, in drawing this paper to a close, to give to it a personal touch. It is my privilege to be serving, in the capacity of business statistician, a corporation manufacturing paint. To the officials of such a company a forecast of the future is of the greatest importance. The president and vice-president are interested in the larger questions of development and expansion, in order that adequate preparation may be made. The general manager wants to know when to increase his invoice, and when to decrease it. He wants to know what are the readings on the paint barometer; whether the morrow will be "fair and bright," or "dark with impending storm." The factory superintendent wants to eliminate rush seasons, to do away with overtime, with all of its high over-head and inefficiency. He must satisfy all demands upon the factory, and any advance information as to what these are likely to be is of great value. The purchasing agent wants to know when to buy raw materials. The man who is spending millions of dollars of the corporation's money must know market conditions, he must buy when the market conditions are right, and he must have the most reliable information obtainable. The sales manager, who has charge of the distribution of the finished product among the various branch houses, must see to it that their invoices are properly maintained. When he ships a carload of house paint to Kansas City, the laws of probability must be depended upon to decide just how much shall go forward in five gallon containers, how much in one gallon, two quart, one quart, one pint, or one-half pint containers. If a mistake is made, it means orders shipped by express to take care of deficiencies, or it means large amounts invested in slow-moving invoices. As I make my monthly rounds of the factory, I am met always with the same question everywhere: "What is going to happen?" In order to answer this question intelligently I must keep informed as to the monthly production of lead and zinc, to the visible supply of flax seed, the production and sale of turpentine, the production and sale of construction timber, the number and value of residence building permits, employment and wages, and the buying power of the country as a whole. With this information in hand, monthly forecasts as to the sale of paint and the price trends of the raw materials may be made. The task, you will appreciate, is not an easy one.

The profession of the business statistician was born of the travail of the business depression of 1920–21. The government realizes that ignorance on the part of the business man was the underlying cause of that depression. It is now doing all it can to supply dependable information. This information must be interpreted. The Mathematical Association of America must aid in preparing and working out a curriculum which will meet the needs of the one who is to make the interpretation. The appointment of a committee who shall study this new demand upon the statistician may be desirable. A new avenue of service to his country is opening before the mathematician. He must not fail.

article, not only gives Pell credit for all the new symbols in Rahn's book, but designates both the English and German editions as being Pell's book. Vacca says, "I intend in this note to give account of a symbolism of the same kind, but more complicated and precise, due to John Pell (1610–1685); and used by him systematically in his work, *Introductio in Algebram,* Londini a. 1668 (a first edition appeared in the German language, Tigurii, 1659). It has not been possible for me to consult the original work of Pell. I limit myself therefore to giving an idea of his method as it is found in the exposition given by Wallis in Vol. 2. p. 233–246 of his *Works* (London a. 1693)." Still more recently, J. Tropfke in one place [1] mentions $\div$ as used by John Pell and William Jones; in another place [2] he ascribes $\div$ to Pell, but mentions Rahn as having used it in 1659.

7. Conclusion. Nowhere have I been able to find any reason for assigning to Pell the new signs found in Rahn's algebra of 1659. There is no evidence that Pell used them before he prepared his additions to the English edition of Rahn's book. That the new signs in Rahn's algebra of 1659 are due to Rahn and not to Pell would seem to follow also from the fact that Pell used Brancker's modifications of the 1659 symbols. The symbol for involution, as used by Rahn, cleverly expressed the idea by its very shape—an Archimedean spiral. It must have been adopted because of its suggestiveness and on that account would not be likely to have been discarded by its inventor, in favor of the omicron-sigma which by its shape does not express the idea of "involviren."

The growth of myth relating to Pell and to Rahn's text is perhaps the less surprising when one recalls that erroneous statements have been current on the actual content of the two editions. As previously stated, the table of composite and prime numbers in the 1659 edition of Rahn's Algebra was said not to go beyond 10,000, when as a matter of fact the table extends to 24,000. Again, it has been said that the English edition (1668) contained a solution of the famous indeterminate equation $x^2 - ay^2 = 1$, when no such solution is actually given. Moreover, this famous equation has been called "Pell's Equation," even though Pell published nothing on it and, as far as known, did not occupy himself with it.

THE THREE-BAR CURVE.

By F. V. MORLEY, New College, Oxford, England.

1. The main attraction of the three-bar curve lies in the simplicity of its mechanical generation. Historically, the "sweet simplicity," as James Watt says, especially took his fancy in the so-called "parallel-motion" by which he drew the curve; and the "delicate smoothness" of that three-bar linkwork gave him more pleasure than any of his other inventions.[3] But though we can construct the curve mechanically with an ease exceeded only by that of the circle—

[1] J. Tropfke, *Geschichte der Elementar-Mathematik*, vol. 2, 2d ed., 1921, p. 20.
[2] J. Tropfke, *op. cit.*, vol. 2, 2d ed., 1921, p. 25.
[3] See Muirhead, *Life of James Watt*, London, 2d ed. (1859), p. 288.

I find it as easy to draw accurately as a conic—no analytical treatment of corresponding simplicity has been attained. The question is really one of providing a kinematical treatment for what is essentially a matter of elementary kinematics.

The simplicity of generation and the difficulty of analysis have attracted many writers,[1] and some very pretty results have cropped up as the outcome of their researches. In 1875, during the first stage in the history of the curve, when there was a remarkable interest in linkages, Roberts made the discovery which Cayley elaborated of the "triple-generation" of the three-bar sextic. The subject slept for some time, but during the last few years Colonel Hippisley announced a "pairing" on the curve which Dr. Bennett shaped into a keystone property—the simple transformation, by isogonal conjugates, of the curve into itself. It seems worth while to give a direct proof of this last property, based on an analysis which gives a connected treatment of the curve.[2] We have first to illustrate the type of argument; then recapitulate the "triple-generation"; and proceed to the transformation as the result of a simple algebraical theorem.

2. We shall adopt the notation and methods of inversive geometry, using that term in Professor Morley's sense as the geometry which illustrates the theory of functions of a complex variable in much the same way that projective geometry illustrates the analytic theory of forms. Here the reference scheme is a *base line,* on which are spread the real numbers ρ, and a unit or *base circle* with centre at 0. On the base circle are spread a series of special complex numbers, called *turns,* and represented by t; they are of the form

$$t = 1^p,$$

where p is any real number. This scale of turns upon the base circle has certain special properties—the product of two turns is a turn; the sum of two turns is not in general a turn; but any root of a turn is a turn. If we connect t, thought of as a point of the base circle, with the $\angle\, 10t$ (called θ), we have

$$t = \cos\theta + i\sin\theta = e^{i\theta}.$$

The conjugate complex number is

$$1/t = \cos\theta - i\sin\theta = e^{-i\theta}.$$

We express this shortly by saying the conjugate of t is $1/t$. Both t and its conjugate, $1/t$, are numbers in the scale of turns; that is, are points of the base circle.

The product of any real number ρ and any turn t is a complex number which we call

$$x = \rho t \tag{2.1}$$

[1] Without giving a complete bibliography, one should mention Samuel Roberts, *Proc. London Math. Soc.*, vol. 2 (1869), p. 125, and *ibid.*, vol. 7 (1876), p. 17; Cayley, *ibid.*, vol. 3 (1870), p. 100, and vol. 7 (1876), p. 136; Sylvester, in several discursive papers, but one particularly to be referred to in *Nature*, vol. 12 (1875), p. 214; Clifford, *Kinematic* (1878), p. 149; Darboux, *Bulletin des Sciences Mathématiques*, ser. 2, vol. iii (1879), p. 109; and of more recent writers, Colonel Hippisley, *Proc. London Math. Soc.*, ser. 2, vol. 18 (1917), p. 136, and *ibid.*, vol. 21 (1922), p. 410; Dr. G. T. Bennett, *ibid.*, ser. 2, vol. 20 (1920), p. 73.

[2] For a condensed exposition of this analysis, see *Proc. London Math. Soc.*, ser. 2, vol. 21 (1921), p. 140.

and we identify this number x with a point of the plane. Drawing the vector $0x$, we say that ρ is the length, and t the direction factor. Thus without losing its representation of a point on the base circle, t may also be thought of as the rotation by which the line $0x$ is obtained from the base line. Equation (2.1), the naming of a point by the vector $0x$, may be looked upon as a parametric equation either of a line or of a circle, according as t or ρ is made a constant. It may readily be generalized. We follow the convention of expressing any fixed point of the plane by a constant complex number denoted by a small letter. Thus

$$x - c = \rho t \tag{2.2}$$

expresses that the vector cx has length ρ, direction t. If t is fixed, (2.2) is the equation of a line in terms of the parameter ρ. If ρ is fixed, it is the equation, with parameter t, of a circle with centre c and radius ρ.

The process of forming the conjugate of a complex number x is thought of as the operation of reflexion in the base line—an operation sending each point x into a new point which we write with a superposed bar, $\bar{x} = \rho/t$. The reflexion affects all points of the plane, the base line being invariant under the operation; it sends the base circle into itself, replacing t by $1/t$; and it sends a vector $x - c = \rho t$ into $\bar{x} - \bar{c} = \rho/t$.

3. The above outline does not take us very far, but we may indicate the form taken in this analysis by the transformation by isogonal conjugates. This transformation has three fixed points, say a_1, a_2, a_3, taken on the base circle. It will be useful to consider the points as roots of the cubic equation

$$a^3 - \sigma_1 a^2 + \sigma_2 a - \sigma_3 = 0,$$

where

$$\begin{aligned} \sigma_1 &= a_1 + a_2 + a_3, \\ \sigma_2 &= a_2a_3 + a_3a_1 + a_1a_2, \\ \sigma_3 &= a_1a_2a_3. \end{aligned}$$

The usual defining property of a pair of points x, y under the isogonal transformation is

$$\angle\, a_2a_1x = \angle\, ya_1a_3 \tag{3.1}$$

and so for the other two sets of angles (Fig. 1). We are therefore concerned with the relative directions of the vectors a_2a_1 and xa_1. If $a_2 - a_1 = \rho_1t_1$ and $x - a_1 = \rho_2t_2$, the relative direction is given by

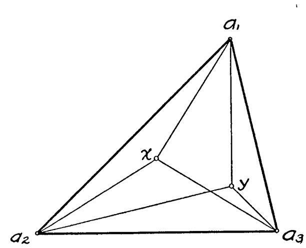


FIG. 1. The Transformation by Isogonal Conjugates.

$$\frac{x - a_1}{a_2 - a_1} = \frac{\rho_2t_2}{\rho_1t_1} = \mu\tau$$

where $\mu = \rho_2/\rho_1$ is real and $\tau = t_2/t_1$ is a turn. Similarly $(a_3 - a_1)/(y - a_1) = \mu'\tau'$ where, by (3.1), $\tau' = \tau$. Hence we have the expression

$$\frac{(x - a_1)(y - a_1)}{(a_2 - a_1)(a_3 - a_1)} \tag{3.2}$$

equal to a real number, μ/μ', and consequently equal to its conjugate. The congugate of (3.2), bearing in mind that $\bar{a} = 1/a$, is

$$\frac{a_1{}^2 a_2 a_3(\bar{x} - 1/a_1)(\bar{y} - 1/a_1)}{(a_2 - a_1)(a_3 - a_1)} .$$

Hence

$$a_1(x - a_1)(y - a_1) = \sigma_3(1 - \bar{x}a_1)(1 - \bar{y}a_1). \tag{3.3}$$

Similarly-formed equations are to be true for a_2, a_3, as well as for a_1. We may include all three equations under the form

$$a(x - a)(y - a) - \sigma_3(1 - \bar{x}a)(1 - \bar{y}a) = 0, \tag{3.4}$$

where, as a takes in succession the specific values a_1, a_2, a_3, we get the equations of type (3.3). Equation (3.4), as a cubic in a, must be identically

$$a^3 - \sigma_1 a^2 + \sigma_2 a - \sigma_3 = 0.$$

Equating coefficients, we have

$$x + y + \sigma_3 \bar{x}\bar{y} = \sigma_1 \tag{3.5}$$

as the equation expressing the transformation by isogonal conjugates.

Equation (3.5) gives for every x a definite partner y, with these exceptions:

(*1*) when x is one of the a's, in which case y may be any point on the opposite side of the base triangle a_1, a_2, a_3. We avoid ambiguity by assigning points on the side, say a_2a_3, according to the directions of lines on a_1—that is, according as x approaches a_1.

(*2*) when $x \to \infty$, in which case the partner may be found by considering the path which x takes to ∞. That is, if $x/\bar{x} = \tau$ as $x \to \infty$, then $y = \sigma_3/\tau$, a point on the base circle.

The fixed points of the transformation are the centres of the four circles touching the sides of the triangle. The pair x, y are foci of a conic inscribed in the triangle; consequently the transformation (3.5) is sometimes called the focal transformation, or simply "focal pairing."

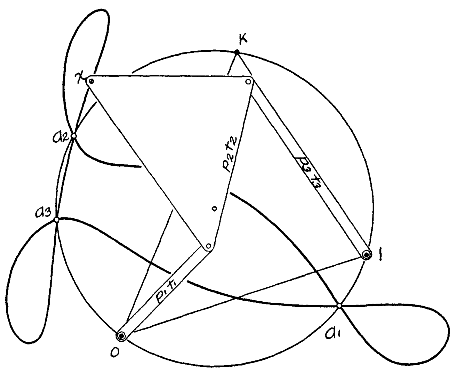


FIG. 2. The Three-Bar Curve.

4. The three-bar curve is generated by a linkwork of three jointed rods (Fig. 2), the two legs having each a pivot at fixed points of the plane. We may choose these points to be 0 and 1. The linkwork has one degree of freedom; any point on either leg traces a circle; but any point on the middle or traversing bar traces a three-bar sextic curve. Watt chose to study the motion of the midpoint of the traversing bar—a special three-bar curve. It is more general and no more difficult to consider the tracing point, x, as attached to the traversing bar by a rigid triangular plate.

If we call the lengths of the bars in order of size ρ_1, ρ_2, ρ_3 and their directions t_1, t_2, t_3, we have immediately two fundamental vector identities,

$$\rho_1 t_1 + \rho_2 t_2 + \rho_3 t_3 = 1 \tag{4.1}$$

and its conjugate equation

$$\rho_1/t_1 + \rho_2/t_2 + \rho_3/t_3 = 1. \tag{4.2}$$

In these equations the ρ's are constant and the t's vary with the motion of the linkwork. The tracing point x may be named from the base point 0 as

$$x_0 = \rho_1 t_1 + k\rho_2 t_2, \tag{4.3}$$

where k is a complex constant, giving the "shape" of the triangular plate on the traversing bar $\rho_2 t_2$. But, named from the point 1, x will be

$$\begin{aligned} x_1 &= x_0 - 1 \\ &= \rho_1 t_1 + k\rho_2 t_2 - \rho_1 t_1 - \rho_2 t_2 - \rho_3 t_3 \\ &= \rho_2 t_2(k - 1) - \rho_3 t_3. \end{aligned} \tag{4.4}$$

It is thus suggested that if we are to name x symmetrically, from base point anywhere, we should use an expression

$$x = c_1\rho_1 t_1 + c_2\rho_2 t_2 + c_3\rho_3 t_3, \tag{4.5}$$

where the three c's are complex constants.

In that case, consider each c as a fixed point of the plane. Since the identity (4.1) holds, we may write (4.5) as

$$x - c_3 = (c_1 - c_3)\rho_1 t_1 + (c_2 - c_3)\rho_2 t_2.$$

This gives the naming of x with respect to c_3. It must therefore be identical with (4.3) when $c_3 = 0$ and $c_1 = 1$. That is, $c_1 - c_3 = 1$ and $c_2 - c_3 = k$. There is then a third point k, which plays a part equivalent to that played by 0 or 1. This point is the vertex of a triangle on 01 similar to the triangle attaching x to the traversing bar. Hence we have the triple-generation theorem, that the curve traced by a three-bar linkwork attached to two fixed points, c_3, c_1, may also be traced by three-bar linkworks attached to two other pairs of fixed points, c_1, c_2, and c_2, c_3.

When Roberts communicated this theorem to Cayley, the latter put it into the form illustrated by Figures 3 and 4. That is, we take any triangle $C_1C_2C_3$ and through any point x within it draw lines parallel to the sides. Let the triangles outlined in Fig. 3 be supposed rigid plates, jointed at x, and let the other lines of the figure represent bars completing three jointed parallelograms. Then, however the system is moved about in its plane, as shown in Fig. 4, the triangle $c_1c_2c_3$ will be always of the same shape; and further, starting as in Fig. 4 from any given position of the three triangular plates, the assemblage may be moved without altering $c_1c_2c_3$ in magnitude. Unless $c_1c_2c_3$ are in their maximum position $C_1C_2C_3$, x is movable and will describe a three-bar curve under the simultaneous guidance of the three three-bar linkworks.

5. Let us call the three fixed points $c_1c_2c_3$ *poles* [1] of the three-bar curve, and consider them as on the base circle. They are then constant turns. We have

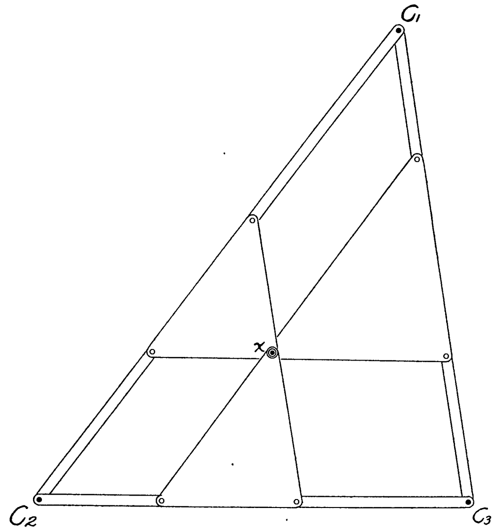


FIG. 3. The Triple Generation Theorem. Maximum Position.

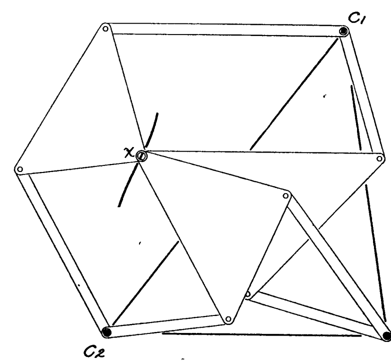


FIG. 4. The Triple Generation Theorem. In Motion.

defined the curve itself by the parametric equation (4.5), which we may write for brevity

$$x = \Sigma \rho_1 c_1 t_1, \tag{5.1}$$

the summation understood to be for three terms. This carries with it the conjugate equation

$$\bar{x} = \Sigma \rho_1 / c_1 t_1. \tag{5.2}$$

Now, according to Colonel Hippisley's suggestion, let us define a new point, y, by the equations

$$y = \Sigma \rho_1 c_1 / t_1, \tag{5.3}$$

$$\bar{y} = \Sigma \rho_1 t_1 / c_1. \tag{5.4}$$

The point y is also on the curve.

Out of the four equations (5.1)–(5.4) we expect some simple relation between x and y. But before proceeding further we may note, from the fundamental identities

$$\begin{aligned} 1 - \rho_1 t_1 &= \rho_2 t_2 + \rho_3 t_3, \\ 1 - \rho_1 / t_1 &= \rho_2 / t_2 + \rho_3 / t_3, \end{aligned} \tag{4.2}$$

that

$$\begin{aligned} \rho_2 \rho_3 (t_2/t_3 + t_3/t_2) &= 1 + \rho_1^2 - \rho_2^2 - \rho_3^2 - \rho_1(t_1 + 1/t_1) \\ &= B_1 - \rho_1(t_1 + 1/t_1), \end{aligned}$$

[1] They have also been called *foci* of the sextic; the change of terminology is not very happy, but is necessary in considering the curve from an inversive rather than from a projective standpoint. The argument is set forth in the *Proc. London Math. Soc.*, ser. 2, vol. 21 (1921), pp. 153, 154.

where B_1 is real and constant. We have then, upon forming the product xy,

$$\begin{aligned} xy &= c_1^2\rho_1^2 + c_2^2\rho_2^2 + c_3^2\rho_3^2 + c_1c_2B_3 + c_2c_3B_1 + c_3c_1B_2 - \Sigma c_2c_3\rho_1(t_1 + 1/t_1) \\ &= A - c_1c_2c_3(\bar{x} + \bar{y}), \end{aligned} \tag{5.5}$$

where A is constant.

From the definitions of x and y we therefore obtain directly the two equations

$$\bar{x} + \bar{y} + xy/s_3 = \bar{C} \tag{5.6}$$

and

$$x + y + \bar{x}\bar{y}s_3 = C, \tag{5.7}$$

where s_3 is written for the product $c_1c_2c_3$ and $\bar{C} = A/s_3$ so that C is still a constant.

The equations (5.6) and (5.7) give a one-to-one correspondence, mutual or involutory, between x and y. Thus the whole curve has a transformation into itself in which the points x, y figure as correspondent points; and the form of (5.7) instantly suggests the transformation by isogonal conjugates given in equation (3.5). In fact, we may identify (5.7) as the transformation by isogonal conjugates with respect to the three points a_1, a_2, a_3 whose symmetric functions are σ_1, σ_2, σ_3, if we put

$$\sigma_3 = s_3 \qquad \text{and} \qquad \sigma_1 = C. \tag{5.8}$$

If this is done, we arrive by a short route at the keystone property of the three-bar curve, the transformation noted first by Dr. Bennett. The points a_1, a_2, a_3 are not the same as the points c_1, c_2, c_3, but they too are on the base circle; and are, as a matter of fact, the double points of the three-bar curve. The relation (5.8), that the sum of the central vectorial angles is the same (mod 2π) for the set of double-points as for the set of poles, was given by Cayley.

THE DYNAMICS OF MONOPOLY.

By G. C. EVANS, The Rice Institute, Houston, Texas.

1. Introduction. It is related to a familiar symptom of "prosperity" that the demand for a commodity is greater when the price is increasing than when it is decreasing. An exaggerated instance of this law of demand, perhaps an undue generalization from the last crisis, is what the wholesale lumber dealers tell us is characteristic of their sales, namely, that when prices are going up the demand is insatiable, but when prices go down it is nil until the price movement stops. It is this phenomenon, far outside the scope of traditional economic theory, which we wish to discuss.

In an earlier paper (*1922*, 371–380), the author investigated the situation in which the cost of producing an amount u of a given commodity per unit time was a quadratic function of the amount produced:

$$q(u) = Au^2 + Bu + C,$$

and the demand for the commodity—the amount which the market would absorb in unit time if the price were p—was a linear function of the price:

$$y = ap + b.$$

The simplest approximation for the situation now to be discussed is to write for the demand function the following linear differential expression:

$$y = ap + b + h\frac{dp}{dt}\cdot \tag{1}$$

The practical case is probably to take $h > 0$, but it is interesting, at least theoretically, to leave the sign of that constant arbitrary, as we shall see. As for the other constants, we assume, as in the earlier paper,

$$a < 0, \qquad b > 0, \qquad A > 0, \qquad B > 0, \qquad C > 0.$$

As before, also, we assume that $u = y$. We limit ourselves to the case of a single producer.

We must now introduce some kind of extremal condition to determine the unknown function $p(t)$. On account of the presence of the quantity $p' = dp/dt$ we cannot simply say, as we did without this quantity, that the profit

$$\pi = py - q(u) = pu - q(u)$$

is to be a maximum, for the problem has now become one of the type indicated in the closing section of the article cited, where the quantities involve a whole interval of time. A simple assumption, descriptive of monopoly, is *to make a maximum the total profit over an interval of time. Let us further assume that at the time $t_1 = 0$ the price is p_1, and at the time t_2 the price is p_2; and in the first instance restrict our investigation to functions $p(t)$ which are continuous with their first derivatives in the intervening stretch of time.* We can imagine that at the initial time the cost function has been changed from some previous formula to the one given above, and the producer desires, on the basis of the new cost function, to arrive at a new equilibrium price at the time t_2, by continuous change, in the most profitable manner. This is a problem of *economic dynamics*.[1]

2. Maximum of an Integral. Our task is to find the function $p(t)$ which makes a maximum the integral

$$\begin{aligned}\Pi &= \int_0^{t_2} \pi(p, p', t)dt \\ &= \int_0^{t_2} \{p(ap + b + hp') - A(ap + b + hp')^2 - B(ap + b + hp') - C\}dt\end{aligned} \tag{2}$$

with $p(0) = p_1$ and $p(t_2) = p_2$. An editor of the MONTHLY—Professor Bennett—has said that one should be obliged to present a certificate of character before

[1] See for instance the classification in Amoroso, *Lezioni di Economia Matematica*, Bologna (1921), ch. 5.

being initiated into the mysteries of the Calculus of Variations, to which study our present investigation belongs, since its fascination is so great that neophytes seek to introduce it into problems which would otherwise be perfectly simple. We shall perform, however, the reverse process and discuss the possible maxima of our integral, Π, in terms of the elementary properties of a function of a single variable.[1]

Let $p(t)$ be the desired function, if there is any which satisfies the conditions, and let $f(t) = p(t) + \psi(t)$ be any other function continuous with continuous derivative and such that $f(0) = p(0), f(t_2) = p(t_2)$. Then $\psi(t)$ will be continuous with its derivative, and $\psi(0) = \psi(t_2) = 0$. Write

$$\xi(t) = p(t) + x\psi(t),$$
$$\frac{d\xi}{dt} = \xi'(t) = p'(t) + x\psi'(t),$$

x being an arbitrary parameter, $0 \leq x \leq 1$. Consider the integral

$$\Pi(x) = \int_0^{t_2} \pi(\xi, \xi', t)dt,$$

which for a given arbitrary $\psi(t)$ or $f(t)$ is a function of the single variable x.

On account of the explicit form of $\pi(\xi, \xi', t)$ it is seen at once, by carrying out the differentiation, that, given $\psi(t)$, the quantity $d\Pi/dx$ exists for every value of x. In fact

$$\begin{aligned} \frac{d\Pi}{dx} &= \int_0^{t_2} \left\{ \frac{\partial \pi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \pi}{\partial \xi'}\frac{\partial \xi'}{\partial x} \right\} dt \\ &= \int_0^{t_2} \left\{ \frac{\partial \pi}{\partial \xi}\psi(t) + \frac{\partial \pi}{\partial \xi'}\psi'(t) \right\} dt. \end{aligned} \tag{3}$$

Hence it is necessary, in order that $x = 0$ shall correspond to the maximum of Π (*i.e.*, that $p(t)$ shall be the desired function), that

$$\frac{d\Pi}{dx} = 0 \qquad \text{when} \quad x = 0. \tag{a}$$

It is sufficient for a maximum if, no matter what $\psi(t)$ has been given, subject to the enunciated conditions, we have $\Pi(1) < \Pi(0)$. But by the law of the mean:

$$\Pi(1) - \Pi(0) = \frac{d\Pi(x)}{dx}$$

for some x, $0 < x < 1$. Hence it is sufficient for a maximum, if given $\psi(t)$ arbi-

[1] It is perhaps unnecessary to remind the reader that a readable introduction to the Calculus of Variations may be found in Goursat, *Cours d'Analyse*, vol. III, Paris (1915). In our problem the t does not happen to occur explicitly in the function under the integral sign.

trarily in its domain we have

$$\frac{d\Pi}{dx} < 0 \qquad \text{for all } x, \quad 0 < x < 1. \tag{b}$$

It happens that the conditions (*a*), (*b*) yield the solution of our problem.

In the expression for $d\Pi/dx$, equation (3), we can now perform on the term involving $\psi'(t)$ an integration by parts and write

$$\frac{d\Pi}{dx} = \left[\frac{\partial \pi}{\partial \xi'}\psi(t)\right]_0^{t_2} + \int_0^{t_2} \left\{\frac{\partial \pi}{\partial \xi} - \frac{d}{dt}\frac{\partial \pi}{\partial \xi'}\right\}\psi(t)dt, \tag{3'}$$

in which the first term of the second member vanishes, since $\psi(0) = \psi(t_2) = 0$. By a short calculation

$$\frac{\partial \pi}{\partial \xi} = 2a\xi(1 - Aa) + (b - 2Aab - Ba) + h\xi'(1 - 2Aa),$$

$$\frac{\partial \pi}{\partial \xi'} = h\xi(1 - 2Aa) - h(2Ab + B) - 2Ah^2\xi',$$

$$\frac{d}{dt}\frac{\partial \pi}{\partial \xi'} = h\xi'(1 - 2Aa) - 2Ah^2\xi'',$$

and therefore

$$\frac{d\Pi}{dx} = \int_0^{t_2} \{2Ah^2(p''(t) + x\psi''(t)) + 2a(1 - Aa)(p(t) + x\psi(t)) + (b - 2Aab - Ba)\}\psi(t)dt. \tag{4}$$

3. The Conditions (*a*) and (*b*). If (*a*) is to be satisfied with $\psi(t)$ arbitrary in its domain, we may deduce from equation (4), after putting $x = 0$, the vanishing of the multiplier of $\psi(t)$ in the integrand itself. Hence

$$2Ah^2p''(t) + 2a(1 - Aa)p(t) + (b - 2Aab - Ba) = 0. \tag{5}$$

In other words, $p(t)$ must be the continuous solution of this equation with continuous derivative which at $t = 0$ takes on the value p_1, and at $t = t_2$ the value p_2, if there is such a solution.

But in this case, for any x, $0 < x < 1$, the quantity $d\Pi/dx$ reduces to

$$\frac{d\Pi}{dx} = x\int_0^{t_2} \{2Ah^2\psi''(t) + 2a(1 - Aa)\psi(t)\}\psi(t)dt$$

$$= x[2Ah^2\psi'(t)\psi(t)]_0^{t_2} + x\int_0^{t_2} \{-2Ah^2(\psi'(t))^2 + 2a(1 - Aa)(\psi(t))^2\}dt,$$

as we partly retrace the integration by parts. This expression is in fact essentially negative, since the term outside the integral vanishes with $\psi(t)$ at 0 and t_2, and the integral itself is negative on account of the inequalities $A > 0$, $a < 0$.

Condition (*b*), as well as condition (*a*), is therefore satisfied if $p(t)$ is the

desired solution of (5). This equation is directly solvable in terms of exponential or hyperbolic functions. Thus if we introduce the constants

$$p_0 = \frac{b - 2Aab - Ba}{-2a(1 - Aa)}, \qquad m^2 = \frac{-a(1 - Aa)}{Ah^2},$$

the equation may be written in the form

$$p'' - m^2 p = -m^2 p_0, \tag{6}$$

and the general solution of the equation is seen by trial to be

$$p(t) = p_0 + C_1 e^{mt} + C_2 e^{-mt} \tag{7}$$

with C_1 and C_2 arbitrary constants and m the positive root of m^2. If we let $r_1 = p_1 - p_0$, $r_2 = p_2 - p_0$, substitution in this formula gives us

$$C_1 = \frac{r_1 - r_2 e^{mt_2}}{1 - e^{2mt_2}}, \qquad C_2 = \frac{r_1 - r_2 e^{-mt_2}}{1 - e^{-2mt_2}}. \tag{7'}$$

There is therefore always a unique solution of the problem proposed.

4. Discussion of Solution. A particular solution of (6) is the constant $p = p_0$. It is interesting to note that this price is the "Cournot monopoly price" obtained when the equation of demand does not involve dp/dt.[1] It continues to be a solution of our problem with $h \neq 0$ if the end values are properly chosen. Moreover the formulæ (7), (7′) show that the only solutions which remain finite as t_2 becomes infinite are those which approach p_0 asymptotically, *viz.*,

$$p = p_0 + r_1 e^{-mt}. \tag{8}$$

No solution of (6) which is not identically equal to p_0 can take on the value p_0 more than once; in fact, as is seen from (7), this will be the value of t, if that value is real, not negative and $\leq t_2$, for which

$$e^{2mt} = -C_2/C_1.$$

The calculation of the slope dp/dt gives an accurate picture of the graph of p as a function of t. From (7),

$$\frac{dp}{dt} = m[C_1 e^{mt} - C_2 e^{-mt}].$$

At the time $t = 0$, this has the value

$$p_1' = m(C_1 - C_2),$$

and the slope will vanish for the single value of t (if that value is real, $0 \leq t \leq t_2$) for which

$$e^{2mt} = C_2/C_1.$$

[1] See the author's paper, already cited.

By calculating the values of $C_1 - C_2$ and C_2/C_1 in terms of r_1 and r_2, the reader will find it interesting to verify the following facts with respect to the graph of price against time.

If r_1 and r_2 have opposite signs, the graph crosses the line $p = p_0$ once and has no horizontal tangent in the interval, the price continuously decreasing or increasing, as the case may be, from p_1 to p_2; if r_1 and r_2 have the same sign, the graph fails to cross the line $p = p_0$ at all, and has one and only one horizontal tangent provided the interval of time is large enough,—*i.e.*, provided

$$\cosh mt_2 \geq \frac{r_1}{r_2} \geq 1/\cosh mt_2.$$

Otherwise there is no horizontal tangent. Whether there is a maximum or a minimum is decided in any case of the existence of a horizontal tangent (since the second derivative does not vanish) by the slope of the graph at $t = 0$, and this is readily seen to have the same algebraic sign as the quantity

$$r_2 - r_1 \cosh mt_2.$$

But a comparison of this quantity with the previous inequality shows that when r_2 and r_1 are both positive the graph has a minimum, and when r_2 and r_1 are both negative the graph has a maximum; in other words, when r_1 and r_2 have the same sign, the graph is contained between p_0 and one of the values p_1 or p_2.

It is worthy of remark that these results are all independent of the sign of h in (1). It is merely a question of which end of the graph contributes most of the profit or least of the loss—since it is not essential that Π itself shall be positive.

5. An Extension of the Domain of $\psi(t)$. Obviously it is usually not possible to keep dp/dt continuous at the times $t = 0$ and $t = t_2$, as we see in the case when we change from one price p_1, which has been constant, to a second price p_2, which will be constant; it is therefore unreasonable to restrict ourselves to the consideration of curves whose derivatives remain continuous *within* the interval of time 0 to t_2. It is interesting to note that the solution already obtained is valid even when this restriction has been removed from the class of functions with which we deal in forming Π.

In fact, let $p(t)$ be the function already found, and $f(t)$ any other taking on the same initial and final values, continuous and with a derivative which remains bounded but may have discontinuities at a finite number of places.[1] Then $\psi(t) = f(t) - p(t)$ will be a function of the same sort. It remains to show that condition (6) will still be fulfilled.

If in the evaluation of the second member of (3) we separate out those terms which do not involve x, they will all cancel among themselves, for they form precisely the quantity $(d\Pi/dx)_{x=0}$ which vanishes as before on account of the

[1] More generally, let $f(t)$, taking on the same initial and final values as $p(t)$, be absolutely continuous and have a derivative which is summable, with its square.

choice of $p(t)$. Hence

$$\frac{d\Pi}{dx} = x\int_0^{t_2}[2a(1 - Aa)\{\psi(t)\}^2 - 2Ah^2\{\psi'(t)\}^2 + 2h(1 - 2Aa)\psi(t)\psi'(t)]dt.$$

But

$$2\int_0^{t_2}\psi(t)\psi'(t)dt = \int_0^{t_2}\frac{d}{dt}\{\psi(t)\}^2dt = 0,$$

since $\psi(0) = \psi(t_2) = 0$, and the rest of $d\Pi/dx$ is essentially negative since $a < 0$, $A > 0$. The point is thus proved.

6. Related Problems. One purpose in writing the present paper, as well as the previous one, has been to show the wide range of problems suggested and solvable by a moderate mathematical equipment, and to encourage others to read in a direction that cannot but be fruitful.[1] The reader may for instance be tempted to apply the methods of this paper to the problem of competition discussed in the earlier article or, by means of a price index, commence the study of the changes in the demand and cost functions resulting from taxation. There are one or two specific questions that are worth mentioning because they lie close at hand.

What can be said about the extremals—that is, solutions of (6)—which start from a given value p_1 at $t = 0$? Which gives the greatest total profit during the arbitrary time t_2? With this form of the question the result depends upon the algebraic sign of h, the question not admitting an answer for all values of t_2 unless h is negative. But for very large changes in price it is not reasonable to assume the law (1) to remain valid. Instead of changing the law, however, it is desirable to limit the variation of price to some region $p_0 \pm M$ where M is some fixed amount. What can be said about the best price curve in this range, starting from p_1, as t_2 becomes indefinitely large?

A last question exhibits the ease with which one may be led into questions of considerable mathematical difficulty and interest. We can imagine as a further refinement to (1) a law in which the quantities a, h are not constants but functions of p, p'. As a likely hypothesis consider the case where a, b in (1) are given constants, but h has one of two values h_1, or h_2, according as p' is positive or negative respectively.

EULER'S OUTPUT, A HISTORICAL NOTE.

By W. W. R. BALL, Cambridge, England.

Professor D. E. Smith has enriched many of the recent numbers of the MONTHLY with extracts from letters in his possession on details of mathematical history. There is perhaps a similar interest in the following extract from a letter in my possession written by De Morgan and dated 17 October 1858. The first part of the extract only puts in a striking way what is familiar to many students;

[1] For example, the works of Cournot, Jevons, Walras, Pareto and Fisher. Those who can read Italian will find interesting the volume of Amoroso, already cited.

the allusion in it to 1000 miles in 1000 hours refers of course to the famous bet, on the issue of which about £100,000 was staked, made by Captain Barclay Allardyce in 1809 that he would walk one mile in each of a thousand consecutive hours, covering nearly six weeks. The story in the second part of the extract rests I believe only on tradition, but I have heard it from other sources and have no reason to doubt its truth.

Extract from De Morgan's Letter.—Euler's life, dating from 1736, the year in which his productions began to appear with rapidity, is a period of 47 years; during the last 17 years he was totally blind, and throughout the whole of it he suffered from the consequences of a fever which deprived him of the sight of one eye at its commencement. Nor was he secluded from the world: he married a second wife, and was the father of 13 children. His life was not free from such calamities as interrupt the course of study. He saw the deathbeds of ten of his children; his house was set on fire and wholly burnt; and an attempt to restore his sight by couching led to an illness which nearly ended his days. He was fond of conversation, of the society of his family, and of music; and was, throughout the whole of his career, at the orders of a royal or imperial patron. Nevertheless, if his memoirs be counted, and if his separate works (not volumes) be allowed for at the average rate of 20 memoirs each, which is an insufficient rating both as to bulk and matter, the result is as follows: Distribute Euler's work through the whole period—the real and average distributions will not much disagree—and there is for each and every fortnight in 47 years a separate effort of mathematical invention, digested, arranged, written in Latin, and amplified, often to a tedious extent, by corollaries and scholia. Through all this mass, the power of the inventor is almost uniformly distributed, and apparently without effort. There is nothing like this, except this, in the history of science: it is the thousand miles in the thousand hours.

When Euler was at Berlin, and there is no reason to suppose it was otherwise at St. Petersburg, he was in the habit of writing memoir after memoir, and placing each, when finished, at the top of a pile of manuscript. The secretaries of the academy helped themselves from time to time, by taking papers from the top of the pile, according to their estimate of the bulk of matter likely to be wanted for reading. The consequence was that, as the pile often increased more rapidly than the demands upon it, the memoirs which happened to be at the bottom remained there for a long time. This explains how various memoirs of Euler were published, though considerable extensions and improvements of the matter contained in them had been previously published [under his name].

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

QUESTIONS.

The following question is suggested by Professor Dadourian's note on the catenary, which is printed below.

51. Can any reader supply approximate formulas for the problem of a cable suspended from two points at different levels?

DISCUSSIONS.

I. NOTE ON THE CATENARY.

By H. M. DADOURIAN, Trinity College.

The object of this note is to derive a number of exact and approximate relations which might be of use in connection with uniform and flexible cables suspended from two points.

Let s denote the length of that portion of the cable which lies between its lowest point and any of its points $P(x, y)$, T be the tension of the cable at $P(x, y)$, T_0 the tension at the lowest point of the cable and w the weight of the cable per unit length. Then it may be shown that the following relations hold:[1]

$$T^2 = T_0^2 + w^2 s^2, \tag{1}$$

$$s^2 = y^2 - a^2 \qquad (a = T_0/w), \tag{2}$$

$$2s = a(e^{x/a} - e^{-x/a}). \tag{3}$$

Let $2D$, $2L$, H, and T_m denote, respectively, the span, the length, the dip below the suspension points, and the maximum tension of the cable. Then, for $s = L$, $y = H + a$. Making these substitutions in (2) we have

$$a = (L^2 - H^2)/2H; \tag{4}$$

and consequently

$$T_0 = (L^2 - H^2)w/2H. \tag{5}$$

Introducing this value of T_0 into (1) and simplifying, we obtain

$$T_m = (L^2 + H^2)w/2H. \tag{6}$$

From (5) and (6) we get

$$L = (\sqrt{T_m^2 - T_0^2})/w. \tag{7}$$

When the tension of the cable is very great; in other words, when the sag is very small, simple expressions may be obtained for L, T and T_m in terms of the easily measurable magnitudes D, H and w.

[1] *Cf.* Dadourian, *Analytical Mechanics*, 2nd ed., p. 109

Expanding the right-hand member of (3) in powers of x, we have

$$s = x + x^3/(a^2 3!) + x^5/(a^4 5!) + \cdots. \tag{8}$$

Therefore we obtain $L = D$ for a first approximation and $L = D + D^3/6a^2$, or $L = D[1 + 2D^2H^2/3(L^2 - H^2)^2]$, for a second approximation. In the correction term of the last equation $L^2 - H^2$ may be replaced by D^2 and the following relation obtained:

$$L = D(1 + 2H^2/3D^2). \tag{9}$$

Thus the increase in length due to sagging is given by

$$\Delta L = 2H^2/3D; \tag{10}$$

that is,

$$\Delta L = D^3w^2/6T_m^2. \tag{11}$$

From (5), (6) and (9) we obtain, after neglecting terms containing H^4/D^4,

$$T_0 = (D^2 + H^2/3)w/2H, \tag{12}$$

$$T_m = (D^2 + 7H^2/3)w/2H.^1 \tag{13}$$

II. An Inequality in Connection with Logarithms.

By A. A. Bennett, University of Texas.

Two of the most obvious questions that present themselves in an introductory logical study of logarithms are: (1) How can one demonstrate that the tables of common logarithms are correct (to the given number of places) even in such simple cases as for instance $\log_{10} 2$ and $\log_{10} 3$? (2) How is one to be convinced in a few steps that the limit of $(1 + x/n)^n$ as n tends toward infinity through positive integral values, exists, and can indeed serve as a definition of e^x (confining ourselves to real values of x), without recourse to the theory of infinite series?

It is of some interest that a single elementary inequality can be used to answer both of these questions and others related to them. Thus two problems arising from the study of a common subject but having otherwise apparently little in common are again related by a common origin for the proof, although this common element is put to different use in the two cases. The inequality is the following,

$$(1 + x/m)^m(1 - x/n)^n < 1, \tag{A}$$

for every real x and every pair of natural numbers m and n not less than the numerical value of x.

This inequality is readily established by induction. Indeed we have, in the first place, $(a - d)/a < a/(a + d)$ for $a > d > 0$, since on clearing of fractions,

[1] The circle of curvature at the lowest point of the catenary has a radius a, and the equations (2), (3) and (8) apply to this circle to the order of approximation used in this paper. Hence also (10) is approximately true when the circle is substituted for the catenary. Consequently graphical methods based on the assumption of a circular arc are admissible to the desired degree of accuracy. The formulas (2) and (4) are of course exact for the catenary, but not for the circle.
Editor.

a and $a + d$ being positive, $a^2 - d^2 < a^2$. Similarly, if we write $a - d$ in place of a, $(a - 2d)/(a - d) < (a - d)/a$ for $a > 2d > 0$. Multiplying these two inequalities of like sign together we have *a fortiori* $[(a - 2d)/(a - d)][(a - d)/a] < [a/(a + d)]^2$, if $a > 2d > 0$. More generally, $(a - md)/a < [a/(a + d)]^m$. Hence $[(a - md)/a]^n < [a/(a + d)]^{mn}$. Continuing in this manner, we obtain

$$[(a - md)/a]^n < [a/(a + nd)]^m, \tag{B}$$

for $a > md > 0$. A slightly more general inequality follows at once from this, namely,

$$[(a - md)/a]^n < [(a + c)/(a + c + nd)]^m, \tag{C}$$

for $a > md > 0$, and $c > 0$. Writing ax/mn in place of d, we have, on condition that $n > x > 0$, the desired inequality (A) for positive real values of x. From the symmetry of the equation, we conclude its validity also for x negative.

Let us first take up the second of the two questions proposed. For convenience in writing we shall use the symbol $E(x, m)$ to denote the expression $(1 + x/m)^m$. We shall also break up the discussion into a succession of numbered propositions, some of which are little more than lemmas. We shall assume that $E(x, m)$ is defined for all real finite values of x and m for which $(1 + x/m) > 0$. This assumes familiarity with the concept of fractional, irrational, and negative exponents as introduced in the subject of elementary algebra.

1. $E(x, m)$, for $m > 0$, is continuous in x and m. The details of the proof will be omitted.

2. For p, q, real numbers for which $1 + qx/p > 0$, $[E(x, p/q)]^q = E(qx, p)$. An analytical identity.

3. For a given positive m and for x ranging over positive values, $E(x, m)$ is an increasing function of x. The proof is obvious if the positive mth root be taken.

4. For a given positive x, and for m ranging over positive values, $E(x, m)$ is an increasing function of m. The proof will be carried through several steps. For positive integral values of m, $E(x, m)$ may be written as a polynomial in x of the form

$$1 + x + (1 - 1/m)x^2/2! + \cdots + (1 - 1/m)(1 - 2/m)\cdots(1/m)x^m/m!.$$

The result of replacing m by $m + 1$ is to increase each of the $m + 1$ terms already secured and to add a new final term which is positive. The theorem is therefore established for positive integral values of m. To compare the values of $E(x, m)$ for two positive rational values of m it is sufficient that we reduce these to a common denominator, and compare the results for the two numerators. In other words, to prove the proposition for rational numbers it will suffice to show that for each fixed positive integral denominator, q, $E(x, p/q)$ is an increasing function of the numerator regarded as ranging over natural numbers. Proposition 2 above serves to prove this case. In view of the continuity mentioned in 1 above, the complete proposition follows.

5. $E(x, m)$ is bounded for any real positive x, as m increases through positive real values. Indeed from (A) we have an infinite set of inequalities, namely, $E(x, m) < 1/E(-x, n)$ for every natural number, n, greater than x. This holds at once if m be a natural number, but by 4 continues to be true for all real positive values of m.

6. The limit of $E(x, m)$ for x given and m approaching infinity exists for every positive real x. Let this be denoted by $E(x)$.

7. If $0 < x < y$, $E(x) < E(y)$.

8. $[E(-x, m)]^{-1} = E(x, m-x)\cdot[1 + x/(m-x)]^x$ for $m > x > 0$. An analytic identity.

9. The limit of $E(-x, m)$ for x given and m approaching infinity exists for every positive real x. Let this be denoted by $E(-x)$. It is sufficient to pass to the limit in each factor of the right-hand member of 8, and to note that neither limit is zero. Since the second factor approaches unity and the first factor approaches $E(x)$, we have

10. $E(-x) = 1/E(x)$.

11. $E(ax) = [E(x)]^a$ for real values of a. We have, from 2, $E(x, m)^a = E(ax, am)$. For a positive, we may let m increase toward infinity and thus prove the proposition in this case. For a negative, we may write $a = -b$, and use 10 above, and the result just obtained for b in place of a.

12. $E(x + y) = E(x)\cdot E(y)$. This follows at once from 11, by writing $y = cx$.

13. $[(m + x)/m]^m \leq E(x) \leq [n/(n-x)]^n$ for every positive m and n for which also $m + x$ and $n - x$ are positive. For x, zero, it is evident that the equality sign must be used throughout. For x, positive, the inequality for the first two members is a restatement of 4 above, after the notions of 5 and 6 are introduced. The inequality between the first and third members for x positive, is a consequence of (A). The above reasons also show that at worst there could be only an equality between the second and third members for certain choices of x and n. For x negative, we may write $y = -x$, and express the relations in terms of y. Inverting, rearranging and making use of 10, we have from the relation, $[(n + y)/n]^n < E(y) \leq [m/(m-y)]^m$, established for y positive, the relation desired with an inequality between the second and third members. We shall not spend the time in showing that the equality holds only for $x = 0$.

The relation, 13, for the case in which $m = n = 1$, is well-known.

The propositions given above not only serve to show that $E(x)$ as here defined may be written in the form of e^x, where $e = E(1)$, but serve to give simple approximations to e^x, even for large values of x, which is more than can be said for the series, owing to the slow convergence of the latter, although of course modifications may be introduced. The arithmetical character of the present treatment is of course its chief interest.

Turning now to the first of the two questions proposed, we shall find it more convenient to use the fundamental inequality in the form (B). Being interested in common logarithms, and in particular in the logarithms of 2 and of 3, we may make use of powers and products of these, as also like expressions involving

10/2, namely 5. The interesting feature of the inequality is that if several available numbers are given near to each other, new expressions are obtained numerically much larger but with ratios more nearly equal to unity. Thus from 4, 3, 2, we have $4/3 < 3/2$, whence $8 < 9$. From $8/9 < 9/10$, we get $80 < 81$. Using the more general form, (C), we have, from $25/24 < 16/15$, the relation $125 < 128$. This one relation when written in the form $1000 < 1024$ suffices to show that the logarithm of 2 is slightly in excess of 0.3. From $(9/10)^2 < 10/12$ we have $243/250 < 1$.

Without going further in this direction, we shall see what we can obtain by use of the ratios, 80/81, 125/128 and 243/250, alone, and their products. To simplify the problem, we shall consider merely products not involving more than two of these fractions as factors. Testing these for relative magnitude, and arranging accordingly, we have the following in increasing order of magnitude:

$$1,\ (250/243)\cdot(125/128),\ (128/125)\cdot(80/81),\ 81/80,\ 128/125,\ 250/243.$$

Multiplying by the common denominators of these taken three in succession at a time, we have the following products to consider:

$$\begin{aligned}
2^6 \times 3^5 \times 5^2 &= 388800,\\
5^8 &= 390625,\\
2^{17} \times 3 &= 393216,\\
2^2 \times 3^9 \times 5 &= 393660.\\
2^{15} \times 5 &= 163840,\\
3^8 \times 5^2 &= 164025,\\
2^{11} \times 3^4 &= 165888.\\
3^9 \times 5^2 &= 492075,\\
2^{11} \times 3^5 &= 497664,\\
2^5 \times 5^6 &= 500000.
\end{aligned}$$

Applying (B) to these, we have

$$\left(\frac{2^6 \times 3^5 \times 5^2}{5^8}\right)^{2591} < \left(\frac{5^8}{2^{17} \times 3}\right)^{1825},$$

$$\left(\frac{5^8}{2^{17} \times 3}\right)^{444} < \left(\frac{2^{17} \times 3}{2^2 \times 3^9 \times 5}\right)^{2591},$$

$$\left(\frac{2^{15} \times 5}{3^8 \times 5^2}\right)^{1863} < \left(\frac{3^8 \times 5^2}{2^{11} \times 3^4}\right)^{185},$$

$$\left(\frac{3^9 \times 5^2}{2^{11} \times 3^5}\right)^{2336} < \left(\frac{2^{11} \times 3^5}{2^5 \times 5^6}\right)^{5589}.$$

Taking logarithms of both sides, collecting terms, and writing x in place of log 2, and y in place of log 3, and hence $1 - x$ in place of log 5, we have the following

four inequalities:

$$
\begin{aligned}
76717x + 14780y &< 30146,\\
-52556x + 20284y &< -6143,\\
32213x - 15644y &< 2233,\\
-97436x - 18601y &< -38206.
\end{aligned}
$$

Unlike the case with equalities, all four of these relations are required in order to provide suitable inequalities for x and y. Combining we obtain the desired relations in the form

$$
\begin{aligned}
.301029+ &< \log_{10} 2 < .3010301-\\
.47712\ - &< \log_{10} 3 < .47712+.
\end{aligned}
$$

In this work there has been no emphasis upon peculiar properties not available for other numbers, beyond utilizing the fact that 2 is a factor of 10. The work is made more complicated but at the same time gives much closer approximations as more independent prime numbers are considered. Had we confined ourselves to 2 and 5, we would have had only such fractions as 4/5, 5/8, 125/128, 15625/16384, 1953125/2097152, etc., while by introducing 7 in addition to 3, we would have had even with small numbers all of the following with which to commence, 2/3, 3/4, 4/5, 5/6, 6/7, 7/8, 8/9, 9/10, 14/15, 15/16, 20/21, 24/25, 27/28, 32/35, 35/36, 48/49, 49/50, 63/64, 80/81.

These methods may be compared with those given in such algebra books as discuss the direct computation of common logarithms from first principles.

For a different treatment[1] of some of the questions in the first part of this article, see M. B. Porter, "The derivative of the logarithm," this MONTHLY (*1916*, 204–206). Some remarkable approximations are obtained in a systematic and very ingenious method by E. B. Escott: "The calculation of logarithms," *Quarterly Journal of Pure and Applied Mathematics* (1910) pp. 157–167; numerous references are given.

III. A MATHEMATICAL TREATMENT OF PERSPECTIVE.

By J. P. BALLANTINE, Columbia University.

The purpose of this paper is to demonstrate that a picture can be observed and appear exactly like the object only if observed from a single point, called the point of vision, and to give a construction for the point of vision in terms of the three vanishing points for three mutually perpendicular sets of parallel lines.

The author hesitates to submit this paper since the theory is so simple as to be almost trivial, but still the matter does not seem to be of general knowledge, since it is possible to find in mathematical text-books of the highest grade reproductions of photographs whose point of vision is from one to two inches from the printed page.

[1] A treatment of the exponential function based on algebraic inequalities, using no infinite series, and having some resemblance to the method forming the first part of the present paper, may be found in G. Chrystal, *Algebra*, Part II, Edinburgh, 1900, pp. 77–80. EDITOR.

Postulate. A plane P contains a drawing of an object in correct perspective as observed from a point A whose position is fixed with regard to the plane, if and only if the object can be so placed that the projection of the object on P from A coincides with the drawing.

Definition. If a set of parallel lines is projected on a drawing plane, the point of intersection of the projections is called the vanishing point for the set of lines.

Theorem. The direction from the point of vision of a drawing to the vanishing point of a set of parallel lines is parallel to the lines.

Theorem. If A and B are vanishing points for two sets of parallel lines making the angle α, then A and B subtend an angle α at the point of vision.

Theorem. If A and B are vanishing points for two sets of parallel lines which are mutually perpendicular, then the point of vision lies on a sphere of which the line AB is a diameter.

Theorem. If A, B, and C are the three vanishing points corresponding to three mutually perpendicular sets of parallel lines, then the point of vision lies at a point of intersection of the three spheres with lines AB, AC, and BC as diameters, and is thereby uniquely determined as a single point in front of the drawing.

If we consider a printed photograph of a building in which three sets of mutually perpendicular lines appear, the three vanishing points can be constructed, and it is surprising to notice how often the point of vision is much closer to the picture than is convenient to observe from. In the case of many drawings of buildings where a strict observance of the rule of two-point perspective as commonly used by artists, in which vertical lines are supposed to be drawn parallel, would make the two upper edges of the building appear to come together in an acute angle, the artist realizes the error and fudges the straight lines into curves so as to make the angle obtuse. Such a practice is an overt admission that the theory used is only an approximation.

RECENT PUBLICATIONS.

EDITED BY D. C. GILLESPIE, Cornell University, Ithaca, N. Y., to whom communications should be sent.

REVIEWS.

The Reorganization of Mathematics in Secondary Education. A report by the National Committee on Mathematical Requirements. Published by the Mathematical Association of America, 1923. x + 652 pages.

For many months after the appearance of the preliminary report of The National Committee on Mathematical Requirements, teachers of mathematics throughout the United States waited more or less patiently for the promised complete report. The fact that the members of this committee are mathematicians and teachers whose reputations are nation-wide led us to look for a valuable report. In its final form it far exceeds all expectations. The possi-

bilities of a committee of this sort, adequately financed, working over a period of years, are fully exemplified in the publication now under discussion.

It consists of six hundred fifty pages besides valuable plates, and is divided into two principal parts. The first part deals with "General principles and recommendations." These are committee reports divided into eight chapters. The second part consists of eight chapters giving reports on special investigations conducted by various individuals for the committee. The first chapter is merely a "bird's-eye view" of the entire report. The real work of the committee begins in the second chapter.

In these days, when state legislation and survey reports are opposing required mathematics in secondary schools, it is highly essential that arguments be presented in justification of the retention of these courses in the high school curriculum. Pupils and parents are constantly asking the question "What's the stuff good for?" It is incumbent upon the teachers of mathematics in colleges and secondary schools satisfactorily to answer this question. This report furnishes ample material for meeting these demands. The second chapter in particular is devoted to a discussion of the valid aims and purposes of instruction in mathematics. These aims are divided into practical, disciplinary, and cultural, each having several sub-heads. One sentence under the topic "The point of view governing instruction" gives the kernel of the entire chapter: "The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment, and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual." The subject of unified mathematics is given a very brief but fair treatment under the heading "The organization of subject matter." The necessity of thorough professional training for teachers is presented in the latter part of the chapter.

In chapters three and four, the committee outlines the work in mathematics to be covered in grades seven to twelve inclusive. The introduction in grades seven to nine of such topics as line, bar, and circle graphs; statistical graphs; simple frequency distributions; intuitive geometry; and numerical trigonometry may at first cause somewhat of a shock to some of our more conservative educators. However, if the chapters and references are carefully studied, the proposed plan will not appear as revolutionary as it may seem at first sight. The writers of these chapters plainly state that they are not able in all cases to advise as to the order of topics recommended, and thus very wisely avoid committing themselves on the subject of unified mathematics. In discussing the work of the last three years, they make the following statement: "In the majority of high schools, at the present time, the topics suggested can probably be given most advantageously in separate units of a three year program. However, the National Committee is of the opinion that methods of organization are being experimentally perfected whereby teachers will be enabled to present much of

this material more effectively in combined courses unified by one or more of such central ideas as functionality and graphic representation." The striking features of the fourth chapter are the arguments in favor of taking the elective courses suggested for this period, and especially for the presentation of elementary calculus.

In the discussion of "College entrance requirements" the attitude of the committee is unusually fair and unbiased. The position is taken that the high school courses in mathematics should prepare for college courses in general, not mathematics courses only. With this in view, a tabulation is given showing the findings of a committee that endeavored to determine what topics are essential for other college courses. It is furthermore conceded that since college entrance requirements exert such a marked influence upon secondary school teaching, they must "reflect the spirit of sound progressive tendencies." The college entrance examinations, so influential in the eastern section of the United States, are criticized in a very able manner, and valuable suggestions are given for their improvement. Detailed lists of minor requirements and major requirements in algebra are carefully worked out. In the following chapter, the essential propositions of plane and solid geometry are tabulated.

Perhaps the most valuable article in the entire report in actually helping teachers to improve their methods of presenting secondary school mathematics is that on "The function concept in secondary school mathematics." Under this topic E. R. Hedrick brings out in a very simple manner the possibilities of showing the points of contact between algebra, geometry, trigonometry and the problems of every-day experience. Numerous illustrations of the interrelations between quantities are brought to the attention of the reader.

The efforts of the committee in the last chapter of part one to establish greater uniformity in the use of terms and symbols are highly commendable. In every case, the aim seems to be to substitute the simple, commonsense term for the more difficult or technical one. Symbols not universally used are not recommended. Every writer of examination questions, text books in mathematics, or contributions to educational journals would do well to read this chapter.

The first article in the second part of the report deals with the old subject of formal discipline. From a study of twenty-nine important experiments in "transfer," Miss Vevia Blair drew up a set of seven apparently logical conclusions. These were very carefully worded and submitted to forty prominent educational psychologists for their approval or disapproval. One psychologist accused Miss Blair of asking suggestive questions, and so framing them that the great majority of the answers must necessarily be in the affirmative. Whether or not this charge is justifiable, Miss Blair is to be congratulated on getting twenty-four of these forty to state their views in plain English on this much discussed question. The replies are exceedingly interesting, and definitely indicate a general belief in a certain degree of transfer of training. Within certain limitations, the transfer of training is held to be a valid aim in teaching, the amount of transfer being largely dependent upon the methods of teaching.

"The theory of correlation applied to school grades" by A. R. Crathorne is a masterful illustration of the possibilities of the modern theory of statistics applied to problems of secondary education. The article is equally valuable to teachers of mathematics and to those interested in statistics in showing practical applications of the correlation coefficient, correlation ratio, partial correlation, and regression equations. The conclusions are indicative of some important relationships between the various high school subjects, and seem to be entirely consistent with the results of other investigations in the same field.

Although the chapter by J. C. Brown on "Mathematical curricula in foreign countries" deals with conditions as they were previous to the war, it is evident that in many respects we might well follow the lead of our European neighbors in secondary school mathematics. Mr. Brown discusses the curricula in thirteen European countries. He then summarizes the work by years, and makes a comparison by graphs and by tabulations. After reading this chapter, one is chagrined at the superior preparation of the European teacher of secondary mathematics. It is evident that greater emphasis is placed upon thoroughness through drill. Both algebra and geometry are introduced much earlier in the course than is the case in this country, and there is a much closer correlation between the subjects. Two sentences quoted from Mr. Brown show clearly his attitude: "European school men believe that a course in mathematics should be planned by those who know some mathematics rather than by educators who are practically ignorant of the subject. The reports do not indicate that the schools of Europe are hearing a demand for weak algebra and anæmic geometry, or even for no work in these subjects."

"Experimental courses in secondary school mathematics"—a report by Raleigh Schorling—occupies slightly over one hundred pages. Owing to the fact that the junior high school is still in the formative period, most of the experimentation is confined to the seventh, eighth, and ninth grades. A detailed description is given of at least one course in each of fifteen prominent schools. In every case, the school has made a distinct contribution under the leadership of one especially trained in mathematics and in education. The headings of some of these reports will best indicate the comprehensiveness of this study: What mathematics is needed by industry?—Correlated mathematics—What mathematics is demanded by commerce?—An early effort to construct a course with little reference to later courses—The problem of plane geometry—A program of coöperative research—No mathematics in the ninth grade—What mathematics should a boy study who can stay in school only ten weeks?—The project method—Progressive correlation. These accounts of what has actually been done to answer such questions as the above constitute the third part of the chapter, and furnish the source material for the conclusions which are summarized in part one and discussed and illustrated in part two. The conclusions are in accord with the recommendations of chapter three. This chapter merely strengthens the former chapter by showing that such courses have actually been offered successfully.

In accordance with the present popularity of the testing movement, the subject of "Standardized tests in mathematics for secondary schools" is given by far the most space of any subject. This report by C. B. Upton occupies one hundred and fifty pages, or nearly one-fourth of the entire publication. It gives a rather minute description of the various tests in arithmetic, algebra, and geometry with an explanation of the methods of using them. In addition to the standardized educational tests in these three subjects, the Rogers test of mathematical ability, and the Thurstone vocational guidance tests are discussed. The chapter closes with a valuable bibliography of the subject. The only logical conclusion to be drawn from this article is that when a subject as new as this (not yet ten years old) has been accorded such an important position by a committee of mathematicians—not an over-zealous group of educational specialists in measurements—it behooves every mathematician in the land to join the procession. Two sentences from the author's conclusion summarize the importance of tests: "They have begun the detailed analysis of mathematical abilities, they have rendered possible improved methods of classification, and diagnosis and prognosis of individual ability can now be made by this help. Analysis, classification, and diagnosis—these three outstanding ends they serve; and the greatest of these is diagnosis."

In chapter fourteen, we are made to feel even more keenly than in chapter eleven the inferiority of our mathematics courses as compared with those of European countries. This chapter is entitled "The training of teachers of mathematics," and is written by R. C. Archibald. The first part is a summary of a more extensive report published by the Bureau of Education in 1918, dealing with the training of teachers of mathematics in foreign countries. The greater part of the chapter is devoted to a summary of the requirements in the various sections of the United States for the certification of teachers, and of an outline of the courses offered by the colleges and universities of these sections. The chapter closes with a proposed tentative ideal for the preparation of teachers which deserves high commendation, but is decidedly too ambitious in its aims to permit hope for realization in the near future.

Reports of investigations by the questionnaire method constitute the next chapter. The first one is by A. R. Crathorne on "Change of mind between high school and college as to life work." About two thousand freshmen from eleven different colleges answered questions indicating changes in their plans. Interesting tabulations of the various answers are given and summaries are made. The figures show that approximately 57 per cent. had definite plans when they entered high school, and that about half of these changed their plans.

An investigation on "Mathematical interests of high school pupils" was conducted by W. F. Downey. Seven thousand pupils from high schools in the east, west, mid-west, and southwest answered questions concerning their likes and dislikes in mathematics, and in other subjects. The summary shows that four-fifths of the pupils like mathematics, and that the popularity of mathematics compares well with that of other subjects.

Mr. Alfred Davis under the subject "The importance of mathematics as indicated by certain questionnaires" tabulates the results of questions sent out by himself and others to citizens and pupils concerning their attitude toward mathematics. Some of these have been reported elsewhere, but none of these reports leave any doubt in our minds as to the favorable attitude of business men, professional men, and pupils toward the study of mathematics.

A bibliography of nearly one hundred pages covering every possible phase of the teaching of mathematics constitutes the last chapter of the report. This alone is of inestimable value to every teacher of mathematics.

Too much cannot be said in commendation of the National Committee for producing such a comprehensive report, and of the Mathematical Association of America for sponsoring it. It should most certainly be upon the desk of every teacher of secondary mathematics, and should be used in every university and college class dealing with the problems of mathematics teaching. The effects of this work in increasing interest in mathematics and improving the quality of teaching cannot be over-estimated.

A. R. CONGDON, University of Nebraska.

Algebras and Their Arithmetics. By L. E. DICKSON. The University of Chicago Science Series. The University of Chicago Press, 1923. 8vo. xii + 238 pages. Price, $2.35, postpaid.

"The chief purpose of this book is the development for the first time of a general theory of the arithmetics of algebras, which furnishes a direct generalization of the classic theory of algebraic numbers. . . . Just as the final stage in the evolution of number was reached with the introduction of hypercomplex numbers (which make up a linear algebra), so also in arithmetic, which began with integers and was greatly enriched by the introduction of integral algebraic numbers, the final stage of its development is reached in the present new theory of arithmetics of linear algebras."

According to this extract from the preface, the book is primarily an exposition of the author's research, and must therefore be judged mainly according to the novelty and importance of his results. Perhaps the fairest way to form an estimate of the book is to compare the results of the author with those obtained by previous investigators in this field.

It often happens in extending an elementary concept to fit a more general situation that different investigators devise different generalizations. Each of these extensions may be a true generalization of the elementary concept and hence entitled to the same name. But in such cases it is usually true that one definition allows more theorems to carry over from the elementary situation than does any of the others, and is therefore by common consent taken as the standard definition.

This is the situation with regard to the generalization of the term *integer* in passing to linear associative algebras over a general field. A. Hurwitz and Du Pasquier have proposed definitions for the integral elements of linear algebras, and have done considerable work in investigating the consequent theorems for certain algebras. These theorems proved to be decidedly meagre, and indeed their definitions left many algebras entirely without integral elements.

Professor Dickson has recently proved that for a certain algebra the integers of Du Pasquier not only lack the property of unique factorability, but that unique factorability cannot be restored by the introduction of ideals no matter how defined.

In the present work Professor Dickson solves the problem of finding a satisfactory definition for the integral elements of a linear associative algebra, and develops many general properties of them in a manner which makes the reader marvel at his genius for clear and profound analysis. He expresses an algebraic field as a linear algebra, and makes a careful study of its integral elements. By a shrewd generalization he is led to his definition for the integral elements of any algebra. The definition seems to be a happy one, for many interesting general theorems flow easily from it, in marked contrast to the experience with the earlier definitions. All algebras which have integral elements according to Du Pasquier's definition continue to have them under Dickson's definition while the converse is not true. There can be no doubt that the book under review opens up a new field for research which is of vast extent.

The new theory has its practical side also, for applications are made to the solution of Diophantine equations. Only one such equation is solved in detail, but the method is shown to be very powerful.

The first chapter is an introduction to the theory of linear algebras. Attention is called to the fact that the set of all p-rowed square matrices is a linear algebra in p^2 units. An interesting paragraph contains the derivation of quaternions as a special case of a matric algebra over the complex field. The theory of linear dependence with respect to a field is also developed in this chapter.

Chapters II to VIII inclusive are devoted to an exposition of the theory of linear algebras over a general field, the theory due essentially to Wedderburn. This is the most complete and satisfactory theory of linear algebras that has yet been formulated, and the present book is the only place where the theory is to be found outside of foreign mathematical journals. Several simplifications are published here for the first time. In fact, the author communicated freely with Professor Wedderburn during the writing of the book. The more recent work of Scorza was also consulted. As a text on linear algebras, therefore, this book is without rival.

Chapter IX is a concise development of the fundamentals of algebraic number theory. Chapter X, which is 59 pages in length, contains the author's theory of the arithmetics of linear algebras, and has already been discussed.

The concluding chapter XI is on the foundations of algebra, and makes interesting reading. A set of postulates for a field is given and several fundamental theorems are derived from them. The theory of indeterminates and polynomials over a field is developed, and the usual theorems on divisibility of polynomials are proved with a minimum of effort. The chapter closes with a discussion of the congruential theory of polynomials, and Galois fields.

In spite of the inherent difficulty of the subject matter, the book has been written with a view of interesting a wide class of readers, and special effort has

been expended to make the presentation clear and elementary. Only a good general knowledge of mathematics is required, and the preliminary chapters are designed to prepare for an understanding of chapter X. The subject matter is so skillfully arranged and so clearly presented that the book is easy to read and to understand. The use of examples to illustrate the difficult parts is also helpful.

It must not be assumed however that the book is an afternoon's light reading. The author once stated that he did not expect anyone to read his books without a paper and pencil, and the reader will do well to bear this in mind The steps are all given and the proofs are accurate, but there is no exhausting detail. Thus in the middle of page 139 we have a situation which may worry some readers, for there is what appears to be an infinite series $c\omega_2 + c'\omega_2' + \cdots$ with no mention of convergence. A little use of the pencil however will show that all but a finite number of the c's are zero.

In the first two appendices are to be found two rather complicated proofs, and in Appendix III a summarized statement of further results on linear algebras and full references to the original papers in the literature. A unique feature is the statement in this appendix of a number of unsolved problems, practically an invitation to the reader to try his hand at research in this field. The enthusiasm of the author for mathematical research is so great that the reader of this book can hardly help feeling inspired by it, and ambitious to do something on his own account.

The book is well printed with attractive type and thus speaks well for the University of Chicago Science Series of which it is the fifteenth number, and for the Editorial Committee of the Series, of which Professor E. H. Moore is chairman.

Perhaps it is not out of place to state that the mathematical world is indebted for this book not only to the author but also to the University of Chicago Press. Needless to say, such a book as this is financially a losing venture and only the generosity of the Press has made its publication possible. But we must not take this generosity too much for granted, for if more books of this caliber are to be printed by our university presses they must receive the financial as well as the moral support of the public.

C. C. MACDUFFEE.

Lezioni di Geometria Analytica. By E. BORTOLOTTI. Bologna, Nicola Zanichelli, 1923. 8vo. Vol. 1, xxxix + 382 pages. Vol. 2, 229 pages.

This book contains many points of interest to American readers. An introduction of thirty pages gives an unusually full historical development of analytical and projective geometry, covering such topics as the geometric algebra of the ancients, the Greek construction of conics by points, the classic work of Apollonius, the decline of Greek geometry and its renaissance in Italy many centuries later, the researches of Desargues, Torricelli, Descartes and Fermat. A student reading this historical exposition will fully recognize as mythical the statement current in some quarters that the magic pen of Descartes brought forth analytical geometry—*prolem sine matre creatam.*

Conspicuous is the total absence of problems for the student to solve. But in the preface the author stresses the importance of problems and states that he reserves them for publication in a separate volume.

A striking feature of the book is the discussion, in the very first chapter, of imaginary elements, of baricentric coördinates, projective coördinates and homogeneous coördinates. The treatment of the point and straight line is given in the second chapter for plane geometry and in the third chapter for that of space. In the latter chapter are considered also (1) the plane, (2) imaginary points, planes and lines in space, (3) the plückerian or tangential coördinates of planes, as well as the homogeneous cartesian and the general projective coördinates of points and planes. A knowledge of determinants is presupposed.

The second part of the book deals with conics. By this time the student at the University of Bologna has become familiar with limits and derivatives in other mathematical courses and can use that knowledge here. The general treatment of the equation of the second degree in x and y is followed by the consideration of the normal equations of the ellipse, hyperbola and parabola. An appendix deals with homography and reciprocity.

The second volume contains the third and fourth parts which deal, respectively, with quadric surfaces and with the elements of the general theory of curves and surfaces.

The presentation of analytical geometry as given here possesses many points of elegance. The divergence from American texts in the selection and the sequence of topics renders Bortolotti's book of special interest to us.

FLORIAN CAJORI.

ARTICLES IN CURRENT PERIODICALS.

ANNALES DE L'ÉCOLE NORMALE SUPÉRIEURE, volume 58, no. 10, October, 1923: "Analogie entre les séries trigonométriques et les séries sphériques au point de vue de leur sommabilité par les moyennes arithmétiques" (continuation) by E. Kogbetliantz, 289–320.

BULLETIN DES SCIENCES MATHÉMATIQUES, second series, volume 47, September, 1923: Review by E. Cartan of T. Levi-Civita and U. Amaldi, "*Lezioni di Meccanica Razionale. Vol. 1: Cinematica principi e statica.*" (Bologne, 1923), 289–290; "Discours sur Marc Seguin, prononcé à Annonay le 10 Juillet 1923" by E. Picard, 291–298; "Le problème de Dirichlet et le potentiel de simple couche" (continuation and conclusion) by G. Bertrand, 298–307; "Sur la dérivation et l'intégration généralisées" by P. Levy, 307–320.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 29, November, 1923: "An electro-magnetic theory of light-darts" by H. Bateman, 385–393; "Some left co-set and right co-set multipliers for any given finite group" by G. A. Miller, 394–398; "The second mean value theorem for summable functions" by M. B. Porter, 399–400; "Analogies between the U_n, V_n of Lucas and elliptic functions" by E. T. Bell, 401–406; "Singularities of curves of given order" by T. R. Hollcroft, 407–414; "The Hurwitz-Courant Funktionentheorie" [review of A. Hurwitz and R. Courant, *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* (Berlin, 1922)] by O. D. Kellogg, 415–417; Reviews: by A. R. Crathorne of C. V. L. Charlier, *Vorlesungen über die Grundzüge der mathematischen Statistik* (2nd ed., Lund), 418–419; by P. J. Daniell of M. Milankovitch, *Théorie Mathématique des Phénomènes Thermiques produits par la Radiation Solaire* (Paris, 1920), 419–420; by F. Cajori of G. Monge, *Géométrie Descriptive* (Paris, 1922), 420; by L. Ingold of L. Bieberbach, *Differentialrechnung* (Leipzig and Berlin, 1922), 421; by H. B. Phillips of M. Moller, *Kraftarten und Bewegungsformen* (Braunschweig, 1922), 421; by C. H. Forsyth of H. E. Soper, *Frequency Arrays* (Cambridge, 1922), 422; by E. P. Adams of *Atomes et Electrons* (Paris, 1923), 422–423; by H. B. Phillips of A. Gray, G. B. Mathews, and

T. M. MacRobert, *Bessel Functions* (London, 1922), 423; by L. C. Karpinski of H. Wieleitner, *Geschichte der Mathematik* (I, Berlin, 1922), 424; and by E. B. Cowley of A. Mac Leod, *Introduction à la Géométrie non-Euclidienne* (Paris, 1922), and of H. Liebmann, *Nichteuklidische Geometrie* (Berlin and Leipzig, 1923), 425; Notes, 425–428; New publications, 429–432.

Mathematics Teacher, volume 16, October, 1923: "Calculus in the high school" by J. M. Kinney, 321–331; "Time is money" by W. E. Breckenridge, 332–334; "Public honors for mathematical contributions" by G. A. Miller, 335–339; "A Dutch text-book of 1730" by L. G. Simons, 340–347; "Magic circles" by Vera Sanford, 348–349; "Systematic procedure in the solution of algebraic problems" by R. R. Goff, 350–355; "The price of wisdom" by G. W. Evans, 356–358; "Japanese problems" by Shige Hiuama, 359–365; "A study in fractions" by J. E. Worthington, 366–373; News and notes, 374–381; New books, 382–384,—November: "Mathematics club program" by A. H. Wheeler, 385–390; "Craig's edition of Euclid: its use and application of the principal propositions given" by Agnes G. Rowlands, 391–397; "The origin of our numerals" by C. P. Sherman, 398–401; "Live problem material in algebra" by D. S. Davis, 402–413; "Measuring achievement in first year algebra" by H. R. Douglass, 414–420; "Advantages of a general course in mathematics for the first two years in high school" by L. A. McCoy, 421–424; "On the precedence of numerical operations" by R. E. Moritz, 425–430; "The place of the Calculus in the training of the high school teacher" by B. Cosby, 431–439; Discussion, 440–442; News and notes, 443–446; New books, 447–448.

Mathematische Zeitschrift, volume 18, nos. 1–2, September, 1923: "Beitrag zu den Grundlagen der kombinatorischen Analysis situs" by E. Bilz, 1–41; "Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u = 0$" by O. Perron, 42–54; "Ein Seitenstück zur Theorie der linearen Transformationen einer komplexen Veränderlichen. I." by E. Study, 55–86; "Ueber die Randwerte einer analytischen Funktion" by F. Riesz, 87–95; "Herleitung des Gausschen Fehlergesetzes aus einer Funktionalgleichung" by G. Polya, 96–108; "Ueber dyadische Brüche" by A. Khintchine, 109–116; "Ueber eine Verallgemeinerung der Parsevalschen Formel" by F. Riesz, 117–124; "Ueber das Eulersche Summierungsverfahren" (II Mitteilung) by K. Knopp, 125–156; "Ueber eine Verallgemeinerung der Eulerschen Reihentransformation" by O. Perron, 157–172.

Messenger of Mathematics, volume 53, no. 1, May, 1923: "On certain puzzle-questions occurring in early arithmetical writings and the general partition problems with which they are connected" by J. W. L. Glaisher, 1–16.

Proceedings of the National Academy of Science of the U. S. A., volume 9, November, 1923: "A generalization of Volterra's derivative" by H. L. Smith, 397–398.

Quarterly Journal of Mathematics, volume 49, No. 4, March, 1923: "On Legendre's coefficients and associated functions with non-integral subscripts, and their connection with the elliptic integrals" by H. B. C. Darling, 289–303; "On certain Riccati differential equations with algebraic integrals" by A. Berry, 303–308; "Some notes on spheroidal wave-functions" by E. G. C. Poole, 309–321; "Singly infinite class number relations" by E. T. Bell, 322–337; "The genesis of Lamé's equation" by E. H. Neville, 338–352; "Chess tournaments and the like treated by the calculus of symmetric functions" by P. A. MacMahon, 353–384.

Transactions of the American Mathematical Society, volume 24, October, 1922 [published, November, 1923]: "On the location of the roots of certain types of polynomials" by J. L. Walsh, 163–180; "A note on the preceding paper" by D. R. Curtiss, 181–184; "Determination of all general homogeneous polynomials expressible as determinants whose elements are homogeneous polynomials" by H. S. Everett, 185–194; "General vector calculus" by J. B. Shaw, 195–244.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the state-

ments. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3055. Proposed by J. L. RILEY, Tarleton Station, Texas.

Given the non-intersecting circles

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0,$$
$$x^2 + y^2 + a_2x + b_2y + c_2 = 0;$$

it is required to find the four common tangents.

3056. Proposed by W. H. RASCHE, Virginia Polytechnic Institute.

Given four points on the surface of a sphere determining a spherical quadrangle of sides a, b, c, d respectively. Let α be the angle between sides d and a, and β the angle between sides c and d. Give the relation between the angles α and β in terms of a, b, c, d. This is a fundamental problem in connection with spherical mechanisms and linkages in general.

3057. Proposed by A. A. BENNETT, University of Texas.

Euler in a letter to Stirling stated (without any hint as to the method of proof) that $\pi/4$ might be represented as an infinite product where the nth factor is the quotient of the nth odd prime when divided by the integral multiple of 4 nearest to it. Prove that

$$\frac{\pi}{4} = \frac{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot \cdots}{4\cdot 4\cdot 8\cdot 12\cdot 12\cdot 16\cdot \cdots}.$$

3058. Proposed by LOUIS WEISNER, University of Rochester.

If n is any integer greater than 1, the number of integers less than n and prime to n of the form $c + xd$ is $\phi(n)/\phi(d)$, where d is a divisor of n and c is an integer less than d and prime to d.

3059. Proposed by DANIEL KRETH, Wellman, Iowa.

Given the perimeter and the radii of the inscribed and circumscribed circles, to construct the triangle and calculate the lengths of its sides.

3060. Proposed by B. H. BROWN, Dartmouth College.

A link-work of 3 bars, OA, AB, and BO' is fastened to a plane at O and O'; $OO' = AB = 2a$, $OA = BO' = \sqrt{2}a$. The locus of P, the mid-point of AB, is a circle when the bars form (with OO') a parallelogram; and a lemniscate when the bars form a contra-parallelogram.[1] In the latter case, show that the locus of Q, the intersection of OA and $O'B$ (produced) is an equilateral hyperbola. Generalize when $OA = BO' = b$. Consider the relation of the points P and Q from the standpoint of the Argand plane.

3061. Proposed by NORMAN ANNING, University of Michigan.

The cubic, $48y = 25x - x^3$ fits the sine curve, $y = \sin(\pi x/6)$ for $x = -3, -1, 0, 1, 3$. For what x $(-3 < x < 3)$ is the fit the poorest?

SOLUTIONS.

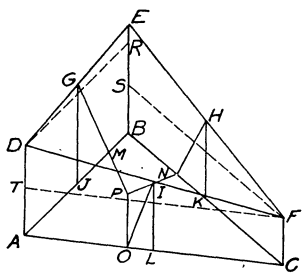
2982 [1922, 313]. Proposed by H. P. MANNING, Providence, R. I.

Solve the following problem without using analytical geometry: A triangular yard has a post at each corner. Given the lengths of the sides of the yard and of the posts, find the length and the position of the foot of a ladder that will just reach to the top of each post without changing the position.

SOLUTION BY B. F. FINKEL, Drury College.

Let ABC be the yard and AD, BE, and CF the posts. Connect the tops of the posts forming the triangle DEF. Let G, H, and I be the mid-points of the sides DE, EF, and FD, respectively. In the plane $ABED$, draw GM perpendicular to DE intersecting AB at M. In like manner,

[1] Link-Work for the Lemniscate, *American Journal of Mathematics*, 1878, vol. 1, p. 386.



determine the point N in the line BC. The point M is equidistant from the points D and E and the point N is equidistant from the points E and F. In the plane of the triangle ABC, erect a perpendicular, MP, to AB at M and a perpendicular, NP, to BC at N. Then all points in MP are equidistant from the points D and E and all points in NP are equidistant from the points E and F. Hence, P, the intersection of MP and NP, is equidistant from the points D, E, and F. Hence, P is the position of the foot of the ladder.

To compute the length of the ladder, let $AB = c$, $AC = b$, $BC = a$, $AD = d$, $BE = e$, and $CF = f$. Draw GJ perpendcular to AB, HK perpendicular to BC, IL perpendicular to AC, DR perpendicular to BE, FS perpendicular to BE, and FT perpendicular to AD.

Then $GJ = (d + e)/2$, $HK = (e + f)/2$, and $IL = (f + d)/2$. From the similar triangles, GJM and ERD, we find $MJ = (e^2 - d^2)/2c$ and $MB = (c^2 + d^2 - e^2)/2c$. Similarly, we find $BN = (a^2 + f^2 - e^2)/2a$.

In the quadrilateral $BMPN$, since the angles at M and N are right angles, the quadrilateral is inscribable in a circle, from which it follows that $BP = MN/\sin B$, but $MN^2 = BM^2 + BN^2 - 2BM\cdot BN \cos B$. Hence,

$$BP^2 = (BM^2 + BN^2 - 2BM\cdot BN \cos B)/\sin^2 B$$

and the length of the ladder, l, is

$$\begin{aligned} l &= \sqrt{BE^2 + BP^2} \\ &= \sqrt{e^2 + \left[\left(\frac{c^2 + d^2 - e^2}{2c}\right)^2 + \left(\frac{a^2 + f^2 - e^2}{2a}\right)^2 - 2\frac{c^2 + d^2 - e^2}{2c}\cdot\frac{a^2 + f^2 - e^2}{2a}\cos B\right]/\sin^2 B} \\ &= \sqrt{e^2 + \frac{a^2(c^2 + d^2 - e^2)^2 + c^2(a^2 + f^2 - e^2)^2 - (c^2 + d^2 - e^2)(a^2 + f^2 - e^2)(a^2 + c^2 - b^2)}{16s(s - a)(s - b)(s - c)}} \end{aligned}$$

where $2s = a + b + c$.

If $d = e = f = 0$, $l = \dfrac{abc}{4A}$, where A is the area of the triangular yard.

Also solved by H. N. Carleton, A. Pelletier, J. B. Reynolds, and F. E. Wood.

2993 [1922, 420]. Proposed by H. C. Bradley, Massachusetts Institute of Technology.

Let ABC be any triangle, and O the center of its circum-circle. Bisect the arcs AB, BC, and CA at F, D, and E. With F, D, and E as centers draw arcs passing in each instance through the adjacent corners of the triangle. Prove that these arcs intersect at the in-center of the triangle ABC.

I. Solution by Theodore Bennett, University of Illinois.

We shall first establish the following lemma: Let $A_1A_2A_3$ be the vertices of any triangle, and let $C_1C_2C_3C_4$ be the centers of the four circles which touch the three sides of $A_1A_2A_3$. Let P_{ij} be the mid-point of the segment C_iC_j. Then the points P_{ij} lie on the circum-circle of $A_1A_2A_3$.

It is clear that any one of the points C_i is the ortho-center of the triangle (T_i) formed by the other three, and the feet of the altitudes of the triangle T_i are the points A_i. Hence, the nine point circle of T_i is the circle passing through $A_1A_2A_3$. From the well-known properties of the nine point circle, it follows that this circum-circle of $A_1A_2A_3$ bisects each segment C_iC_j, and thus passes through the six points P_{ij}.

Now consider the triangle ABC, with in-center I, and circum-circle O. Construct the circle K, passing through ABI. The lines AI and BI are two of the bisectors of the angles at A and B; let the other bisectors of these angles be AJ and BJ. Then it is obvious that the center of K is the mid-point of IJ. But I and J are two of the points C_i in our lemma, and therefore the center of K lies on the circle O, and is obviously F, the mid-point of the arc AB. Similarly the circles described on BC and CA pass through I, and the theorem is proved.

It should be noted that there are two possible positions for each of the points F, E, and D, giving rise to six circles, which pass by threes through the four points C_i, each of these six circles containing two of the points C_i.

II. Solution by H. Halperin, College Station, Tex.

The lines AD, BE, and CF are the bisectors of the angles A, B, and C of the triangle ABC and hence intersect at O', the in-center of the triangle. Consider triangle AFO'. The angle $FO'A$ is measured by $\frac{1}{2}$(arc AF + arc DC) and the angle $O'AF$ is measured by $\frac{1}{2}$(arc FB + arc BD). But arc AF = arc FB and arc DC = arc BD, hence $\angle FO'A = \angle O'AF$ and hence $FO' = FA = FB$ and O' lies on the arc AB described with F as center. Similarly it is proved that it lies on the arcs BC and CA, described with D and E as centers, respectively.

Also solved by Michael Goldberg, William Hoover, A. Pelletier, A. V. Richardson, and W. W. Weber.

In his article "On the I-Centers of a Triangle" (*1918*, 241–246), Professor Altshiller-Court gave a simple proof of the theorem contained in this problem and references to previous proofs.

2994 [1922, 420]. Proposed by R. M. MATHEWS, Wesleyan University.

Can the following construction be made without the use of a regulus? Construct a line which meets four given skew lines.

Solution by Elijah Swift, University of Vermont.

Yes, the line required can be found by the usual Euclidean constructions,—passing a plane through a point and a line, and in the plane using a straightedge and one fixed circle.

Let the lines be α. β, γ, and δ. Take any point, P, on α and pass a plane through P and β cutting δ in B and a second plane through P and γ cutting δ in C. Then the line PB cuts α, β, and δ and the line PC cuts α, γ, and δ. If we can find a position of P for which B and C coincide, the line $PB = PC$ corresponding will cut all four given lines.

If we let the point, P, take different positions, P_1, P_2, etc., we obtain two ranges of points on δ, B_1, B_2, etc., and C_1, C_2, etc. These ranges are projective, for they are the intersections of δ with pencils of planes through β and γ, respectively, and corresponding planes of the two pencils cut α in the same point, P. The problem now reduces to finding the self-corresponding points of two projective ranges on the same line determined by three corresponding pairs (B_1, C_1), (B_2, C_2), (B_3, C_3), and this may be solved by the well-known construction with a fixed circle. Since there may be two, one or no such points, there may be two, one or no lines cutting all four given skew lines.

2995 [1922, 420]. Proposed by S. A. COREY, Des Moines, Iowa.

Give a geometric proof of each of the identities

(*a*) $\cos(a + 2mx) = \cos a - 2\sin x[\sin(a + x) + \sin(a + 3x) + \cdots + \sin(a + (2m - 1)x)]$

(*b*) $\sin(a + 2mx) = \sin a + 2\sin x[\cos(a + x) + \cos(a + 3x) + \cdots + \cos(a + (2m - 1)x)]$

where m is a positive integer.

Solution by A. V. Richardson, Bishop's College.

Let $P'OP$ be the horizontal diameter of a circle in the positive direction with center O and radius $OP = 1$ Take on the circumference the points A, A_2, A_4, $\cdots$, A_{2m} so that $\angle POA = a$, $\angle AOA_2 = \angle A_2OA_4 = \cdots = \angle A_{2m-2}OA_{2m} = 2x$; and denote by Q_{2i} the projection of A_{2i} on the diameter $P'P$. Now AA_2 makes with OP (the positive direction) the angle $90° + a + x$, as is readily seen by dropping the perpendicular OE_1 upon AA_2. Hence

$$QQ_2 = AA_2\cos(90° + a + x) = -2EA_2\sin(a + x) = -2\sin x\sin(a + x)$$

Similarly $Q_{2i-2}Q_{2i} = -2\sin x\sin[a + (2i - 1)x]$. Now the identity $OQ_{2m} = OQ + QQ_2 + Q_2Q_4 + \cdots + Q_{2m-2}Q_{2m}$ gives at once $\cos(a + 2mx) = \cos a - 2\sin x\sin(a + x) - 2\sin x\sin[a + 3x] - \cdots - 2\sin x\sin[a + (2m - 1)x]$. By projecting on the vertical diameter we obtain the expansion for $\sin(a + 2m\,x)$.

2996 [1922, 420]. **Proposed by E. J. OGLESBY, Flushing, N. Y.**

Given $u_1 = .2500$, $u_2 = .4113$, $u_3 = .4785$, $u_4 = .4965$, find x when $u_x = .4311$.

SOLUTION BY S. A. COREY, Des Moines, Iowa.

This problem admits of more than one solution. If we assume that u_x is an analytic function of x whose fourth and higher derivatives are zero, a solution may be obtained as follows.

Let $y = x - 1$. Then we may write $u_y = .2500 + ay + by^2 + cy^3$, where the values of a, b and c may be found from the equations obtained by giving u the values stated. Substituting these values of a, b and c ($.2233\frac{1}{6}$, $-.0695$, $.0074\frac{5}{6}$) in the above equation, we obtain a cubic whose only real root is $y = 1.203$. Whence $x = 2.203$.

Also solved by H. HALPERIN, A. PELLETIER, J. L. RILEY, ELIJAH SWIFT, and the PROPOSER.

2997 [1922, 420]. **Proposed by M. ZAMETKIN, Jamaica, New York.**

Given $a = \sin 5°$, $b = \sin 49°$, and $c = \sin 87°$, prove that

$$\sin 73° = \frac{a^2 - b^2 + ac}{4a(a^2 - b^2 + ac) - (a - b + c)}.$$

SOLUTION BY MAURICE BAUDIN, Miami University.

It is found by successive reductions that

$$a^2 - b^2 + ac = -\sin 39° \sin 49°;$$

and then that

$$a - b + c = \frac{a^2 - b^2 + ac + bc}{a + b} = \frac{\sin 49° \sin 24°}{\sin 68°} = 4 \sin 49° \sin 8° \sin 52°.$$

Hence we have

$$\frac{a^2 - b^2 + ac}{4a(a^2 - b^2 + ac) - (a - b + c)} = \frac{\sin 39°}{4(\sin 5° \sin 39° + \sin 8° \sin 52°)} = \sin 73°,$$

since $\sin 39° = 4 \sin 13° \sin 47° \sin 73°$, and $\sin 5° \sin 39° + \sin 8° \sin 52° = \sin 13° \sin 47°$.

Also solved by H. HALPERIN, A. PELLETIER, and A. V. RICHARDSON.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Professor L. A. HOWLAND, head of the department of mathematics at Wesleyan University, has been made acting president of that institution.

Assistant Professor MARY CURTIS GRAUSTEIN, of Wellesley College, has returned to the college after two years' leave of absence.

At the University of Vermont, Mr. H. G. MILLINGTON has been promoted to an assistant professorship of mathematics, and has been transferred from the College of Arts and Sciences to the College of Engineering. Mr. G. H. NICHOLSON, of Harvard University, has been appointed instructor of mathematics in the Arts College.

At Colgate University, Mr. H. A. DOBELL has been granted leave of absence for study at the University of Pennsylvania, and Mr. J. C. POLLEY, of Yale University, and Mr. G. O. HOHL, of Lebanon Valley College, have been appointed instructors.

At the Carnegie Institute of Technology, Professor O. T. GECKELER has been made head of the department of mathematics; Assistant Professor H. S. LIGHTCAP has been promoted to an associate professorship, and Mr. R. P. JOHNSON, Mr. E. A. WHITMAN, and Mr. E. A. STARR have been promoted to assistant professorships. Mr. J. H. SIMESTER of the University of Toronto has been appointed instructor.

At Swarthmore College, Mr. L. J. COMRIE has been promoted from the position of research assistant in the Observatory to an assistant professorship of mathematics and astronomy. Professor J. A. MILLER lectured October 25 on the Solar Eclipse of 1923 before the Franklin Institute.

At Lafayette College, Associate Professor W. M. SMITH has been granted a six months' leave of absence for study at Oxford and Cambridge. Mr. E. H. WELLS, formerly of Wooster University, has been appointed instructor.

At St. John's College, Annapolis, Maryland, Professor B. H. WADDELL and Assistant Professor T. L. GLADDEN have resigned, and Mr. G. A. BINGLEY, instructor of mathematics at the United States Naval Academy, has been appointed assistant professor.

Mr. F. W. WINTERS, of Harvard University, has been appointed assistant professor of mathematics at Miami University.

Mr. R. H. MACCULLOUGH has been appointed head of the department of mathematics at Defiance College (Ohio).

Miss NORMA STELFORD, of the State Normal School at Superior, Wisconsin, has accepted a position on the staff of the Northern Illinois State Teachers College, DeKalb, Illinois.

Professor H. P. KEAN, of Illinois Wesleyan University, has been appointed assistant professor of mathematics at Wittenberg College, Springfield, Ohio.

Dr. H. P. PETTIT, of the University of Illinois, has been appointed professor of mathematics and head of the department at Illinois Wesleyan University.

At Purdue University, Dr. W. E. EDINGTON has been promoted to an assistant professorship, and Mr. C. E. BARR has been appointed instructor of mathematics.

Dr. MARIE WHELAN, of Johns Hopkins University, has been appointed professor of mathematics at Olivet College (Michigan).

At the Michigan State Normal College, Ypsilanti, Dr. THEODORE LINDQUIST, formerly head of the department of mathematics, State Teachers' College, Emporia, Kansas, has been appointed professor of mathematics, and Associate Professor J. F. BARNHILL has been promoted to a full professorship.

At Beloit College, Mr. R. E. HUFFER, formerly instructor at Michigan Agricultural College, has been appointed assistant professor; Mr. WILLIAM OLIVER has been appointed instructor in applied mathematics, and Miss IRENE ELDRIDGE has been given a temporary appointment as instructor.

At Kentucky Wesleyan College, Mr. W. B. HUGHES has been appointed to succeed Professor R. G. DEMAREE as head of the department.

At Erskine College, Due West, South Carolina, Mr. H. A. WISE of the University of South Carolina has been appointed to succeed Professor W. C. HALLIDAY as professor and head of the department.

At Lander College, Greenwood, South Carolina, Professor W. W. WEBER, formerly head of the department of mathematics at Southern College, Lakeland, Florida, has been appointed to succeed Professor J. D. SALLEY, and Miss LAURA D. SARGENT has been appointed associate professor of mathematics to succeed Mr. ERROL MARTIN.

Professor C. S. COX, formerly head of the department of mathematics and physics at Birmingham Southern College, has been appointed to the same position at Southern College, Lakeland, Florida.

At the University of Georgia, Associate Professor D. F. BARROW has been promoted to a full professorship; Instructors A. H. STEVENS and J. P. HILL have resigned, and Mr. FOREST CUMMING and Mr. E. M. EVERETT, both of the University of Georgia, have been appointed instructors.

At the Georgia School of Technology, instructors E. R. C. MILES, R. L. DRISCOLL, P. L. ARMSTRONG, E. B. WILSON, and G. T. TRAWICK have resigned, and the following have been appointed instructors: J. G. EVANS, W. W. PURKS, J. G. GRIFFIN, T. P. BRANCH, J. W. TAYLOR.

At Iowa State College, Assistant Professor E. C. KIEFER has resigned to become head of the department of mathematics at James Millikin University. Mr. B. A. ROGERS, formerly professor of physics at the Connecticut College of Agriculture, has been appointed instructor.

At Washington University, St. Louis, Assistant Professor THEODORE DOLL has resigned to accept a position with the American Bridge Company, and will be located at Gary, Indiana. Instructor M. E. MEYERSON has resigned to take up work in commercial chemistry. Dr. JESSICA M. YOUNG has been promoted to an assistant professorship of mathematics and astronomy. Mr. EDMOND SIROKY, of the Terrell Croft Engineering Company, has been appointed assistant professor of applied mathematics. Mr. H. A. HOOVER, of Iowa State College, has been appointed instructor of mathematics.

Associate Professor S. LEFSCHETZ, of the University of Kansas, has been promoted to a full professorship of mathematics.

Mr. A. S. HATHAWAY has been appointed professor of mathematics at Friends University, Wichita, Kansas.

Associate Professor W. T. STRATTON of Kansas State Agricultural College has been promoted to a full professorship of mathematics.

At Creighton University, Omaha, Nebraska, Professor PERK has gone to St. Louis University for higher studies. Mr. PATRICK REGAN, of St. Louis University, and Mr. L. E. ROMBAUT, of Notre Dame University, have been appointed assistant professors of mathematics. Mr. ALPHONSE SCHMITT of St. Louis University has been serving as professor since September, 1922.

Professor D. A. LEHMAN, of Goshen College, has been appointed professor of mathematics at Bluffton College.

At the University of North Dakota, Instructor J. D. LEITH has been granted leave of absence for the current academic year and is doing graduate work at Columbia University. Mr. S. F. BIBB of the University of Chicago and Mr. J. F. GATES of the University of North Dakota have been appointed instructors.

At the Colorado Agricultural College, Instructor C. W. YOUNG has been promoted to an assistant professorship, and Mr. A. G. CLARK, instructor of mathematics at the University of Wyoming, has been appointed to an assistant professorship.

At the University of Oklahoma, Professor S. W. REAVES, head of the department of mathematics, has been appointed acting dean of the College of Arts and Sciences. Associate Professor J. O. HASSLER has been promoted to a full professorship; Assistant Professor N. A. COURT has been promoted to an associate professorship. Instructors ELLA MANSFIELD and H. G. LIEBER have resigned, the latter to continue his graduate studies at Columbia University.

At Oklahoma City College, Mr. C. M. ALLEN, formerly head of the department of mathematics, has been transferred to the Education Department. He is succeeded as head of the department of mathematics by Mr. G. E. MEADER, formerly assistant professor of Education at Oklahoma University.

At Oklahoma Agricultural and Mechanical College, Mr. C. B. GRUMBINE has been appointed professor of mathematics and Mr. W. C. PAYNE assistant professor.

Associate Professor J. W. CALHOUN, of the University of Texas, has been promoted to a full professorship of applied mathematics.

Mr. W. M. WHYBURN, of the University of Texas, has been appointed professor of mathematics at South Park College, Beaumont, Texas.

At the Agricultural and Mechanical College of Texas, Assistant Professor W. L. HUGHES and Instructor F. W. FRARY have resigned; Mr. F. W. SPARKS has been appointed assistant professor, and Dr. W. P. UDINSKI and Mr. W. D. BATEN have been appointed instructors.

The following appointments to instructorships of mathematics at American colleges and universities are announced: At Case School of Applied Science, Mr. V. I. BENANDER, of Harvard University; at the University of Florida, Mr. H. W. CHANDLER of the University of Minnesota; at Hamline University, Mr. E. I. MICKELSON of the University of Minnesota; at Illinois College, in mathematics and physics, Mr. G. W. SCHNEIDER, succeeding Mr. W. G. GUILD, now continuing his graduate study at the University of Illinois; at John Tarleton Agricultural College, Miss MARY MARRS; at Keuka College, Mr. C. I. KELCHNER; at Meredith College, Miss MARGARET WYATT succeeding Mrs. E. M. HIGHSMITH; at the University of Rochester, Dr. LOUIS WEISNER of Columbia University; at Rockford College, Miss MARTHA P. MCGAVOCK, of Wellesley College; at the University of Tennessee, Mr. AUGUSTUS LISK, of the University of Kentucky; at Tulane University, Mr. C. G. LATIMER and Mr. T. F. COPE; at Union College, Mr. H. N. ROWE, succeeding Mr. R. D. BENNETT, who resigned for further study at the University of Chicago; at Virginia Polytechnic Institute, Mr. T. A. HATCHER; at Washington State College, Mr. O. M. AKEY, of Ohio State and Stanford Universities, succeeding Mr. R. B. KENNEDY; at West Virginia University, Mr. CLAIRE HARKINS.

LADY SHAW, wife of SIR NAPIER SHAW, died September 22, 1923. She was at one time lecturer in mathematics at Newnham College, Cambridge.

Professor M. A. BAILEY, head of the department of mathematics at the New York Training School for Teachers since 1899, died on November 25th at the age of sixty-seven. After headmasterships at the high schools in Winsted, Conn., and Keene, N. H., he went to the State Normal School at Emporia, Kansas, as organizer and supervisor of the mathematics department. For the last twenty-four years he prepared the curriculum in mathematics for the New York schools.

The degree of doctor of laws was conferred on Professor R. C. ARCHIBALD on the occasion of the inauguration of President G. J. TRUEMAN at Mount Allison University, October 18.

Mr. HUGH PORTER, who has been principal of the senior high school and dean of the junior college at Wichita Falls, Texas, has returned to the North Texas State Normal College as professor of mathematics.

Professor J. A. WHITTED has resigned his position at Hedding College and has become professor of mathematics at Ohio Northern University.

Associate Professor W. P. RUSSELL of Pomona College is spending a year's leave of absence in study at Harvard.

Mr. C. B. WALSH, headmaster of Woodmere Academy, Woodmere, N. Y., died on July 13, 1923, at the age of thirty-nine years. Mr. Walsh was a charter member of this Association and was, for a time, president of the Association of Teachers of Mathematics of the Middle States and Maryland.

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to D. C. GILLESPIE, Cayuga Heights, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924
NUMBER 3, MARCH

PUBLISHED BY THE ASSOCIATION
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others

SEVENTH ANNUAL MEETING OF THE MISSOURI SECTION.

The seventh annual meeting of the Missouri Section was held at the University of Missouri, Columbia, Missouri, on Friday and Saturday, November 30 and December 1, 1923, in connection with the meeting of the Southwestern Section of the American Mathematical Society.

The attendance was thirty-two including the following twenty-one members of the Association: E. F. Allen, C. H. Ashton, H. Blumberg, W. C. Brenke, Theodosia T. Callaway, E. W. Chittenden, Mary E. Decherd, O. Dunkel, E. S. Haynes, E. R. Hedrick, L. Ingold, C. G. Jaeger, S. Lefschetz, J. V. McKelvey, U. G. Mitchell, P. R. Rider, E. Stephens, E. B. Stouffer, J. S. Turner, R. A. Wells, W. D. A. Westfall.

There was an enjoyable informal reception on Friday evening at the home of Professor E. R. Hedrick. There were two sessions of the Missouri Section on Saturday, both being presided over by the chairman, Professor Hedrick. The members of the two sections were the guests at a luncheon at the Daniel Boone Tavern on Saturday given by the Missouri Chapter of the Pi Mu Epsilon Fraternity. At the business meeting it was decided to hold the 1924 meeting in Kansas City at the time of the meeting of the State Teachers' Association. The following officers were elected for 1924: Chairman, R. R. FLEET, William Jewell College; Vice-chairman, R. A. WELLS, Park College; Secretary-Treasurer, P. R. RIDER, Washington University.

The following four papers were read:

(1) "Suggestions toward a comparative pedagogy of mathematics" (by invitation) by Professor HENRY BLUMBERG, University of Illinois.

(2) "An elementary discussion of the roots of the cubic" by Professor OTTO DUNKEL, Washington University.

(3) "A class of surfaces applicable to a sphere" by Mr. C. G. JAEGER, University of Missouri.

(4) "Service mathematics" by Professor THEODOSIA T. CALLAWAY, Stephens Junior College.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. The suggestions arise largely out of comparison of current mathematical pedagogy with pedagogic ideas and methods in music, art, literature, physics, biology, philosophy, etc. Professor Blumberg discussed these suggestions under three heads: (*a*) Initiation into mathematics. (*b*) Colleagueship of teacher and student. (*c*) Freedom from various accepted conceptions and prejudices. The paper is to appear in full in the future.

2. In this paper Professor Dunkel considers the cubic in which the second term has been removed and discovers the character of its roots by means of synthetic division, this process replacing the use of the derivative in some form

of treatment. In this manner the significance of the discriminant of the cubic is determined in an elementary manner.

3. In Mr. Jaeger's paper the equation of a two-parameter class of surfaces applicable to the sphere were obtained by solving Codazzi's equation after first making certain limitations on the second fundamental quantities, D, D', and D''.

4. Professor Callaway reported on service mathematics "a series of tests and exercises designated to locate and remove the mathematical difficulties of students in courses in elementary clothing and foods." A careful examination of standard texts in these subjects was made with a view to determining definitely what mathematical concepts and processes are used in such courses and the degree of difficulty to which each process is carried. The tests and exercises are based on the results of this study.

P. R. RIDER, *Secretary-Treasurer.*

THE ALGEBRA OF CORRELATION.

By DUNHAM JACKSON, University of Minnesota.

1. Introduction. The increased attention that is given to statistical methods by workers in the most diverse fields of science, as well as by mathematicians themselves, appears to be one of the most significant features of contemporary study. In a presentation of the mathematical aspects of the subject, there is room for considerable discretion as to the amount of mathematical knowledge to be demanded of the reader at the start. Even if the student finds a treatment suited to his preparation, he may be helped by a supplementary account which throws light on the matter from a slightly different angle, or clarifies the logical relations of the various parts by leading up to them in a slightly different order. It is on the basis of such considerations that the following exposition of some of the elementary theorems on correlation has been prepared. The reader who is acquainted with the existing literature of the subject will find nothing here that is new,[1] but he may find one item or another of his reading thrown into clearer relief. The mathematical prerequisites are covered by the ordinary freshman courses in trigonometry and college algebra. Some use will be made of the simplest parts of analytic geometry, but such facts as are needed in this connection, beyond the content of elementary courses in algebra, will be developed in the text.

2. Definition of the coefficient of correlation. Suppose that measurements have been made of two quantities pertaining to each of a set of n individuals of some sort. Let the observed values of the first quantity for the n individuals

[1] In particular, this article covers some of the same ground as the excellent paper of Professor Huntington in a recent volume of this MONTHLY, but the treatment is varied to such an extent that there is little actual duplication. The reader is referred to Professor Huntington's paper for a bibliography, which makes it unnecessary to include similar references here. See E. V. Huntington, "Mathematics and statistics, with an elementary account of the correlation coefficient and the correlation ratio," this MONTHLY (*1919*, 421–435).

respectively be $X_1, X_2, \cdots, X_n$, and let the corresponding values of the second quantity be $Y_1, Y_2, \cdots, Y_n$. The "individuals" may be persons, or groups of persons, or inanimate objects, or periods of time, etc.; the measured quantities may be the grades scored by the students of a class on a pair of tests, or the numbers of ballots cast in the counties of a state at a primary and at a regular election, or the lengths and breadths of a group of shells, or the amounts of rainfall and amounts of grain production in a given area in a series of years. To a reader experienced in the order of ideas involved, these vague indications will serve as an introduction to the definitions that are to be formulated. To one without such experience, they will probably convey little meaning. But that is of secondary consequence for the time being, inasmuch as a reader of either class will come into possession of all that is essential for the mathematical discussion by saying:

Let $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ be any n pairs of real numbers. They may be all distinct, or there may be repetitions among them; it will be assumed merely, for a reason which will be presently apparent, that the X's are not all equal to each other, and that the Y's are not all equal among themselves.

The *dramatis personæ* being thus assembled, let

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n), \qquad \overline{Y} = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n),$$

and for each value of k from $k = 1$ to $k = n$ let

$$x_k = X_k - \overline{X}, \qquad y_k = Y_k - \overline{Y},$$

so that the x's and y's are the deviations of the X's and Y's from their respective arithmetical means.

The *coefficient of correlation* between the X's and the Y's is defined by the formula

$$r = (\Sigma x_k y_k)/\sqrt{(\Sigma x_k^2)(\Sigma y_k^2)}, \tag{1}$$

the summation in each case extending[1] from $k = 1$ to $k = n$. In accordance with a convention which is practically universal, the radical sign by itself is understood to represent always the *positive* square root, a minus sign being written explicitly if the negative root is meant. The denominator of the expression for r is therefore always positive, and the algebraic sign of r is the same as the algebraic sign of the numerator.

It has been emphasized that the X's and the Y's, for the purpose in hand, may be any real numbers whatever, subject only to the condition that neither X_k nor Y_k is constant throughout. This condition is equivalent to saying that neither Σx_k^2 nor Σy_k^2 shall be zero, and is imposed for the sake of insuring that the denominator in the expression for r shall not vanish. The quantities x_k and y_k, on the other hand, are less thoroughly arbitrary, since it follows immediately from their definition that

$$\Sigma x_k = \Sigma y_k = 0. \tag{2}$$

[1] This understanding with regard to the sign of summation will be maintained throughout the paper.

Subject to this restriction, however, and to the condition that neither set shall consist exclusively of zeros, they may also be taken arbitrarily.[1]

The reader who is accustomed to work in a particular field of application may be prone to object, as the discussion proceeds, that an argument formulated in such general terms is too abstract to carry conviction. Such a critic is urged to bear in mind that the symbols employed represent *numbers*, and that the statement of an algebraic theorem is a statement that certain numerical relations will be verified by calculation in any specific instance. The assertion that *if you perform* certain arithmetical operations, *you will get* certain results, is a prediction with regard to a phenomenon at least as concrete as those with which many a psychological investigation is concerned; and if the calculation is performed by mechanical operations on a calculating machine, the verification is as objectively manifest to the senses as in any physical experiment. The pure mathematician may or may not be disposed to claim that his conclusions have a higher kind of authority than those of the experimental scientist,[2] but he can at any rate be confident that his results have every kind of substantial reality that belongs to any branch of knowledge.

3. Homogeneity of r. The value of the expression for r is unchanged [3] if x_k is replaced throughout by cx_k, c being positive and independent of k. For the numerator and the denominator are both multiplied by c, which cancels from the quotient. This property is of fundamental importance. It means that the units of measurement for the two sets of observed quantities can be chosen independently of each other. If the two sets of quantities are of the same kind, the units need not be the same in both cases;[4] and, what is more important, if the quantities are of different kinds, so that the units are not comparable at all, the coefficient nevertheless may have a definite meaning.[5]

4. Numerical limits of r. If it happens that the numbers x_k and y_k are so related that $y_k = cx_k$, for all values of k, where c is a positive number independent of k, it is seen that

$$\Sigma x_k y_k = c\Sigma x_k^2, \qquad \sqrt{\Sigma y_k^2} = c\sqrt{\Sigma x_k^2}, \qquad r = 1.$$

If $y_k = -cx_k$, the number c itself again being positive, then $\Sigma x_k y_k = -c\Sigma x_k^2$. On the other hand, as a result of the understanding that the radical sign always represents the positive square root,

$$\sqrt{\Sigma y_k^2} = +c\sqrt{\Sigma x_k^2}.$$

[1] No other restriction is necessary, because any numbers whose algebraic sum is zero may be regarded as constituting their own deviations from their own arithmetical mean.

[2] The present writer would be exceedingly reluctant to make any such claim.

[3] The same conclusion would hold if the corresponding expression were formed with X_k and Y_k instead of x_k and y_k; that is, it is independent of the restriction (2).

[4] For example, in correlating two sets of examination grades, it is not necessary to have the two examinations marked on the same scale.

[5] Of course the value of the coefficient will be affected by a change in the method of measurement of one of the quantities, such as the substitution of an area for a length in estimating the size of an object, or the assignment of different relative weights to the questions on an examination.

In this case, therefore, $r = -1$. The coefficient of correlation is plus or minus 1, according as y_k is positively or negatively proportional to x_k. In terms of the original quantities X_k and Y_k, the condition of proportionality means that

$$Y_k - \overline{Y} = \pm c(X_k - \overline{X}), \qquad Y_k = \pm cX_k + (\overline{Y} \mp c\overline{X}),$$

the last parenthesis being independent of k; that is, Y_k is a linear function of X_k.

It will be shown that if y_k is not proportional to x_k at all, the value of r is always *between* $+1$ and -1.

The proof will be made to depend on a well-known fact about the roots of a quadratic equation. Suppose an expression of the form $ax^2 + bx + c$ (for example, $x^2 + 2x + 2$) is written down with such coefficients that the value of the expression is always positive, no matter what (real) value is given to x. Since every real value of x makes the expression positive, no real value of x makes it zero. That is, the quadratic equation $ax^2 + bx + c = 0$ has no real root, and the discriminant $b^2 - 4ac$ must be negative.

Now suppose the numbers x_k and y_k are not proportional. If t is an arbitrary real number, the quantities $y_k - tx_k$ are not all zero for any one value of t, for their simultaneous vanishing would be precisely the condition of proportionality of y_k and x_k. Form the expression $\Sigma(y_k - tx_k)^2$. Being a sum of squares, this expression can never be negative. But it can also never be zero, for its terms can not all be zero simultaneously, and so some of them at least must be positive. If the expression is expanded in the form

$$\Sigma y_k^2 - 2t\Sigma x_k y_k + t^2 \Sigma x_k^2,$$

it is seen to be *a quadratic expression in t, which is positive for all real values of t.* It comes under the rule of the preceding paragraph, if x, a, b, c are replaced by t, Σx_k^2, $(-2\Sigma x_k y_k)$, and Σy_k^2 respectively. So $4(\Sigma x_k y_k)^2 - 4(\Sigma x_k^2)(\Sigma y_k^2)$ must be a negative quantity; that is,

$$(\Sigma x_k y_k)^2 < (\Sigma x_k^2)(\Sigma y_k^2).$$

But the members of the last inequality are the squares of the numerator and the denominator respectively of the formula for r. It is recognized that $r^2 < 1$, and hence that r must be between[1] $+1$ and -1.

5. **Correlation by rank.** In an actual problem, the calculation of r may call for a considerable amount of numerical work. When the original data are not accurate enough to justify such an elaborate calculation, a formula is sometimes used which is considerably easier to evaluate. The individuals measured are arranged in order according to the observed values of X, the one with the largest X being placed first, the one with the next largest X, second, and so on. Then they are again arranged in order—generally, of course, in a different order—

[1] In this section, as in § 3, no use has been made of the condition (2). The proof and the conclusion apply equally well to the quantity which is obtained if x_k and y_k in the formula defining r are replaced by arbitrary numbers X_k and Y_k. That is, the quantity so obtained is equal to ± 1 if X_k and Y_k are proportional, and is between $+1$ and -1 in all other cases.

according to the observed values of Y. The difference between the rank-numbers assigned to an individual in the two arrangements is denoted by D_k, and the *coefficient of correlation by rank* is

$$\rho = 1 - 6(\Sigma D_k^2)/(n^3 - n). \tag{3}$$

It will now be shown that this is not really a new kind of correlation coefficient, but only a special case under the definition already given. Let $X_1', \cdots, X_n'$, $Y_1', \cdots, Y_n'$ be the rank-numbers in the two arrangements. That is, X_1' is equal to 1, 2, 3, $\cdots$, according as X_1 ranks first, second, third, $\cdots$, in order of algebraic magnitude, among all the numbers X_k, and so on.[1] It will appear that ρ is the coefficient of correlation between the numbers X_k' and the numbers Y_k', according to the definition[2] of § 2. To put it in another way, if ρ is allowed to denote this coefficient of correlation, it will be found that ρ is given by the formula (3), where $D_k = X_k' - Y_k'$.

Let x_k' and y_k' be the deviations of X_k' and Y_k' from their respective means. Then, by definition,

$$\rho = (\Sigma x_k' y_k')/\sqrt{(\Sigma x_k'^2)(\Sigma y_k'^2)}, \tag{4}$$

the relation (3) being temporarily in abeyance, as a theorem to be proved.

The numbers X_k' in the aggregate are merely the numbers 1, 2, 3, $\cdots$, n, arranged in some order. Their mean is consequently $\frac{1}{2}(n + 1)$. This can be found by applying the formula for an arithmetical progression,

$$\frac{1}{n}(1 + 2 + 3 + \cdots + n) = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2},$$

or by pairing 1 with n, 2 with $n - 1$, and so on, and taking the mean of each pair. The mean of the Y_k''s is the same, for the same reason. A first inference is that

$$D_k = X_k' - Y_k' = \left(x_k' + \frac{n+1}{2}\right) - \left(y_k' + \frac{n+1}{2}\right) = x_k' - y_k'.$$

The numbers x_k' are the numbers

$$n - \frac{n+1}{2}, \quad (n-1) - \frac{n+1}{2}, \quad \cdots, \quad 1 - \frac{n+1}{2}, \tag{5}$$

in some order, and the y_k''s are the same numbers, in the same or (generally) in a different order.

A further inference is that $\Sigma x_k'^2 = \Sigma y_k'^2$. Hence the two factors under the radical sign in (4) are equal, and, if the subscripts are omitted for simplicity,

$$\sqrt{(\Sigma x'^2)(\Sigma y'^2)} = \Sigma x'^2 = \Sigma y'^2 = \tfrac{1}{2}(\Sigma x'^2 + \Sigma y'^2),$$

[1] If two or more of the X_k's are tied, it is customary to divide the corresponding rank-numbers among the individuals concerned, using fractions if necessary; the theory is somewhat complicated thereby, and this case will be left aside.

[2] Hence, in particular, the extreme values of ρ are $+ 1$ and $- 1$, as in any other application of the r-formula. As far as the limit $+ 1$ is concerned, this is also evident directly from (3).

the last expression being set down because of its use in the ensuing reductions. By a succession of steps, somewhat artificial in appearance, perhaps, but exceedingly simple, it is seen that

$$\rho = \frac{2\Sigma x'y'}{\Sigma x'^2 + \Sigma y'^2} = \frac{\Sigma x'^2 + \Sigma y'^2 - \Sigma x'^2 + 2\Sigma x'y' - \Sigma y'^2}{\Sigma x'^2 + \Sigma y'^2}$$

$$= 1 - \frac{\Sigma(x'^2 - 2x'y' + y'^2)}{\Sigma x'^2 + \Sigma y'^2} = 1 - \frac{\Sigma D^2}{\Sigma x'^2 + \Sigma y'^2} = 1 - \frac{\Sigma D^2}{2\Sigma x'^2}.$$

The resemblance to (3) is now apparent; it remains to evaluate the denominator $2\Sigma x'^2$.

It would not be a problem of excessive difficulty to find the sum of the squares of the numbers (5) directly. The work can be much simplified, however, by an observation which is important in other connections as well. Let X_k, $\overline{X}$, and x_k have the general significance assigned to them in § 2. Then [1]

$$\Sigma X_k^2 = \Sigma(x_k + \overline{X})^2 = \Sigma x_k^2 + 2\overline{X}\Sigma x_k + n\overline{X}^2,$$

or, *because of the fact that* $\Sigma x_k = 0$,

$$\Sigma x_k^2 = \Sigma X_k^2 - n\overline{X}^2.$$

In the present instance,

$$\Sigma x_k'^2 = \Sigma X_k'^2 - n\left(\frac{n+1}{2}\right)^2$$

$$= 1^2 + 2^2 + \cdots + n^2 - n\left(\frac{n+1}{2}\right)^2.$$

It is well known that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6};$$

the proof is a simple exercise in mathematical induction, regularly presented in courses in college algebra, and need not be detailed here. Hence

$$\Sigma x'^2 = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4} = \frac{n^3 - n}{12} = \frac{n(n^2-1)}{12},$$

and

$$\rho = 1 - \frac{\Sigma D^2}{2\Sigma x'^2} = 1 - \frac{6\Sigma D^2}{n(n^2-1)}.$$

6. Spearman's Footrule. A still simpler formula involving rank-numbers is known as Spearman's Footrule.[2] In the notation of the preceding section, it is

$$R = 1 - \frac{3\Sigma|D_k|}{n^2 - 1},$$

---

[1] More generally, if h is any number whatever,

$$\Sigma(x_k + h)^2 = \Sigma x_k^2 + nh^2.$$

[2] C. Spearman, 'Footrule' for measuring correlation, *British Journal of Psychology*, vol. 2 (1906–08), pp. 89–108. This paper calls attention also to the formula of the preceding section.

the symbol $|D_k|$ denoting of course the absolute value[1] of D_k. This R will not be discussed here, except to the extent of showing that it is *not* a special case of the coefficient r defined by (1); it lacks some of the characteristic properties of the latter, and does not lend itself so readily to algebraic treatment. Its maximum value, to be sure, is 1, found in the case of perfect agreement in rank, when all the numbers D_k are zero. But the minimum value for a given number of entries is not -1. In the case of four individuals, for example, comparison of the arrangements (1, 2, 3, 4) and (4, 3, 2, 1) for $(X_1', \cdots, X_4')$ and $(Y_1', \cdots, Y_4')$ respectively gives $R = -\frac{3}{5}$, and it is readily seen[2] that this is the minimum. Furthermore, the minimum value of R does not necessarily imply complete reversal of order; the same value $-\frac{3}{5}$ is found by comparing (1, 2, 3, 4) with (4, 3, 1, 2) or (3, 4, 1, 2). It is suggested[3] that negative values "large enough to be appreciable" be avoided "by ranking one of the series in the reverse order," with a note that "this reversal of ranks usually modifies the result, but only to an extent that is much smaller than the probable error and therefore negligible." It may be added that a negative result, perhaps too small to be practically significant, may remain even after the reversal. Comparison of (1, 2, 3, 4) with (2, 4, 1, 3) gives $R = -\frac{1}{5}$, and comparison with the inverted arrangement (3, 1, 4, 2) gives $-\frac{1}{5}$ again. The "footrule" has been found useful in practice, but, as Spearman himself emphasizes, its possibilities and limitations are to be recognized by experience, not by inference from what is known about the coefficients r and ρ. The formulas and tables expressing ρ in terms of R depend on an assumption as to the distribution of the data, and are not a matter of universal demonstration.

7. Geometrical interpretation of r. The meaning of the general coefficient r can be made clearer by a geometrical representation. Let the pair of numbers (x_k, y_k) be regarded as the coördinates of a point, with respect to a pair of rectangular axes. Then the n pairs $(x_1, y_1), \cdots, (x_n, y_n)$ are represented by n points distributed somewhere in the plane; these points may be all distinct, or some of them may coincide. They do not all lie on the y-axis, because of the assumption that the x's are not all zero, and it can be said similarly that they do not all lie on the x-axis. Since $\Sigma x_k = \Sigma y_k = 0$, they are so arranged that their center of gravity, if they are regarded as material particles of equal weight, falls at the origin. Apart from these conditions, they may be any n points in the plane. It has been seen that if $r = 1$, there is a positive constant c such that $y_k = cx_k$ for all values of k. This means that the points all lie on the straight line $y = cx$, passing through the origin and running into the first and third quadrants. If $r = -1$, the points lie on a line $y = -cx$, containing the origin

[1] The formula is commonly written with $6\Sigma g$ in place of $3\Sigma|D_k|$, where Σg stands for the sum of the "gains" in rank observed in passing from one arrangement to the other, the "losses" being neglected. As the sum of the "gains" is necessarily equal numerically to the sum of the "losses," and $\Sigma|D_k|$ is the arithmetical sum of all together, $\Sigma|D_k|$ is equal to $2\Sigma g$.

[2] There is no loss of generality in taking $X_1' = 1$, $X_2' = 2$, $X_3' = 3$, $X_4' = 4$, and then the 24 possible arrangements for the numbers Y_k' can be tried out in succession.

[3] Spearman, *loc. cit.*, p. 96, footnote.

and going from the second quadrant into the fourth. If r is neither $+1$ nor -1, they do not lie on any one line.[1] There is a relation between the magnitude of r and the degree of their approach to a linear arrangement. This will be clearest in terms of a still further specialized notation, to be introduced in the following paragraph. It may be observed in passing that if the points (X_k, Y_k) are plotted instead of (x_k, y_k), they will lie on a straight line if $r = \pm 1$, but this line will not generally pass through the origin.

Let σ and τ stand for the quantities

$$\sigma = \sqrt{(\Sigma x_k^2)/n}, \qquad \tau = \sqrt{(\Sigma y_k^2)/n}.$$

These are the *standard deviations* of the x's and of the y's respectively. Let

$$s_k = x_k/\sigma, \qquad t_k = y_k/\tau.$$

It follows from § 3, or directly by substitution of σs_k for x_k and τt_k for y_k in (1), that

$$r = (\Sigma s_k t_k)/\sqrt{(\Sigma s_k^2)(\Sigma t_k^2)}.$$

But $\Sigma s_k^2 = \Sigma t_k^2 = n$, so that the last formula becomes simply

$$r = \frac{1}{n}\Sigma s_k t_k.$$

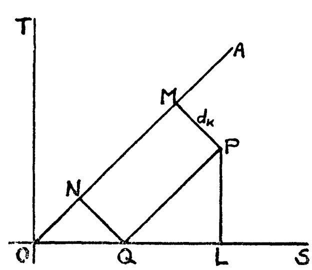
By what amounts to a repetition of a part of the algebraic work of § 5, it is seen that

$$\begin{aligned} r &= \frac{1}{2n}\Sigma(2s_k t_k) = \frac{2n - (n - 2\Sigma s_k t_k + n)}{2n} \\ &= 1 - \frac{\Sigma s_k^2 - 2\Sigma s_k t_k + \Sigma t_k^2}{2n} = 1 - \frac{1}{2n}\Sigma(s_k - t_k)^2. \end{aligned} \tag{6}$$

Similarly,

$$r = \frac{-2n + (n + 2\Sigma s_k t_k + n)}{2n} = -1 + \frac{1}{2n}\Sigma(s_k + t_k)^2.$$

As the sums of squares can not be negative, these formulas show[2] once more that r is a number belonging to the interval from -1 to $+1$. They also lend themselves to an immediate geometric interpretation, if the points having the coördinates (s_k, t_k) are plotted with respect to a pair of coördinate axes.



Let P in the figure be an arbitrary one of the points (s_k, t_k), and let OA be the line $t = s$, bisecting the angle between the axes OS and OT. The coördinates of P are $OL = s_k$ and $LP = t_k$. Let the perpendicular distance PM be denoted by d_k. The manner of construction of the rest of the figure will be sufficiently clear without verbal description. It

[1] If they are in a straight line at all, that line contains their center of gravity, and so must pass through the origin.

[2] Cf. Huntington, *loc. cit.*, p. 424.

is seen that

$$d_k = \overline{PM} = \overline{QN} = \overline{OQ}/\sqrt{2},$$
$$\overline{OQ} = \overline{OL} - \overline{QL} = \overline{OL} - \overline{LP} = s_k - t_k,$$
$$d_k = (s_k - t_k)/\sqrt{2}.$$

Other situations of the point P will give a somewhat different figure, but the result will be the same in all cases, except possibly for algebraic sign. So the relation (6) means that

$$r = 1 - \frac{1}{n}\Sigma d_k^2;$$

the coefficient of correlation is less than 1 *by an amount equal to the mean of the squares of the distances of the points* (s_k, t_k) *from the line* $t = s$. The condition $r = 1$, in this notation, means that all the points lie on the 45° line. Similarly, r exceeds -1 by an amount equal to the mean square distance from the other 45° line, $t = -s$.

8. Least squares: the arithmetical mean. The correlation coefficient is found to possess additional meaning in the light of the method of least squares. It is not the purpose of the present article to discuss the theoretical justification of that method, but only to show how its application works out in one or two instances. It is perhaps simplest to begin with an illustration which does not involve the idea of correlation, but is of fundamental importance in itself.

Let $X_1, X_2, \cdots, X_n$ be any n given real numbers, distinct or not. Let it be required to find a number x, so that the quantity $\Sigma(X_k - x)^2$ shall have the smallest possible value. The answer is given by an observation which will be used again in the following section.

The most general expression of the second degree in x, the left-hand member of the general quadratic equation, is $ax^2 + bx + c$. It will be sufficient here to consider the case that $a > 0$. Then the expression can be written in the form

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a};$$

the second member reduces immediately to the first on simplification. One term on the right is independent of x, and the other is positive or zero for all real values of x, since a is positive. So the whole expression takes on its smallest value when the square term is zero, that is, when $x = -b/(2a)$, and the resulting minimum value is

$$\frac{4ac - b^2}{4a} = c - \frac{b^2}{4a}. \qquad (7)$$

The quantity to be minimized in the particular problem that was proposed is

$$\sum_{k=1}^{n}(X_k - x)^2 = \sum_{k=1}^{n} X_k^2 - 2x\sum_{k=1}^{n} X_k + nx^2.$$

This can be identified with $ax^2 + bx + c$ by taking $a = n > 0$, $b = -2\Sigma X_k$, $c = \Sigma X_k^2$. The minimizing value of x is $x = -b/(2a) = (\Sigma X_k)/n$; that is, the minimum is given by the arithmetical mean. It will be noticed that this important fact is independent of any restrictive assumption as to the distribution of the numbers X_k.

9. **Least squares: lines of regression.** To come back to the pairs of observations which have formed the main subject of the paper, let it be required to determine a coefficient λ so that the line $y = \lambda x$ shall give the best possible representation of the arrangement of the points (x_k, y_k), in the sense that the expression $\Sigma(\lambda x_k - y_k)^2$ shall be as small as possible. The quantity to be minimized is $\lambda^2 \Sigma x_k^2 - 2\lambda\Sigma x_k y_k + \Sigma y_k^2$, which is in the form $a\lambda^2 + b\lambda + c$, with $a = \Sigma x_k^2 > 0$, $b = -2\Sigma x_k y_k$, $c = \Sigma y_k^2$. The minimum is given by

$$\lambda = -\frac{b}{2a} = \frac{\Sigma x_k y_k}{\Sigma x_k^2} = \frac{\Sigma x_k y_k}{\sqrt{(\Sigma x_k^2)(\Sigma y_k^2)}} \cdot \sqrt{\frac{\Sigma y_k^2}{\Sigma x_k^2}} = r\frac{\tau}{\sigma}\,.$$

The line $y = \lambda x = r(\tau/\sigma)x$ is called a *line of regression.*[1]

For the sake of simplifying the expression for the minimum value of the sum of squares, as given by (7) of the preceding section, let $p = (1/n)\Sigma x_k y_k$. Then

$$\Sigma x_k^2 = n\sigma^2, \qquad \Sigma y_k^2 = n\tau^2, \qquad \Sigma x_k y_k = np, \qquad r = \frac{p}{\sigma\tau}\,.$$

Consequently

$$c - \frac{b^2}{4a} = \Sigma y_k^2 - \frac{(\Sigma x_k y_k)^2}{\Sigma x_k^2} = n\tau^2 - \frac{np^2}{\sigma^2} = n\tau^2\left(1 - \frac{p^2}{\sigma^2\tau^2}\right) = n\tau^2(1 - r^2).$$

If the square root of one nth of the minimum sum of squares is denoted by τ_1, then

$$\tau_1^2 = \tau^2(1 - r^2).$$

There is another line of regression, $x = \mu y$, determined by the condition that $\Sigma(\mu y_k - x_k)^2$ shall be a minimum. The value of μ is found to be $r\sigma/\tau$, and σ_1, defined as equal to the square root of one nth of the minimum sum of squares in this case, is given by the equation

$$\sigma_1^2 = \sigma^2(1 - r^2).$$

A broader question, in appearance at least, would be that of determining the best approximating equation of the form $y = \lambda x + \lambda'$, in the sense of the method of least squares. The same method of treatment would show, without much more difficulty, that λ' must be zero, so that the line of regression really gives the solution of this problem as well. The details of the proof need not be given here.[2]

---

[1] Cf. Huntington, *loc. cit.*, pp. 427–428. The notation is somewhat different.

[2] If λ is given *any* fixed value, say $\lambda = \lambda_0$, the best representation that can be found by varying λ' alone is obtained by taking $\lambda' = 0$; the formal proof depends on the fact that $\Sigma x_k = \Sigma y_k = 0$.

10. Partial correlation. In the simplest case of three variables, a coefficient of partial correlation gives a measure of the degree of dependence of two of the variables on each other, apart from such relationship as arises out of their common dependence on the third. This vague indication is of course only descriptive; its substance will appear from the formal work that follows.

Let $X_1, \cdots, X_n$; $Y_1, \cdots, Y_n$; and $Z_1, \cdots, Z_n$ be three sets of observations on n individuals, and let $x_1, \cdots, x_n$; $y_1, \cdots, y_n$; and $z_1, \cdots, z_n$ be their deviations from their respective means. Let

$$\Sigma x_k^2 = n\sigma^2, \qquad \Sigma y_k^2 = n\tau^2, \qquad \Sigma z_k^2 = n\omega^2,$$

$$r_{12} = \frac{\Sigma x_k y_k}{\sqrt{(\Sigma x_k^2)(\Sigma y_k^2)}}, \qquad r_{13} = \frac{\Sigma x_k z_k}{\sqrt{(\Sigma x_k^2)(\Sigma z_k^2)}}, \qquad r_{23} = \frac{\Sigma y_k z_k}{\sqrt{(\Sigma y_k^2)(\Sigma z_k^2)}},$$

so that σ, τ, ω are the standard deviations of the three sets respectively, and r_{12}, r_{13}, r_{23} are the coefficients of correlation of two of the sets at a time, calculated in the usual way. It is seen at once that

$$\Sigma x_k y_k = nr_{12}\sigma\tau, \qquad \Sigma x_k z_k = nr_{13}\sigma\omega, \qquad \Sigma y_k z_k = nr_{23}\tau\omega.$$

Let μ_1 and μ_2 be the coefficients which minimize the sums of squares $\Sigma(x_k - \mu z_k)^2$ and $\Sigma(y_k - \mu z_k)^2$ respectively. By the results of the preceding section, $\mu_1 = r_{13}\sigma/\omega$, and $\mu_2 = r_{23}\tau/\omega$. Let

$$u_k = x_k - \mu_1 z_k, \qquad v_k = y_k - \mu_2 z_k.$$

The u's and v's may be regarded as what is left of the x's and y's, after the utmost possible has been done to eliminate any (linear) dependence on the z's. At any rate, the definition is to be given in terms of the quantities u_k and v_k, however they may be characterized verbally.

The coefficient of partial correlation between the X's and the Y's is simply the coefficient of correlation between the u's and the v's, defined in the ordinary way:

$$r' = (\Sigma u_k v_k)/\sqrt{(\Sigma u_k^2)(\Sigma v_k^2)}.$$

Its extreme values are therefore $+1$ and -1, as in the case of any other coefficient of correlation.

It is possible to express r' in such a form that its numerical determination does not require the actual calculation of the numbers u_k and v_k. By the preceding section,

$$\Sigma u_k^2 = n\sigma^2(1 - r_{13}^2), \qquad \Sigma v_k^2 = n\tau^2(1 - r_{23}^2).$$

Furthermore,

$$\begin{aligned}\Sigma u_k v_k &= \Sigma(x_k - \mu_1 z_k)(y_k - \mu_2 z_k) \\ &= \Sigma x_k y_k - \mu_1 \Sigma y_k z_k - \mu_2 \Sigma x_k z_k + \mu_1\mu_2 \Sigma z_k^2,\end{aligned}$$

or, by substitution of the equivalents given above for μ_1 and μ_2 and the various sums in the last member,

$$\begin{aligned}\Sigma u_k v_k &= nr_{12}\sigma\tau - r_{13}\frac{\sigma}{\omega}\cdot nr_{23}\tau\omega - r_{23}\frac{\tau}{\omega}\cdot nr_{13}\sigma\omega + r_{13}\frac{\sigma}{\omega}\cdot r_{23}\frac{\tau}{\omega}\cdot n\omega^2 \\ &= n\sigma\tau(r_{12} - r_{13}r_{23}).\end{aligned}$$

Consequently

$$r' = (r_{12} - r_{13}r_{23})/\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}.$$

The more complicated cases of partial correlation follow the same order of ideas, but would require a more elaborate notation for their systematic treatment.

TWO MODELS IN STATISTICAL MECHANICS.

By A. J. LOTKA, Johns Hopkins University.

1. Introduction. Those thermal effects which we commonly observe with our gross senses (assisted on occasion by more or less refined thermometric and other instruments), and which we treat, in terms of such observations, by the method of thermodynamics, are very successfully interpreted, by the method of statistical mechanics, as the observed effects of " concealed motions " of imperceptible particles (molecules). So, to quote only the simplest possible example, the pressure which a gas exerts upon its container is, in this interpretation, recognized as the momentum lost, per unit of time, by the molecules which impinge upon, and are reflected from, the walls of the container.

While the conclusions of statistical mechanics in this and in many other respects stand in excellent harmony with observation and with the thermodynamic setting of physical laws, yet a critical scrutiny reveals certain apparent conflicts which are not resolved without effort. These conflicts are closely bound up with the fact that, in the thermodynamic interpretation, natural processes are essentially unidirectional in time, whereas in mechanics the forward and the backward directions in time are on an equal footing. The motion of a " purely mechanical system " is periodic; for example, one glance at a sine curve, representing the motion of a pendulum, immediately makes it plain to the eye that, cut in two at a crest, the curve is symmetrical to the right and left; motion represented by such a curve is symmetrical as regards $+ t$ and $- t$, when suitable choice of origin is made.[1] A pendulum alone would never enable us to distinguish between yesterday, to-day and to-morrow. But in the processes considered in thermodynamics the case is different. If two bodies in contact in a non-conducting enclosure are observed on three days A, B, C, to show respectively a temperature contrast of 10, 20 and 40 degrees, we know that the day A must be later than B, and B later than C. The progress of heat conduction is *not* indifferent to the sign of t. The physicist commonly expresses this by saying that for any isolated system the *entropy* can only increase. So, for example, when a body M_1 at temperature θ_1 loses a quantity Q of heat by simple *conduction* to another body M_2 at temperature θ_2, the body M_1 is said to lose entropy Q/θ_1 and the body M_2 to gain entropy Q/θ_2. The net gain in entropy $(Q/\theta_2 - Q/\theta_1)$ for the entire system is necessarily positive[2] since θ_1 is necessarily greater than θ_2.

[1] For a rigorous discussion of this point, see Poincaré, *Thermodynamique*, 1908, p. 441.

[2] The entropy of an isolated system increases, not only in the equalization of temperature differences by conduction, but in all *spontaneous* processes, as, for example, in diffusion. A com-

Here, then, thermodynamics and statistical mechanics seem at first sight to be in conflict. The entropy of an isolated system always increases. If the system were composed of a number of particles in mechanical motion, its behavior would be periodic, and its entropy would decrease as often as increase.

The complete analytical examination of the question thus raised we owe chiefly to Boltzmann,[3] who showed that purely mechanical systems do, indeed, in certain circumstances display a property analogous to entropy. He showed, moreover, that the entropy of a gas in a given state stands in a simple relation to the probability of that state, and that the statistical interpretation of the law of increasing entropy amounts to the statement that an isolated system passes always from less probable to more probable states until equilibrium is reached.

A critical understanding of Boltzmann's argument is not reached without considerable effort, and models illustrating on a large scale some of the moot points regarding the behavior of a swarm of molecules may prove a welcome aid. Two such model processes are therefore here submitted.

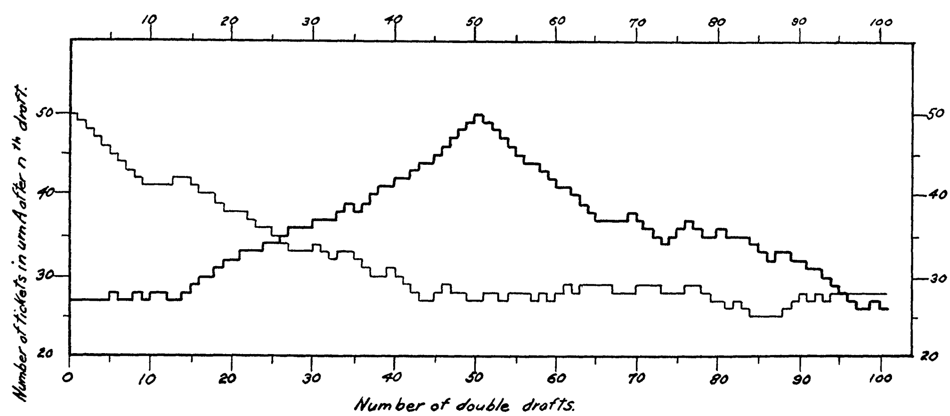


FIG. 1. Staircase curve obtained by plotting history of contents of urn A. Lighter line represents number of black balls in A after n drafts, starting with 50 black balls, and tracing history forward in time. Heavier line represents history of urn A traced forward and *backward* in time from the fiftieth draft.

2. The two models. The first of these two model processes is exemplified in Fig. 1. The more lightly drawn curve in this figure was obtained by drawing blindly balls from two urns A and B, the former containing initially only black balls, the latter only white balls. After each draft the balls drawn were returned to *opposite* urns. This process, which may be regarded as a crude imitation of the mutual diffusion of two gases, was repeated 100 times, and the number of black balls in urn A after each draft was plotted. The curve illustrates in obvious manner the passage of the system from less probable to more probable states, in analogy with the increase in entropy that accompanies the diffusion of two

plete exposition of the significance of entropy cannot be given in any parenthetic statement; for detailed treatment of this matter the reader must be referred to standard works on the subject, such as E. Buckingham's *Theory of Thermodynamics*.

[3] *Vorlesungen über Gastheorie*, 1896, vol. 1, pp. 42 *et seq.*

gases into each other (at constant total volume). The curve further shows small fluctuations, small deviations from strictly even distribution of the black balls in the two urns, persisting indefinitely after sensible equilibrium is reached. This also is in correct analogy with gaseous diffusion. Statistical mechanics admits such small departures, such momentary very slight *decreases* in entropy, when the system is very near equilibrium. Experimentally also these small fluctuations have been demonstrated by Svedberg,[1] Smoluchowski[2] and others. Thermodynamics is silent regarding these minute variations; it does not presume to give information regarding phenomena in the realm of molecular dimensions, but concerns itself only with the grosser *average* manifestations directly observable by the senses.

So far the model process described has only reaffirmed familiar examples,[3] but we have not exhausted its possibilities. We have noted that minor departures from equal distribution of black and white balls between the two urns continue to occur indefinitely. On theoretical grounds we should expect also, on very rare occasions, large departures from the mean, even to the extent of 50 balls starting from equal distribution in the two urns, and gradually assembling, all of them, in one single urn A. Can we hope to convert this theoretical expectation into actuality of observation? Not by frontal attack. Urns and balls would likely be worn to dust before that miraculous draft occurred. But our model will yield to persuasion. A simple artifice will give us a view of that lonely peak rising to the level of 50 from among a long stretch of small fluctuations.

Improbable things are happening all around us, but for the most part they are of no practical significance, and we fail to notice them. The miraculous draft which assembles in one urn 50 balls initially spread evenly in the two urns is happening under our eyes, but we fail to distinguish the balls, except as to *blackness and whiteness*. Let us slightly recast our model. Let us number the balls, or, since this is more convenient, use tickets numbered from 1 to 50 for the initial charge of urn A, and fifty more tickets, numbered from 51 to 100, for the initial charge of urn B. We now make a series of drafts as before, but this time we keep records of all the tickets by number. In an actual experiment 50 such drafts were made. At the end of the series we noted the contents of each urn, either by direct inspection or by consulting the record of tickets drawn. Starting with the urn contents just as they were at the end of the first series, we then made a second series of 50 drafts. The identity of all the tickets in urn A at the beginning of the second series being known, their previous as well as their subsequent history could be traced from the records. The history thus revealed in the experiment here recorded is shown in the more heavily drawn curve of Fig. 1. As will be seen, this contains a peak reaching up to the extreme possible limit of 50 tickets. It might be thought that this does not represent a rare event, inasmuch as we can produce at will as many of these peaks as we please. But

[1] Th. Svedberg, *Die Existenz der Moleküle*, Leipzig, 1912, p. 148.

[2] M. v. Smoluchowski, *Bull. Acad. Cracovie*, 1916, p. 218.

[3] Compare P and T. Ehrenfest, *Physikalische Zeitschrift*, 1907, vol. 8, p. 311.

any skeptic who has the patience can convince himself that such peaks are truly rare, if he will run two series, as described, the first of one million, and the second of another million drafts. It is safe to predict that he will encounter but one peak running up to 50. Nevertheless, the model makes it quite clear that such peaks are not only possible, but do actually occur. In point of fact it can be shown that, starting with black and white balls evenly distributed, 25 of each in each urn, we may expect to see all black balls in one urn about once in $3\frac{1}{2}$ million years, allowing one draft every second.

This illustrates very well how the apparent conflict between the verdict of thermodynamics and that of statistical mechanics can be harmonized. Thermodynamics says, two gases from two connected containers will mix completely, and will stay mixed forever. Statistical mechanics says, the gases will first mix, and then in N years they will unmix again, but N is a number large beyond all human comprehension, and for all practical purposes infinite.

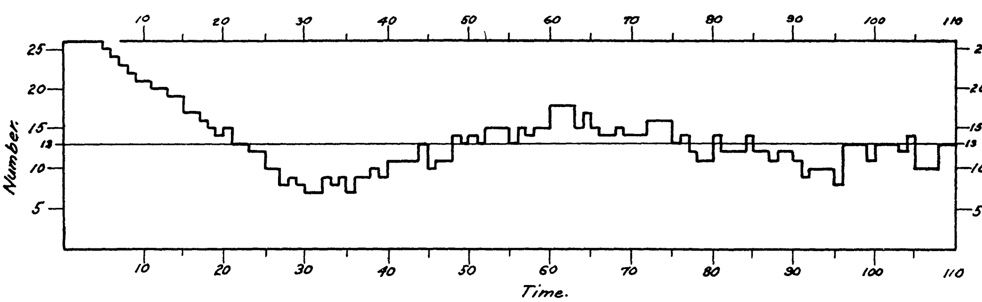


FIG. 2. Staircase curve obtained by plotting number of pendulums (out of a total of 26) on left of median line at successive epochs. The unit of the time scale is one tenth of a second.

As has already been indicated, the systems treated in statistical mechanics are, strictly speaking, periodic. The question accordingly arises how their periodic motion can exhibit phenomena analogous to increase in entropy. The second model illustrates this point very clearly. Twenty-six pendulums of periods $T = .5, .6, .7, .8, \cdots 2.9, 3.0$ seconds are started simultaneously from the median position to the left, and are then allowed to oscillate undisturbed. Count is made, at the end of every tenth of a second, of the number of pendulums on the left of the median. In this way the staircase curve, Fig. 2, was obtained (in this case by computation, not by observation). It will be seen that in the fragment of a period covered by the record, this exhibits all the characteristics of a " passage from a less probable to a more probable distribution," though, in point of fact, we know that the system has a perfectly definite period of about 7385 years, and moves in an absolutely determinate manner. The appearance of " chance " in this perfectly determinate mechanical process is brought into still greater prominence if we plot the deviations, from the mean, of the number of pendulums found on the left of the median lines, at successive counts. We thus[1] obtain the points

[1] Discarding the first 13 out of 425 counts, as being obviously unusual and strongly influenced by initial conditions.

indicated by small circles in Fig. 3. The polygonal diagram in Fig. 4 represents the corresponding frequencies for the simpler system of four pendulums, with the periods $T = .5, .6, .7, .8$ seconds; these are found to lie, very nearly indeed, on points corresponding to the coefficients of the expansion of $(1 + 1)^4$. A little reflection shows that if this relation holds for one series of pendulums of the kind here considered, it should also hold when one more pendulum is added to the series, and so on. We may test this conclusion for the case of 26 pendulums. Instead of plotting the coefficients of $(1 + 1)^{26}$, however, it is more convenient, and practically equivalent, to plot the Gaussian error curve with a standard deviation of $\sqrt{26/4}$ (= 2.549). This has been done in Fig. 3 and, as will be seen, the fit is good, considering the smallness of the sample (412 observations, extending over 41.2 seconds out of a total period of 7385 years).

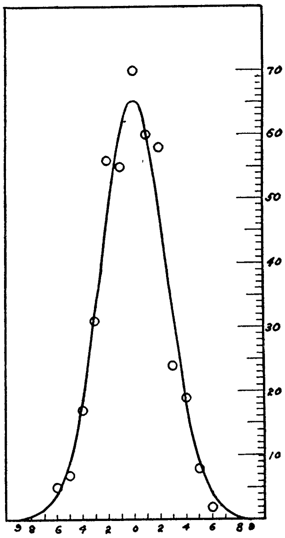


FIG. 3. Frequency of deviations from mean in number of pendulums (out of a total of 26) to left of median, in the case represented by Fig. 2.

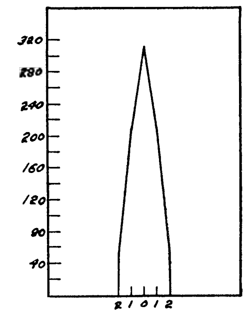


FIG. 4. Frequency of deviations from mean, as in Fig. 3, but plotted for 4 pendulums making 840 counts to cover one entire period

What do our model curves teach us regarding the alleged asymmetry of time? It is at once obvious that in the model of Fig. 1 the heavy curve makes no essential distinction between the forward and the backward direction in time. We know that high peaks are very rare, and therefore, if we find the system in a state corresponding to a high point of the curve, we know that we are very probably near the crest of a peak, so that we are either actually descending, or very soon shall be. But all this is equally true whether we read the curve from left to right or from right to left. Similar remarks apply to Fig. 2, the details of which may be left to the reader to work out. The truly significant fact is that high peaks occupy only a vanishingly small portion of the total base line, so that if the system is in a very improbable state, it is extremely likely that it will very soon pass into a much

more probable one—but also, it is extremely likely that it was in a very much more probable state a short time ago. The proposition holds in either sense, making no distinction between $+t$ and $-t$. Whence then comes our intuitive, subjective sense of the asymmetry of time? Perhaps it is in some way connected with the integration constants; for it is only the *differential* equations that do not respect the sign of t. Or perhaps the fundamental directedness of the world events is not expressed in the equations of ordinary dynamics, but strikes its roots deeper into the underlying quantum mechanics.[1] As yet we stand, here, upon uncertain ground of speculation.

In conclusion attention may be drawn to a psychological and biological significance of the models here presented. It appears at first sight as if there were a fundamental difference in character between the first and the second model, since it is essential for the operation of the urn model that the drawing be done *blindly*, so as to give *chance* a part in the process; whereas the pendulum model we operate with our eyes open, apparently in full consciousness of what is going on. Chance seems to play no part here, the system is mechanically determinate.

But there is a blindness which is not of the eye, and there is a vision that surpasses optical vision. The same struggle for existence which has developed in man the organ of sight, to depict for him the external world, to furnish him with a map on which to base his plan of campaign, is also developing his internal vision, whereby he extends his world-picture beyond the powers of the bodily eye. Whether I peep into the urn and manipulate the drafts by the light of my eyes; or whether, in the light of my knowledge of mechanics, I adjust the pendulums to equal lengths and phases; or again, whether, in the more serious affairs of life, I employ these same faculties to diverse ends, the effect is the same. In greater measure or less these organs and faculties emancipate me from the bonds of the fortuitous and make me, in this sense, a controller of events. Their function is to substitute choice for chance, to introduce aimed collisions in place of random encounters.

THE NUMBERS OF REPRESENTATIONS OF INTEGERS IN CERTAIN FORMS $ax^2 + by^2 + cz^2$.

By E. T. BELL, University of Washington.

1. Introduction. The number $N(n = ax^2 + by^2 + cz^2)$ of sets (x, y, z) of integers $x, y, z \gtreqless 0$ satisfying $n = ax^2 + by^2 + cz^2$ is called the number of representations of n in the form $ax^2 + by^2 + cz^2$. When $a = b = c = 1$ this number was determined by Gauss.[2]

[1] *Cf.* G. Breit. Are Quanta Unidirectional? *Phys. Rev.*, vol. 22, 1923, p. 313.

[2] *Disquisitiones Arithmeticæ*, Arts. 292, 293 (Leipzig, 1801). *Cf.* Dickson, *History of the Theory of Numbers*, vol. 2, Preface, pp. ix, x.

The only other specific results for $N(n = ax^2 + by^2 + cz^2)$ appear to be an unproved assertion of Liouville (1869) for the case $(a, b, c) = (1, 2, 3)$, and incomplete results for $(a, b, c) = (1, 1, 2)$ (Torelli, 1878), and $(a, b, c) = (1, 2, 2)$ (Stieltjes, 1883).[1]

Assuming Gauss' theorem and the classic $N(n = x^2 + y^2) = 4\xi(m)$, where $n = 2^\alpha m$, $\alpha \geqq 0$, m odd, $\xi(m)$ = the number of $4k + 1$ divisors of m minus the number of $4k + 3$ divisors, we shall obtain *complete* enumerations

$$N\ (n = ax^2 + by^2 + cz^2)$$

for each of the nine forms

$$\begin{array}{lll} x^2 + y^2 + 2z^2, & x^2 + 4y^2 + 8z^2, & x^2 + 8y^2 + 8z^2, \\ x^2 + 2y^2 + 2z^2, & x^2 + y^2 + 8z^2, & x^2 + 4y^2 + 4z^2, \\ x^2 + 2y^2 + 4z^2, & x^2 + 2y^2 + 8z^2, & x^2 + y^2 + 4z^2, \end{array}$$

which, it is easily seen, with Gauss' $x^2 + y^2 + z^2$, include all possible cases of $N(n = ax^2 + by^2 + cz^2)$ in which each of a, b, c is a power, not higher than the third, of 2. The type of theorem indicated in § 14 is of particular interest.

2. The Theorem of Gauss. The new results of this paper will be more interesting to any who have not specialized in arithmetic if we first describe in a general way the nature of Gauss' theorem, which is not at all obvious.

The totality of binary quadratic forms $ax^2 + 2bxy + cy^2$ with integer coefficients a, b, c having their determinants $b^2 - ac$ equal to a constant $D < 0$ is segregated into classes, all those forms, called (properly) equivalent, which are such that any one can be derived from any other by a linear substitution

$$x' = \alpha x + \beta y, \qquad y' = \gamma x + \delta y, \qquad \text{where} \qquad \alpha\delta - \beta\gamma = 1,$$

and $\alpha, \beta, \gamma, \delta$ are integers, being put into the same class. The number of classes for each D is finite. For any D there exists a finite number of forms, called reduced, whose coefficients satisfy certain inequalities, and in each class there is at least one reduced form, which evidently is equivalent to each of the forms in its class and can therefore be taken as the representative of all. If the outer coefficients of the representative form are not both even, the corresponding class is called odd, otherwise even. The numbers of odd, even classes for the negative determinant $- n$ are written $F(n)$, $F_1(n)$, and $E(n) \equiv F(n) - F_1(n)$. Either from arithmetic or elliptic series it is proved that

$$\begin{array}{lll} 3E(8n + 3) = 2F(8n + 3), & E(8n + 7) = 0, & E(4n) = E(n), \\ E(4n + 1) = F(4n + 1), & E(4n + 2) = F(4n + 2), & F(4n) = 2F(n). \end{array}$$

By convention a class equivalent to $a(x^2 + y^2)$ contributes $\frac{1}{2}$ to F or to F_1; one equivalent to $a(2x^2 + 2xy + 2y^2)$ contributes $\frac{1}{3}$ to F_1; $F(0) = 0$, $E(0) = \frac{1}{12}$.

The theorem of Gauss states that

$$N(n = x^2 + y^2 + z^2) = 12E(n). \tag{1}$$

[1] For references and summaries *cf.* Dickson, *loc. cit.*, vol. 2, pp. 294, 295; vol. 3, pp. 133, 216.

3. Notation, Series. Henceforth $m_i\,(i = 1, 2, \cdots)$ denote odd integers $\gtrsim 0$, $n_i\,(i = 1, 2, \cdots)$ even or odd integers $\gtreqless 0$; $n \geqq 0$ is even or odd, $m > 0$ is odd, $\alpha \geqq 0$ is even or odd. When m, n, m_i, n_i occur under a Σ, the summation is with respect to all values of the m, n, m_i, n_i consistent with the above definitions. The functions $\vartheta_\alpha(q)$ are defined by

$$\vartheta_3(q) = \Sigma q^{n_i^2}, \qquad \vartheta_2(q) = \Sigma q^{m_i^2/4}, \qquad \vartheta_0(q) = \vartheta_3(-q).$$

From these definitions and the theorem on $N(n = x^2 + y^2)$ quoted in § 1 it is easy by comparing coefficients of like powers of q to verify

$$\vartheta_0(q)\vartheta_3(q) = \vartheta_0^2(q^2), \quad (2) \qquad \vartheta_3^2(q^2) - \vartheta_0^2(q^2) = 2\vartheta_2^2(q^4), \quad (5)$$

$$2\vartheta_2(q^8)\vartheta_3(q^8) = \vartheta_2^2(q^4), \quad (3) \qquad \vartheta_3^2(q) + \vartheta_0^2(q) = 2\vartheta_3^2(q^2), \quad (6)$$

$$\vartheta_3^2(q^4) - \vartheta_2^2(q^4) = \vartheta_0^2(q^2), \quad (4) \qquad \vartheta_3^2(q^4) + \vartheta_2^2(q^4) = \vartheta_3^2(q^2), \quad (7)$$

$$2e^{i\pi/4}\vartheta_0(q^8)\vartheta_2(q^8) = \vartheta_2^2(iq^4), \qquad (i = \sqrt{-1}); \tag{7.1}$$

or, if preferred, (2) — (7.1) can be taken as well known from elliptic functions.

Now obviously the algebraic equivalent of (1) is

$$\vartheta_3^3(q) = 12\Sigma q^n E(n). \tag{8}$$

If in (8) q be replaced by $-q$, there follow by elementary algebra[1] the important identities of Kronecker and Hermite:

$$\vartheta_0(q^4)\vartheta_2^2(q^4) = 4\Sigma q^{4n+2}(-1)^n F(4n+2), \tag{9}$$

$$\vartheta_0^2(q^4)\vartheta_2(q^4) = 4\Sigma q^{4n+1}(-1)^n F(4n+1), \tag{10}$$

$$\vartheta_2(q^4)\vartheta_3^2(q^4) = 4\Sigma q^{4n+1} F(4n+1), \tag{11}$$

$$\vartheta_2^2(q^2)\vartheta_3(q^2) = 4\Sigma q^{2n+1} F(4n+2), \tag{12}$$

$$\vartheta_2^3(q^4) = 8\Sigma q^{8n+3} F(8n+3). \tag{13}$$

As all results concerning the nine forms to be discussed are immediate consequences of (2) — (13), it follows that for these forms the complete enumeration is implicit in the classical theorems for 2 and 3 squares. It will not be necessary to preserve all the elementary algebraic details for each case, but the reader who cares to supply them will find much more information concerning the forms than that recorded here. I have verified the results numerically.

4. $N(n = x^2 + y^2 + 2z^2) \equiv N(n)$. Multiply (7) by $\vartheta_3(q^2)$, and use (8), (12) on the right of the resulting identity:

$$\Sigma q^{n_1^2+n_2^2+2n_3^2} = 4\Sigma q^{2n+1}F(4n+2) + 12\Sigma q^{2n}E(n).$$

Equating coefficients of like powers of q, we find

$$N(2n+1) = 4F(4n+2), \qquad N(2n) = 12E(n).$$

[1] The extremely simple details are given in full in a note to appear in the *Bulletin of the American Mathematical Society*.

5. Reduction Formulas. Evidently

$$N(pn_1 = pax^2 + pby^2 + pcz^2) = N(n_1 = ax^2 + by^2 + cz^2).$$

Let p be prime, and n_1 prime to p. Every integer n is of the form $p^\alpha n_1$. In order that $p^{\alpha+1}n_1 = x^2 + pby^2 + pcz^2$ shall have solutions it is necessary that $x \equiv 0 \bmod p$; say $x = px_1$. Hence

$$N(p^{\alpha+1}n_1 = x^2 + pby^2 + pcz^2) = N(p^\alpha n_1 = px_1^2 + by^2 + cz^2),$$

and clearly the suffix may be dropped from x_1. Obviously

$$N(n = ax^2 + by^2 + cz^2)$$

is invariant for all permutations of x, y, z. We shall rearrange x, y, z when necessary so that $a \lesseqgtr b \lesseqgtr c$.

In the present discussion $p = 2$, so that $n_1 = m$. As an example of the reduction,

$$N(2^{\alpha+1}m = x^2 + 2y^2 + 2z^2) = N(2^\alpha m = x^2 + y^2 + 2z^2).$$

When $N(m = x^2 + 2y^2 + 2z^2)$ is known, the further evaluation of

$$N(n = x^2 + 2y^2 + 2z^2)$$

is referred by the above reduction to § 4.

6. $\boldsymbol{N(n = x^2 + 2y^2 + 2z^2) \equiv N(n)}$. Since $m_i^2 \equiv 1 \bmod 8$, $4n_i^2 \equiv 0$ or $4 \bmod 8$, we have $2(x^2 + y^2) \equiv 0, 2, 4 \bmod 8$, and therefore

$$\begin{aligned} N(8n + 1 = x^2 + 2y^2 + 2z^2) &= N(8n + 1 = m_1^2 + 8n_2^2 + 8n_3^2), \\ N(8n + 5 = x^2 + 2y^2 + 2z^2) &= N(8n + 5 = m_1^2 + 2m_2^2 + 2m_3^2), \\ N(8n + 3 = x^2 + 2y^2 + 2z^2) &= 2N(8n + 3 = m_1^2 + 2m_2^2 + 8n_3^2). \end{aligned}$$

Now $N(8n + 1)$ is the coefficient of q^{8n+1} in $\Sigma q^{m_1^2} \times \Sigma q^{8n_2^2+8n_3^2}$; $N(8n + 5)$ is the coefficient of q^{8n+5} in $\Sigma q^{m_1^2} \times \Sigma q^{2m_2^2+2m_3^2}$. These direct us to

$$\vartheta_0(q^2)\vartheta_3(q^2)\vartheta_2(q^4) = \vartheta_0^2(q^4)\vartheta_2(q^4),$$

written down from (2), and as in § 4 this gives

$$\begin{aligned} N(4n + 1 &= m_1^2 + 2n_2^2 + 8n_3^2) \\ &- N(4n + 1 = m_1^2 + 2m_2^2 + 2n_3^2) = 4(-1)^n F(4n + 1). \end{aligned}$$

Separating cases of n even or odd we get $N(8n + 1)$, $N(8n + 5)$, which combine into the result for $N(4n + 1)$ stated below. Similarly (3) multiplied by $\vartheta_2(q^4)$ gives the value of $N(8n + 3)$. The rest follow immediately by applying § 5 to § 4, and we have finally

$$\begin{aligned} N(4n + 1) &= 4F(4n + 1), \qquad N(4n + 2) = 4F(4n + 2), \\ N(8n + 3) &= 8F(8n + 3), \qquad N(8n + 7) = 0, \\ N(2^{\alpha+2}m) &= 12E(2^\alpha m). \end{aligned}$$

7. $N(n = x^2 + 2y^2 + 4z^2) \equiv N(n)$. The first of the following comes from the identity obtained by multiplying (7.1) throughout by $\vartheta_0(iq^4)$, the rest by applying § 5 to § 6:

$$N(2n+1) = 2F(4n+2), \qquad N(8n+2) = 4F(4n+1),$$
$$N(8n+4) = 4F(4n+2), \qquad N(16n+6) = 8F(8n+3),$$
$$N(16n+14) = 0, \qquad N(2^{\alpha+3}m) = 12E(2^\alpha m).$$

8. $N(n = x^2 + 4y^2 + 8z^2) \equiv N(n)$. From (4) multiplied throughout by $\vartheta_0(q^4)$ we get the first of the following; the formulas for even numbers come from § 5 applied to § 4:

$$N(4n+1) = 2F(8n+2), \qquad N(4n+2) = N(4n+3) = 0,$$
$$N(8n+4) = 4F(4n+2), \qquad N(2^{\alpha+3}m) = 12E(2^\alpha m).$$

9. $N(n = x^2 + y^2 + 8z^2) \equiv N(n)$. If $2^{\alpha+1}m$ is represented in this form, x, y are both even or both odd. The case in which they are both even is reduced by § 5 to § 4. Of the rest below, $N(4n+1)$ is from (3) multiplied throughout by $\vartheta_0(q^4)$, $N(8n+2)$ is from (3) multiplied by $\vartheta_3(q^8)$.

$$N(4n+1) = 4F(8n+2), \qquad N(4n+3) = N(8n+6) = 0,$$
$$N(8n+2) = 8F(4n+1), \qquad N(8n+4) = 4F(4n+2),$$
$$N(2^{\alpha+3}m) = 12E(2^\alpha m).$$

10. $N(n = x^2 + 2y^2 + 8z^2) \equiv N(n)$. The second of the following is from the identity obtained by multiplying (3) by $\vartheta_2(q^4)$; the first is from the first in § 11; those for even numbers come from applying § 5 to § 7.

$$N(8n+1) = 4F(8n+1), \qquad N(8n+3) = 4F(8n+3),$$
$$N(8n+5,7) = 0, \qquad N(4n+2) = 2F(4n+2),$$
$$N(16n+4) = 4F(4n+1), \qquad N(16n+8) = 4F(4n+2),$$
$$N(32n+12) = 8F(8n+3), \qquad N(32n+28) = 0,$$
$$N(2^{\alpha+4}m) = 12E(2^\alpha m).$$

11. $N(n = x^2 + 8y^2 + 8z^2) \equiv N(n)$. The first comes from (2) multiplied by $\vartheta_2(q^2)$; to get the formulas for even numbers apply § 5 to § 6.

$$N(8n+1) = 4F(8n+1), \qquad N(8n+3,5,7) = N(4n+2) = 0,$$
$$N(16n+4) = 4F(4n+1), \qquad N(16n+8) = 4F(4n+2),$$
$$N(32n+12) = 8F(8n+3), \qquad N(32n+28) = 0,$$
$$N(2^{\alpha+4}m) = 12E(2^\alpha m).$$

12. $N(n = x^2 + 4y^2 + 4z^2) \equiv N(n)$. The enumerations for this form and the next are read off at once from (9)–(13) and by using § 5 on (1).

$$N(4n) = 12E(n), \qquad N(4n+1) = 4F(4n+1),$$
$$N(4n+2) = N(4n+3) = 0.$$

13. $N(n = x^2 + y^2 + 4z^2) \equiv N(n)$. As indicated we find

$$N(4n) = 12E(n), \qquad N(4n+1) = 8F(4n+1),$$
$$N(4n+2) = 4F(4n+2), \qquad N(4n+3) = 0.$$

14. Case when the number to be represented is a square. Many of the foregoing theorems can be remarkably simplified when the number to be represented is a square or a power of two times a square. In a paper to appear elsewhere I have shown that if $n = 2^\alpha m$, then $F(n^2) = 2^{\alpha-1}S(m)$, $E(n^2) = \frac{1}{2}S(m)$, where, if $m = \Pi p_i^{\beta_i}$ is the prime factor resolution of m,

$$S(m) = \Pi[p_i^{\beta_i} + \{1 - (-1)^{\frac{1}{2}(p_i-1)}\}\zeta_1(p_i^{\beta_i-1})],$$

and $\zeta_1(n) =$ the sum of all the divisors of n.

Hence, for example, on referring to § 11, we have

$$N(2^{2\alpha+4}m^2 = x^2 + 8y^2 + 8z^2) = 12E(2^{2\alpha}m^2) = 6S(m),$$

expressing $N(2^{2\alpha+4}m^2)$ without class number functions. The remaining theorems of this kind implicit in §§ 4–13 can be read off by inspection.

CONICAL LOCI ASSOCIATED WITH THE MOTION OF A RIGID BODY ABOUT A POINT.

By E. L. REES, University of Kentucky.

1. The writer gave recently in the MONTHLY (*1923*, 290–296) a vector treatment of the motion of a rigid body in a plane. In the present paper the theory relating to the conical loci associated with the motion of a rigid body with one point fixed is treated by the same vector methods, the advantages of which are more strikingly illustrated here than in the previous article.[1]

2. Instantaneous Axis of Rotation. The following fundamental theorem may be proved vectorially in a number of ways.[2]

The motion of a rigid body one point of which is fixed is at each instant a rotation about an instantaneous axis passing through the fixed point.

3. Polar Cones. The instantaneous axis in the course of its motion generates a cone in space and a cone in the body called the *polar cones*. We shall call the space locus of the inst. axis and its body locus the *space cone* and the *body cone* respectively.

Let $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ be the axes fixed in the body with origin at the fixed point. The angular velocity vector w when referred to the moving trihedral is given by $\mathbf{w}' = \mathbf{w}\cdot\mathbf{i}'\mathbf{i}' + \mathbf{w}\cdot\mathbf{j}'\mathbf{j}' + \mathbf{w}\cdot\mathbf{k}'\mathbf{k}' = \Sigma\mathbf{w}\cdot\mathbf{i}'\mathbf{i}'$.

Differentiating with respect to t, we get

$$\dot{\mathbf{w}}' = \sum\dot{\mathbf{w}}\cdot\mathbf{i}'\mathbf{i}' + \sum\mathbf{w}\cdot\dot{\mathbf{i}}'\mathbf{i}'.$$

[1] A synthetic treatment of the subject of this paper is given in Schoenflies' "*Geometrie du Mouvement*," Paris, 1893, pp. 52–81.

For a vector treatment (Burali-Forti notation) of velocities and accelerations of points of a rigid body the reader is referred to an article by Ziwet and Field in the MONTHLY (*1916*, 371–381).

[2] See this MONTHLY (*1918*, 127) for a vector proof of this theorem; see also Gibbs-Wilson "*Vector Analysis*," pp. 131–132.

Now all of the terms of the second summation vanish since $i' = w \times i'$, etc. Therefore $\dot{w}' = \dot{w}$, which shows that the cones are tangent to each other along the inst. axis, since the tangent plane of each is determined by w and $\dot{w}$, and that the angles of corresponding pairs of consecutive elements are equal. Hence the theorem:

The body cone rolls without slipping on the space cone.

It is obvious that in general the elements of the body cone are the only lines of the body whose trajectory cones have cuspidal lines (lines instantaneously stationary) and that the locus of these cuspidal lines is the space cone.

4. **Cone of Inflections.** Each line of the body through the fixed point generates a cone. We shall now find the instantaneous locus of those lines which are lines of inflection of their trajectory cones. These lines are determined by the vectors p (position vectors of the points of the body) which satisfy the condition $[p\dot{p}\ddot{p}] = 0$. Since $\dot{p} = w \times p$ and $\ddot{p} = \dot{w} \times p + w \times (w \times p)$, this condition leads to the equation

$$[w\dot{w}p]p^2 - (w \cdot p)^3 + w^2 w \cdot p p^2 = 0,$$

which represents a cone of the third order called the *cone of inflections.* Regarding $|p|$ as constant, and differentiating this equation with respect to u, a parameter which fixes the position of p on the cone, we have

$$[w\dot{w}p_u]p^2 - 3(w \cdot p)^2(w \cdot p_u) + w^2 p^2 w \cdot p_u = 0.$$

If u has the value for which the element, determined by p, coincides with the inst. axis, then $w \cdot p_u = 0$, and it results that $[w\dot{w}p_u] = 0$. Consequently *the cone of inflections is tangent to the polar cones along the inst. axis.*

5. **Cone of Cusps.** Each plane of the body through the fixed point in the course of its motion envelops a cone. Let us now find the locus of the characteristics of the planes through the fixed point which are cuspidal lines of these envelopes.

The characteristic of a plane $a \cdot r = 0$, where a is normal to the plane fixed in the body and of unit length, is the intersection of this plane and the plane $\dot{a} \cdot r = 0$. It thus passes through the fixed point and has the direction of $a \times \dot{a}$. To be a cuspidal line this characteristic must be instantaneously stationary, the condition for which is

$$(a \times \dot{a}) \times (a \times \dot{a})_t = (a \times \dot{a}) \times (a \times \ddot{a}) = 0.$$

But this is equivalent to $[a\dot{a}\ddot{a}] = 0$. Hence *the normals to the planes which are tangent to their envelopes along cuspidal lines are the elements of the cone of inflections.* The envelope of these cuspidal tangent planes, at a given instant, is a cone.[1] This cone and the cone of inflections are related as follows: each cone is the envelope of the planes perpendicular to the other, and each is the locus of the lines through the fixed point normal to the other.

[1] This cone is called by some authors the cone of cusps, but it seems more appropriate for our purposes to use this name for the conical locus of the cuspidal lines of the plane envelopes as indicated below.

As just seen, the characteristics of the planes, which are cuspidal lines of the envelopes, are determined by the vectors $\boldsymbol{a} \times \dot{\boldsymbol{a}}$, where $\boldsymbol{a}$ is subject to the condition $[\boldsymbol{a}\dot{\boldsymbol{a}}\ddot{\boldsymbol{a}}] = 0$. Let $\boldsymbol{q} = \mu\boldsymbol{a} \times \dot{\boldsymbol{a}}$. By resolving $\boldsymbol{w}$ along $\boldsymbol{a}$ and $\boldsymbol{q}$ we obtain

$$\boldsymbol{a} = \frac{\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{q}^{-1}\boldsymbol{q}}{\boldsymbol{w}\cdot\boldsymbol{a}}.$$

This value of $\boldsymbol{a}$, when substituted in

$$[\boldsymbol{a}\dot{\boldsymbol{a}}\ddot{\boldsymbol{a}}] = \boldsymbol{q}\cdot(\dot{\boldsymbol{w}} \times \boldsymbol{a} + \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{a})) = 0,$$

gives

$$[\boldsymbol{w}\boldsymbol{q}(\dot{\boldsymbol{w}} + \boldsymbol{w}\cdot\boldsymbol{q}^{-1}\boldsymbol{w} \times \boldsymbol{q})] = 0,$$

or

$$[\boldsymbol{w}\dot{\boldsymbol{w}}\boldsymbol{q}]\boldsymbol{q}^2 + (\boldsymbol{w}\cdot\boldsymbol{q})^3 - \boldsymbol{w}^2\boldsymbol{w}\cdot\boldsymbol{q}\boldsymbol{q}^2 = 0.$$

Thus the locus of the cuspidal lines is a cone of the third order. We shall call this cone the *cone of cusps*.[1]

On differentiating the last equation with respect to u and letting $\boldsymbol{q}$ have the direction of the inst. axis (note that the inst. axis is an element of the cone of cusps) it results that $[\boldsymbol{w}\dot{\boldsymbol{w}}\boldsymbol{q}_u] = 0$, which shows that *the cone of cusps is tangent to the polar cones along the inst. axis.*

The cone of cusps and the cone of inflections are symmetric to each other with respect to their common tangent plane which is also the common tangent plane of the polar cones; for, their equations show that to each element of either cone there corresponds an element of the other which makes the same angle with the inst. axis and the supplementary angle with the vector $\boldsymbol{w} \times \dot{\boldsymbol{w}}$. Thus the elements of the two cones are associated in symmetric pairs.

The polar axis (see next Art.) of an element, determined by $\boldsymbol{p}$, is normal to this element, since $[\boldsymbol{p}\dot{\boldsymbol{p}}\ddot{\boldsymbol{p}}] = 0$, and has the direction of $\boldsymbol{p} \times \dot{\boldsymbol{p}}$. We shall now show that *the locus of the polar axes of the elements of the cone of inflections is the cone of cusps.* Substituting $\boldsymbol{q} = \mu\boldsymbol{p} \times \dot{\boldsymbol{p}}$ in the equation of the cone of cusps, we find that the resulting equation is exactly the equation in $\boldsymbol{p}$ of the cone of inflections, so that if $\boldsymbol{p}$ is an element of the cone of inflections, $\boldsymbol{q}$ is an element of the cone of cusps as was to be proved. We may say then that *the corresponding elements of the cone of inflections and the cone of cusps are perpendicular to each other and are coplanar with the inst. axis* (line of tangency of the cones).

6. Orthogonal Cone. The equation of the normal plane of the trajectory of a point P is $(\boldsymbol{r} - \boldsymbol{p})\cdot\dot{\boldsymbol{p}} = \boldsymbol{r}\cdot\dot{\boldsymbol{p}} = 0$. It thus passes through the fixed point. The polar axis of this trajectory is the intersection of $\boldsymbol{r}\cdot\dot{\boldsymbol{p}} = 0$ and $\boldsymbol{r}\cdot\ddot{\boldsymbol{p}} = 0$, and therefore passes through the fixed point and has the direction of $\dot{\boldsymbol{p}} \times \ddot{\boldsymbol{p}}$. But since $\dot{\boldsymbol{p}} = \boldsymbol{w} \times \boldsymbol{p}$ and $\ddot{\boldsymbol{p}} = \dot{\boldsymbol{w}} \times \boldsymbol{p} + \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{p})$, the direction of $\dot{\boldsymbol{p}} \times \ddot{\boldsymbol{p}}$ is independent of the magnitude of $\boldsymbol{p}$. Therefore the polar axis is the same for the trajectories of all points of a line through the fixed point and is called the *polar axis* of this line.

We now proceed to find the locus of lines whose polar axes are perpendicular

[1] In the inverse motion the cone of inflections and the cone of cusps interchange rôles. See Schoenflies' p. 74.

to the inst. axis. The condition here is $[wp\ddot{p}] = 0$. Consequently,

$$[ww \times p\dot{w} \times p + w \times (w \times p)] = 0$$

is the equation of our locus which is therefore a *quadratic cone.* Substituting $\dot{w} = \dot{w}w_1 + w\dot{w}_1$ in this equation, we get $[ww \times p\dot{w}_1 \times p - wp] = 0$, which, assuming $w \neq 0$, is equivalent to $w \times p \cdot [(\dot{w}_1 \times w_1 + w) \times p] = 0$. *This cone is therefore the locus of the intersection of perpendicular planes through the inst. axis and the line determined by* $\dot{w}_1 \times w_1 + w$. From the manner of its generation it is called an *orthogonal cone.*

The last equation may be written in the form $(p \times \dot{p}) \cdot (\dot{w}_1 \times w_1 + w) = 0$, from which follows the theorem:

The tangent planes of the trajectory cones of the elements of the orthogonal cone form a pencil which has for its axis the element (determined by the vector $\dot{w}_1 \times w_1 + w$) *of the orthogonal cone diametrically opposite the inst. axis.*

The plane of w and $\dot{w}_1 \times w_1 + w$, which is a diametral plane of the orthogonal cone, is normal to the common tangent plane of the polar cones, and *the orthogonal cone is tangent to the polar cones along the inst. axis.* This cone is of course also tangent to the cone of inflections and to the cone of cusps along their line of tangency.

7. Cone of Lines with Stationary Polar Axes. If the polar axis of a line, determined by p, is instantaneously stationary, we have

$$(\dot{p} \times \ddot{p}) \times (\dot{p} \times \ddot{p})_t \equiv (\dot{p} \times \ddot{p}) \times (\dot{p} \times \dddot{p}) = 0,$$

which is equivalent to $[\dot{p}\ddot{p}\dddot{p}] = 0$. Since the derivatives of p are linearly homogeneous in p, this equation represents a *cone of the third order,* which is *the conical locus of lines with stationary polar axes.* Since the polar axes for points of this cone are stationary the osculating circles of the trajectories of these points are also stationary, or in other words, the cone of lines with stationary polar axes is the locus of points for which the osculating circles have contact of the third order with the trajectories. This cone is also the locus of points at which the osculating planes of their trajectories are momentarily stationary.

8. Cone of Stationary Polar Axes. The polar axis corresponding to p is determined by $q = \dot{p} \times \ddot{p} = [w\dot{w}p]p + (w \times p)^2 w$. Multiplying by $w \times \dot{w} \cdot$ and $w \times$ respectively, we get $[w\dot{w}q] = [w\dot{w}p]^2$ and $w \times q = [w\dot{w}p]w \times p$, so that

$$p = \frac{(w \times q) \times w^{-1}}{\pm \sqrt{[w\dot{w}q]}} + \lambda w.$$

Substituting this expression for p in the equation $q = \dot{p} \times \ddot{p}$, we find

$$\lambda = \frac{[w\dot{w}q]w^{-1} \cdot q - (w \times q)^2}{\pm \sqrt{[w\dot{w}q]^3}},$$

so we may express p in terms of q as follows:

$$p = \frac{[w\dot{w}q]q - (w \times q)^2 w}{\pm \sqrt{[w\dot{w}q]^3}}\,.$$

Substituting this final expression for p in the equation of the cone of lines with stationary polar axes we obtain, after simplifying, an equation of the third degree in q. Hence *the locus of stationary polar axes is a cone of the third order.*

QUESTIONS AND DISCUSSIONS.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS.

I. A PROPERTY OF THE ISOGONAL CENTERS OF A TRIANGLE.

By H. M. LUFKIN, Dunkirk, N. Y.

The following is an interesting property of the isogonal centers of a triangle which I believe is new.

It is known that the I_+ point (*cf.* my paper in this MONTHLY, *1923*, 127–131) for any triangle having an angle greater than 120° is outside the triangle, and when no angle is greater than 120° it is inside. If we consider a pair of triangles having two sides of one equal to two sides of the other, but the included angles bearing the relation, that in the first it is $120° - \varphi$, and in the second $120° + \varphi$, it can easily be proved that the sum of the distances from the I_+ point to the vertices of the first triangle is equal to the sum of the distances from the I_+ point of the second triangle to the acute angles diminished by the distance to the obtuse angle.

II. CIRCUMSCRIBED AND INSCRIBED TETRAHEDRA.

By ALBERT A. BENNETT, University of Texas.

In this MONTHLY (*1923*, 178), Professor Coolidge mentions as "Problem 8" among "Some unsolved problems in solid geometry," the following: "What relation must exist between spheres in order that it may be possible to inscribe a tetrahedron in one which is circumscribed to the other?"

One might anticipate applications of elliptic or of hyperelliptic functions as interesting as those in the plane case; that is, until one looks at the problem more closely and notes the presence of too many degrees of freedom. The fact is that the problem is far from poristic. Analytic methods are unwieldy unless they follow closely along the lines of possible synthetic methods. The important theorem in this relation is capable of simple expression as follows.

Theorem: Given any sphere, S, and any point, C, within the sphere. For any two distinct points, P and Q, whose join does not pass through C, there is a length, R, such that for every sphere, S', with center C, and radius less than R, there will be exactly two distinct tetrahedra inscribed in S and circumscribed about S' and having P and Q for two of the vertices. (The term "circumsribed" is used in its narrow sense, as distinguished from "escribed.")

In the following synthetic proof, the explicit formulation of R in terms of S, C, P, and Q, is not made, but the magnitude of the radius is restricted by the order relations in resulting constructions. Consider a small sphere, S', with C as center, just how small, will appear later. Draw two tangent planes, α and β, both of which pass through P and Q and tangent to S' at T_A and T_B, respectively. Consider the self-inverse projectivity π_P, defined between points of the circle, A, in which α cuts S, and points of the circle, B, in which β cuts S, as follows. Two points, one of A and one of B correspond under π_P (neither being P nor Q) if and only if the plane determined by these two points and P is tangent to the sphere, S'. The special cases of P and Q are easily handled by continuity. There will be of course two planes tangent to S' through P and any given point of A. However α is itself one such plane, and the other which is thereby uniquely determined is the one employed in the projectivity. The projectivity, π_Q, is defined analogously by the use of Q in place of P. A pair of points, one on A and one on B, which correspond under both π_P and π_Q will serve as vertices with P and Q of the tetrahedron desired, provided that the tetrahedron thus obtained, whose faces are tangent to S', actually contains S' in its interior. The product transformation, $\pi_P\pi_Q$, will be for each of the circles, A and B, a projectivity of the figure into itself, and will have two real, one real, or two imaginary fixed points. It is necessary to show that two real solutions are always obtained, actually containing S' within the tetrahedron, provided only that S' be taken with less than a determinate radius.

We shall make use of certain particularly labelled points and it is suggested that the reader make his own illustrations, labelling the proper points as directed. Denote by D_{AP} the other intersection with A of the line PT_A. In a similar manner, define D_{AQ}, D_{BP}, D_{BQ}. Of the two planes through P tangent to S' and perpendicular to the plane PQC, let that one which does not separate Q from C, cut circle A again in E_{AP}. Define in similar manner, E_{BP}, E_{AQ}, E_{BQ}. The points, P, D_{AQ}, D_{AP}, Q, will occur on A in this cyclic order, since the lines used to determine D_{AQ} and D_{AP} intersect at T_A which is within the circle A. The point E_{AP} is between P and D_{AP}, and may be made to come as close to the latter as may be desired if the radius of S' be taken sufficiently small. Our restriction upon this radius will now be expressed by the conditions that the points on A and B occur in the following cyclic orders: on A, we are to have the order, P, D_{AQ}, E_{AQ}, E_{AP}, D_{AP}, Q, and similarly on B. The transformation, π_P, when used from A to B, carries P as a point of A into D_{BP} of B, E_{AP} of A into E_{BP} of B, D_{AP} of A into P as a point of B, Q as a point of A into Q as a point of B. In particular, π_P carries all the points (including D_{AQ}, E_{AQ}, E_{AP}) of the arc PD_{AP}

into points of the arc $D_{BP}P$ (including E_{BP}, E_{BQ}, D_{BP}) of the circle B, but in such a way that the points of the large arc PE_{AP} of A go into the points of the small arc, $D_{BP}E_{BP}$ of B, while points of the small arc, $D_{AP}E_{AP}$ of A go into points of the large arc, PE_{BP} of B. The transformation π_Q, may be discussed in a similar manner as to its effect from B to A.

Consider now what happens when a point of A is carried from A to B by π_P, and this is in turn carried back to A by π_Q. The points of the arc PE_{AP} of A are first carried into those of the small arc, $D_{BP}E_{BP}$ of B. But this arc lies entirely within the arc QE_{BQ} of B, which latter is carried by π_Q into $D_{AQ}E_{AQ}$. Thus the result of the two transformations performed in succession is to carry PE_{AQ} into an arc which lies entirely within it. It follows from the continuity of the projectivity, that there will be some point of arc $D_{AQ}E_{AQ}$ which is carried into itself. Since this point will lie between D_{AQ} and E_{AQ}, and will correspond under both transformations with a point of B between D_{BP} and E_{BP}, this pair of corresponding points furnishes an actual solution of the problem, and does not correspond to an escribed tetrahedron. Likewise the small arc $D_{AP}E_{AP}$ is carried as a result of the two transformations applied successively into the arc QE_{AQ}, which completely envelopes it. It therefore also contains an invariant point which furnishes a second actual solution of the problem, as promised.

III. On Algorithms for the Solution of the Linear Congruence.

By H. S. Vandiver, Cornell University.

The purpose of this note is to call attention to an algorithm for the solution of the linear congruence,

$$ax \equiv 1 \pmod{m}, \tag{1}$$

a and m being positive integers prime to each other, which differs from the well-known method depending on the expansion of m/a as a continued fraction. It appears from an examination of Dickson's *History of the Theory of Numbers*, volume II, chapter 2, that the only papers cited there which deal to any extent with the method I shall describe are due to Binet and Sardi. (Explicit references below.)

1. Assume first that $m = p^\alpha$, where p is prime. Consider

$$k_1 m = aq_1' + r_1'$$

k_1 being any integer $\neq 0$, and $|r_1'| < a$. If r_1' is divisible by p, put

$$k_1 m = a(q_1' \mp 1) + r_1' \pm a,$$

where $|r_1' \pm a| < a$. Now $r_1' \pm a$ is prime to p, for if we assume $r_1' \pm a \equiv 0 \pmod{p}$, then $a \equiv 0 \pmod{p}$, contrary to hypothesis. Hence we may write in any case

$$k_1 m = aq_1 + r_1,$$

where $|r_1| < a_1$ and r_1 is prime to m.

Similarly we have

$$k_2 m = r_1 q_2 + r_2,$$

Example 3. Solve $1723x \equiv 1 \pmod{4028}$.

Without setting down the actual figures, we see immediately that the remainder when 4028 is divided by 1723 is close to 1/3 of 1723. Hence we set

$$\begin{aligned} 3 \times 4028 &= 1723 \times 7 + 23, \\ 4028 &= 23 \times 175 + 3, \\ 4028 &= 3 \times 1343 - 1; \end{aligned}$$

and we have

$$1723(7 \cdot 175 \cdot 1343) \equiv 1 \pmod{4028}.$$

In order to compare the amount of computation in this process with that involved in the use of the continued fraction algorithm, we set down the relations required in the latter:

$$\begin{aligned} 4028 &= 1723 \times 2 + 582, \\ 1723 &= 582 \times 3 - 23, \\ 582 &= 23 \times 25 + 7, \\ 23 &= 7 \times 3 + 2, \\ 7 &= 2 \times 3 + 1. \end{aligned}$$

x is the numerator of

$$2 + \frac{1}{3 -}\,\frac{1}{25 +}\,\frac{1}{3 +}\,\frac{1}{3}.$$

In most cases that I have examined, the comparison is about the same as in this example.

RECENT PUBLICATIONS.

EDITED BY D. C. GILLESPIE, Cornell University, to whom communications should be sent.

REVIEWS.

From Determinant to Tensor. By W. F. SHEPPARD. Oxford University Press (American Branch, New York). 1923. 12mo. 127 pages. Price $2.85.

This book might be characterized as a remarkably simple and lucid introduction to the subject of tensors with particular emphasis upon the notation as used by Eddington. The operations upon tensors which are considered are the obvious algebraic ones including that of obtaining the inner product, and the operation of differentiation. No mention is made of the Christoffel symbols and the physical significance of tensors is only suggested. On the completion of the reading of this book, the student will have reached a stage toward which Eddington, for example in his *Mathematical Theory of Relativity,* devotes approximately ten pages. This book is not verbose, does not discuss proofs nor enter into philosophical speculations. The difference in the number of pages may be accounted for in part by the choice for this volume of a small format, and in part by the chapter devoted to an application of tensors to statistics, but chiefly by the detailed character of the explanations and warnings involved in developing

the subject from elementary notions. This may be most easily seen perhaps from the fact that about half of the book is spent in developing the essentials of the theory of determinants and matrices from a point of view slightly more abstract than usual.

As an elementary exposition, this text is highly successful. It contains no exercises and makes no claim to be an embodiment of advanced research or a reference book for theorems or formulas. A fair notion of the content, purpose and general character of the book can be obtained from the author's preface most of which will now be quoted.

"The tensor calculus used in the mathematical treatment of relativity, and concisely explained by Professor A. S. Eddington in his '*Report on the Relativity Theory of Gravitation*,' is, like the various kinds of vector calculus, a system of condensed notation which not only conduces to economy in the writing of symbols, but, what is more important, enables spatial and physical relationships to be grasped as a whole without having to be built up from a number of components which really represent views of different parts of space. Three-dimensional geometry or physics is troublesome enough; the addition of a fourth dimension made the need of a condensed notation imperative.

"Professor Eddington has recently pointed out that the tensor notation and methods can be applied, with happy results, to other and more elementary classes of problems than those for which they were originally devised; and this book is an attempt to put his somewhat compressed exposition into a form in which it will appeal to a larger circle of readers. The book, therefore, is not intended as an introduction to the mathematical theory of relativity—though I hope it may be of some use for that purpose—but rather as an exercise in the elementary application in the methods which, apart from any practical use, possess a special beauty of their own.

"The new notation is not introduced until the fifth chapter. The properties of determinants, which serve as the starting point for the application of the notation, are familiar to the mathematician; but, as I hope the book may be read by some who are not entirely at ease with determinants, I have commenced with four chapters on the elementary theory of the subject. . . .

"What I have called double sets will be recognized by the advanced student as matrices, and many of the propositions will be found to be familiar. But the tensor calculus may fairly claim that, in bringing into closer relation various branches of mathematical study, previously regarded as distinct, it gives them a new life."

ALBERT A. BENNETT.

Géométrie Générale Synthétique Moderne. By ÉMILE BALLY. Paris, Gauthier-Villars et Cie. 1922. Paper, 8vo. viii + 218 pages.

Here is a book from the island of Martinique "dedicated to the friends of geometry . . . written by an amateur for amateurs . . . with a desire to please as well as to convince," available, the author hopes, for readers of intelligence, whether mathematicians or not, who have some taste for abstract reasoning. The author has had to omit extensive references to the work of others, his library being made up of the work of Darboux (*Classe remarquable de courbes et de surfaces*), of Dumant (*Surfaces cubiques*), of Duporcq (*Géométrie moderne*) and the *Encyclopédie des Sciences mathématiques*. As if the lack of tools were not sufficient handicap, the author has had to adjust his work to the higher cost of printing, so that the present volume contains only the preliminary chapter on ordinal arithmetic, chapter I, on the foundations of general geometry, chapters XII, XIII, XIV, on the study of the hexagon and allied configurations, together

with a short appendix containing certain corrections and additions to his work on the synthetic geometry of unicursal curves of the third class and fourth order published in 1920.

The first chapter on ordinal arithmetic contains an exposition of the theory of the number system and of the continuum of points in a line. Just how necessary this chapter may have been for the rest of the work is difficult to judge without examination of the ten missing chapters. The subject is so well treated elsewhere, however, that in the opinion of the reviewer it might have been replaced to advantage by some of the chapters on geometry which have been crowded out.

Chapter I undertakes to give a synthetic development of n-dimensional geometry. It is difficult to read, even for one who is fairly familiar with the subject, on account of the long list of unfamiliar terms. This coining of new terms seems to be characteristic of workers in synthetic geometry. Of the seventy or more new expressions invented by Desargues only the one: "involution" seems to have survived. A glance at the headings of each division of this chapter conveys little to the ordinary mathematician. One must get familiar with the meaning of such terms as "polynarite," "soutiens," "formes axées," "anaxes," "feuillées," "iso-similaire," "punctidualilinéaire," "dualisimilaire." The chief result of the first division is: "Between the polynarities m, and n, of two elements and the polynarities i and j of their intersection and junction exists always the fundamental relation $m + n = i + j$." Translated into ordinary language, this is the familiar theorem about linear spaces S_r: "If S_{r_1} and S_{r_2} intersect in an S_a they lie in an $S_{r_1+r_2-a}$.

Again one is at a loss to judge the necessity of chapter I for the remainder of the book without some knowledge of the contents of the missing chapters. Chapters XII, XIII, and XIV which have to do with configurations in the plane do not need so formidable an introduction. Chapter XII furnishes a detailed study of the various points and lines connected with a hexagon. An elaborate notation is developed to indicate the sets of lines and points. The various kinds of grouping of points and lines are given names and enumerated. All this without supposing the six points to be related in any way. In the final chapter, the hexagon is made to satisfy certain conditions equivalent to the supposition that its vertices lie on a conic. The usual theorems of alignment and concurrence which hold for Pascal lines, Steiner points, Kirkman points; Salmon-Cayley lines, etc., are given (in unfamiliar language) together with theorems affecting certain other elements, not so well known, which the author names for himself, "the points and lines of Bally."

The author has undertaken a useful piece of work in bringing together the scattered results connected with the configuration of six points on a conic and if the reader were not confronted at each step by "textures" and "skeletons" and other strange things it would be all that M. Bally has hoped for it.

D. N. LEHMER.

Les Applications Élémentaires des Fonctions Hyperboliques à la Science de l'Ingénieur Électricien. By A. E. KENNELLY. Paris, Gauthier-Villars, 1922. 153 pp.

This book covers the scope and purport of a series of public lectures delivered in France by the author while serving as the first exchange professor representing a group of American Universities in certain French Universities. The subject matter is practically the same as a previous work in English by the same author —viz: "*The Application of Hyperbolic Functions to Electrical Engineering Problems*" published in 1912 by the University of London Press. It is unfortunate that no reference is made in the preface to this earlier publication. This earlier work is fuller and more complete and would be preferred by American students.

Professor Kennelly was among the first to advocate strongly the wider use of the complex quantity in electrical engineering problems. In considering problems involving long electrical lines where it is necessary to take into account the uniform linear distribution of the resistance, leakage conductance, inductance and capacitance the rigorous solution leads to hyperbolic functions.

The author bases his treatment upon the fact that the theories of continuous currents and alternating currents are essentially the same; all continuous current formulas holding for alternating current circuits when complex numbers are substituted for real numbers. The practical application of hyperbolic functions to engineering problems was handicapped by the lack of suitable tables until this need was supplied by Professor Kennelly with "*Tables of Complex Hyperbolic and Circular Functions*" published in 1914 by the Harvard University Press. Reference to this work should also have been made in the preface.

In developing the idea of hyperbolic functions the author defines the magnitude of a hyperbolic angle, first by the ratio of the hyperbolic arc distance described to the length of the radius vector, and second by the area of the hyperbolic sector; and then traces the relationship between the circular and hyperbolic functions. The functions of a complex angle are derived from a mixed circle and hyperbola diagram. For the mathematician, the main interest in the book would be as a source for problems showing the application of elementary hyperbolic functions to certain problems in electrical engineering.

JOSEPH H. CANNON (University of Michigan).

Mathematical Theory of Finance. By T. M. PUTNAM. New York, John Wiley and Sons, 1923. 8vo. 10 + 117 pages. Price $1.75, postpaid.

It is stated in the preface: "The scope and method of the book have been designed for a three-hour course for one semester, such as is prescribed in the College of Commerce in the University of California."

For a short course the book is an excellent text. The author has used good judgment both in including important formulas and in excluding those of minor importance. In particular, the cumbrous and unnecessary formulas for the time n are conspicuously absent. But when, as on page 18, a problem is given to find n, the problem is easily solved; and moreover, a proper interpretation is given to the fraction which appears.

The real significance of formulas is pointed out by frequent verbal interpretations; and the interpretations given are very good. Perhaps a more simple interpretation of the formula for capitalized cost than that given on page 29 is obtained by noting that the interest on a unit of money for k years is $(1 + i)^k - 1$; and thus the reciprocal of this expression will yield one unit of money every k years.

Formulas for nominal discount are not developed; but on page 12 a problem involving nominal discount is solved by use of nominal interest. Nominal discount is perhaps more important and more easily understood than the force of interest, treated on pages 8 and 9. But—granting that the force of interest should appear even in a brief course—it is to be noted that the author defines the force of interest clearly as the limiting value of j, "for a given effective rate, i" $\cdots$ "as m increases without bound." Much confusion arises unless it is recognized that the i is given, *i.e.*, fixed.

The chapter headings are: interest, annuities, amortization—sinking funds, bonds, probability, life annuities, elementary principles of life insurance.

In a brief chapter on probability, designed for practical purposes, it is unnecessary to draw a sharp line between abstract and empirical probability; and this is not done. Unfortunately, the formal definition for probability on page 62 neglects mention of equally likely cases. However, in the explanation that precedes and follows, this restriction is made clear.

Eleven tables are given, including tables for the usual monetary functions, to seven places of decimals, with sixteen different rates of interest, also the American Experience Mortality Table, and Commutation Columns based thereon at $3\frac{1}{2}\%$ interest for D_x, M_x, N_x, the N_x summing from age x on.

The book has been well printed, and is pleasing to the eye. Very few misprints were noticed—the c in the last line on page 16, and the $p + p$ in line 11, page 69, would trouble no one.

In conclusion, the text throughout is characterized by an excellent choice of material and by lucid explanations.

E. L. Dodd.

Chance and Error. By Marsh Hopkins. London, Kegan Paul, Trench & Co., 1923. 223 pages. Price $3.00.

As is stated in the preface "This little book shows that the vagaries of chance are the result of the interference of yes and no. . . . It was written with the object of extending the usefulness of this very important subject to those whose knowledge of mathematics is limited."

The book is made up of five hundred and fifty examples and their solutions. Some of the chapter headings are: games whose expectation is zero, direct observations, indirect observations, statistics, target practice, errors in three dimensions, monte carlo, variable chances.

Exposition of any well-directed theory is lacking. Each example is stated, briefly explained, and solved. Most explanations are satisfactorily clear to one who has a previous knowledge of least squares and statistics; for one not so schooled, I doubt if the book will be satisfactory, certainly not inspiring.

New names are given to old concepts. For example, we find on page 44, "The *median error* is the error that is as likely to be exceeded as not." I see no necessity for a new name for this concept, the most approved expression for which is the *probable error*. A term which the author uses frequently but never carefully defines is the *average error*.

The basis for the author's logic is found at the beginning of chapter one. Some of the definitions are very obscure. For example, "An element of any thing is very small when its magnitude may be disregarded." And "Anything that tends to occur and cannot occur is imaginary."

The book should prove of some value to the "home" student in Probability. Since Whitworth's "*Choice and Chance*" is now out of print, this book may in a measure fill the void.

C. H. RICHARDSON.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Isis, volume 5, no. 13, October, 1922: "Michigan mathematical papyrus no. 621 (with facsimile)" by L. C. Karpinski, 20–25; "Notes on the knowledge of latitudes and longitudes in the Middle Ages" by J. K. Wright, 75–98; "Quadripartitum Ricardi Walynforde de sinibus demonstratis" by J. D. Bond, 99–115; "Mathematical signs of equality" by F. Cajori, 116–125. Volume 5, no. 14, May, 1923: "Richard Wallingford's Quadripartitum" (English translation) by J. D. Bond, 339–363; "Beiträge zur arabischen Trigonometrie" by C. Schoy, 364–399; "Ein wichtiger Satz über die Ellipse des Fagnano und seine Ergänzung" by K. Bopp, 400–402; "Entwickelungslinien in der Geometrie" by K. Bopp, 406–408.

Journal of the Washington Academy of Science, volume 12, no. 19, November 19, 1922: "Values of sine and cosine θ to 33 places of decimals for various values of θ expressed in sexagesimal seconds" by C. E. Van Orstrand and M. A. Shoultes, 424–436. Volume 13, no. 8, April 19, 1923: "A remarkable formula for prime numbers" by Paul R. Heyl, 150–151.

Science Progress, volume 48, no. 69, July, 1923: "Recent advances in science: Mathematics" by F. P. White, 1–5; "Indeterminate equations of the third degree" by L. J. Mordell, 39–55.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I. [*1923*, 39.]

The printed program of the Mathematics Club of Brown University for 1923–1924 makes the following announcements:

November 2, 1923: "William Rowan Hamilton" by Dempster Hobron '24; "Some properties of the cycloid" by Charlotte Perry '25; "Magic squares" by Frederick Kilbourne, Jr. '26.

December 14: "The map coloring problem" by Mildred Carlen '24; "The history of trigonometry" by Grace Hanson '25.

January 18, 1924: "The five Platonic bodies and higher polyhedra derived from them" by Albert Wheeler, North High School, Worcester, Mass.

February 29: "Blaise Pascal" by Avis Sugden '26; "Cover the red spot" by Frances Wright, Gr.; "Mathematico-chess recreations" by Clarence Bennett, Gr.
March 28: "Repeating decimals" by Professor Mary Curtis Graustein of Wellesley College.
May 2: "The history of mathematics at Brown University" by Elizabeth Stafford, Gr.; "The integraph" by George Sauté '24; "Curiosities in numbers" by Frederick Wood '26.
May Picnic.

THE MATHEMATICS CLUB OF COOPER UNION, New York City. [*1923*, 40.]

The following officers served for the year 1922–1923: President, Peter Kosting '25; vice-president, Barnett Emmerich '23; secretary, Fred Miller '26; faculty advisers, Professor H. W. Reddick and H. W. Barcus, instructor. A membership fee of twenty-five cents for the year provided a fund for a prize to be awarded at commencement to the member of the first-year class having the highest average in mathematics. This prize, a polyphase duplex slide-rule, was won by Fred Miller '26.

Meetings were held on alternate Mondays as follows:
October 23, 1922: "Explanation of the slide-rule" by H. H. Barcus.
November 6: "History of our number system" by Peter Kosting '25.
November 20: "Mathematical fallacies" by Fred Buhrendorff '25.
December 4: "Theorems on collinear points" by Fred Miller '26.
December 18: "Flatland" by David Samson '24.
January 8, 1923: "Hyperspace" by Professor Reddick.
January 22: "Inscribing a cylinder of maximum volume in a cone" by Alexander Gotsdanker '24.
February 5: "Construction of regular polygons" by Fred Van der Voort '26.
February 19: "Demonstration of calculating machines" by Mr. Coxhead, of the Mercedes Calculating Machine Co.
March 5: "Sailing faster than the wind" by Fred Buhrendorff '25; "Solving the right triangle without tables" by Barnett Emmerich '25.
March 19: "Mathematical anecdotes" by Jacob Boorstein '25.
April 2: "An original approximate method of trisecting an angle" by William J. Pickett, instructor.
April 16: "Geometric proofs of the law of tangents" by Isidore Fankucken.

(Report by Mr. Miller.)

THE MATHEMATICS CLUB OF HUNTER COLLEGE, New York City. [*1922*, 354.]

The officers of the Mathematics Club of Hunter College for the year 1922–1923 were: President, Sara Malkin '23; vice-president, Isabel Graves '23; secretary, Esther Alfert '24; treasurer, Bessie Schoenfeld '23; faculty adviser, Miss H. Kunte.

The following topics were presented at the meetings:
The three problems of antiquity—"Duplication of the cube—Cissoid of Diocles" by Lillian Lesser '24; "Trisection of the angle—Conchoid" by Bessie Schoenfeld '23; "Squaring the circle—Quadratrix" by Miriam Jacobi '23.
"Geometry of the compass" by Sylvia Rosenstein '26.
"The construction of magic squares" by Isabel Graves '23.
"Japanese and Chinese mathematics" by Sara Malkin '23.
"How we reckon time" by Harriet Griffin '25.
"Linkages and their applications" by Eugenie Schein '24.
"Mathematics from Ahmes to the Renaissance" by Professor Emma M. Requa.

(Report by Miss Alfert.)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF NEBRASKA, Lincoln, Neb. [*1920*, 320.]

The papers read at meetings in 1922–1923 were as follows:
November 8, 1922: "Some card tricks and the explanation" by Dean C. Engberg.
December 13: "History of the development of mathematics" by Dean A. L. Candy.

January 10, 1923: "Life of Archimedes" by Dean Candy; "What do the letters of algebra stand for?" by Daisy Portenier; "Short cuts in arithmetic" by R. G. Sturm.
February 14: "The Youth Movement in Holland" by Pit Roest.
March 14: "Paper folding" by Gerold Almy; "Archimedes—his method" by Dean Candy.
April 11, 1923: "Einstein's theory" by E. Z. Stowell; "Development of logarithms" by Dean Candy.
May 9: "The story of the ten digits (illustrated by lantern slides)" by Dean Candy.

(Report by Vivian Hanson, Secretary.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3062. Proposed by HARRY LANGMAN, New York City.

Let A be the vertex of a hyperbola. Draw APQ, any line through A cutting the curve again in P and either asymptote in Q. Draw PR parallel to the other asymptote, cutting the first in R. Show that the length RQ is constant and give a simple mechanical construction of the curve based on this property.

3063. Proposed by H. A. BENDER, University of Illinois.

Show that

$$\sum_{r=1}^{r=b}(p^a-1)(p^a-p)\cdots(p^a-p^{r-1})\frac{(p^b-1)(p^{b-1}-1)\cdots(p^{b-r+1}-1)}{(p^r-1)(p^{r-1}-1)\cdots p-1}=p^{ab}-1,\quad(a\geqq b)$$

is an identity in p.

3064. Proposed by BURRELL MORGAN, Panther, W. Va.

One of the parallel sides of a trapezoid containing $17\frac{1}{2}$ acres is 80 rods and the two non-parallel sides are 22 and 48 rods respectively; what is the length of the remaining side?

3065. Proposed by A. S. WIENER, Cornell University.

Prove the following identity:

$$\begin{vmatrix} a^{m+1}+c^m(a+b), & b(bc^{m-1}+a^m), & b(a^{m-1}c+a^m+c^m) \\ c(ab^{m-1}+a^m+b^m), & b^{m+1}+a^m(b+c), & c(a^{m-1}c+b^m) \\ a(ab^{m-1}+c^m), & a(bc^{m-1}+b^m+c^m), & c^{m+1}+b^m(c+a) \end{vmatrix}$$

$$\times\begin{vmatrix} a^{n+1}+c^n(a+b), & b(bc^{n-1}+a^n), & b(a^{n-1}c+a^n+c^n) \\ c(ab^{n-1}+a^n+b^n), & b^{n+1}+a^n(b+c), & c(a^{n-1}c+b^n) \\ a(ab^{n-1}+c^n), & a(bc^{n-1}+b^n+c^n), & c^{n+1}+b^n(c+a) \end{vmatrix}$$

$$=8\begin{vmatrix} a^{m+n+2}+c^{m+n}(a^2+b^2), & b^2(b^2c^{m+n-2}+a^{m+n}), & b^2(a^{m+n-2}c^2+a^{m+n}+c^{m+n}) \\ c^2(a^2b^{m+n-2}+a^{m+n}+b^{m+n}), & b^{m+n+2}+a^{m+n}(b^2+c^2), & c^2(a^{m+n-2}c^2+b^{m+n}) \\ a^2(a^2b^{m+n-2}+c^{m+n}), & a^2(b^2c^{m+n-2}+b^{m+n}+c^{m+n}), & c^{m+n+2}+b^{m+n}(c^2+a^2) \end{vmatrix}.$$

3066. Proposed by B. F. FINKEL, Drury College.

What is the amount of work done in pulling a spool of thread, weight w, up an inclined plane whose length is l and inclination α, the spool to be pulled up the plane by taking hold of the outer end of the thread and allowing the thread to unwind? We assume that we may neglect the weight of the thread unwound, that friction is large enough to prevent slipping, and that the axis of the spool remains horizontal.

3067. Proposed by J. H. MURPHY, Pittsburgh, Pa.

On the base of a right triangle, whose altitude, a, is greater than the base, b, is constructed a triangle whose vertex angle is α. What are the lengths of the two variable sides of this triangle when that part of its area outside of the right triangle is a maximum?

3068. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that

$$\sum_{n=1}^{n=\infty} u_n/n! = (e-1)^2/2$$

if $u_1 = 0$, $u_n = 2u_{n-1} + 1$, $(n = 2, 3, 4, \cdots)$.

3069. Proposed by J. ROSENBAUM, Milford, Conn.

Given the mid-points of the sides of a quadrilateral inscribed in a circle and the radius of the circle, to construct the quadrilateral.

SOLUTIONS.

2998 [1922, 420]. Proposed by F. M. GARNETT, Augusta, Georgia.

A cube has removed from it a right pyramid whose base is a face of the cube and whose altitude is the altitude of the cube. How far from the base of the cube must a plane be passed parallel to the removed face so as to divide the remaining volume of the cube into two equal parts?

SOLUTION BY THEODORE BENNETT, University of Illinois.

Let an edge of the cube be a. The volume of the pyramid is $\frac{1}{3}a^3$, and the remaining volume is $\frac{2}{3}a^3$. Cut the remaining solid by a plane parallel to the base and a distance x above it. The area of the section is clearly

$$a^2 - (a-x)^2 = 2ax - x^2.$$

Hence the volume which we have cut off is

$$\int_0^x (2ax - x^2)dx = ax^2 - \frac{x^3}{3}\,.$$

We wish to determine x so that

$$ax^2 - \frac{x^3}{3} = \frac{a^3}{3}\,.$$

The solutions of this equation are

$$a(1 - 2\sin 10°), \qquad a(1 - 2\sin 50°), \qquad a(1 + 2\sin 70°).$$

The first of these represents the solution of our problem, being approximately $.65a$.

NOTE BY THE EDITORS:—The same result is obtained if the cube is replaced by any cylinder or prism of altitude a and area of base A. It is simpler to compute the volume of the other portion of the figure.] If y is the distance of the plane from the vertex, the volume of this portion is the difference between that of a cylinder of base A and of a cone of base $A(y/a)^2$, hence

$$Ay - \frac{Ay^3}{3a^2} = \frac{Aa}{3} \qquad \text{or} \qquad y^3 - 3a^2y + a^3 = 0.$$

The roots of this equation are $2a \sin 10°$, $2a \sin 50°$, $-2a \sin 70°$, as we see at once by comparing it with the identity $4\sin^3\theta - 3\sin\theta + \sin 3\theta = 0$. The first root is the solution of the problem.

Also solved by H. N. CARLETON, W. F. DANTZSCHER, PHILIP FITCH, MICHAEL GOLDBERG, H. HALPERIN, H. A. ROBINSON, and W. W. WEBER.

2999 [1923, 41]. Proposed by M. B. PORTER, University of Texas.

Given n positive numbers: $a_1, a_2, a_3, \cdots, a_n$, then

$$\Sigma a_i \Sigma a_j^{-1} > n^2 \qquad (i, j = 1, 2, 3 \cdots n)$$

unless $a_i = a_j$, for which case equality occurs; show by passing to limits that, if $\varphi(x) > 0$ and is continuous, $\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)}$ takes on its minimum for $\varphi(x) =$ constant over the interval a to b.

Partial Solution by W. M. Whyburn, University of Texas.

The inequality may be proved by observing that

$$\Sigma a_i \Sigma a_j^{-1} = n + \Sigma \left(\frac{a_i}{a_j} + \frac{a_j}{a_i} \right), \tag{1}$$

where in the summation on the right $i \neq j$ and there are $n(n-1)/2$ terms. If $a_i \neq a_j$, then $(a_i - a_j)^2 = a_i^2 + a_j^2 - 2a_ia_j > 0$; and, since the a's are greater than zero, we have $a_i/a_j + a_j/a_i > 2$. Hence, if not all the a's are equal, we deduce from (1)

$$\Sigma a_i \Sigma a_j^{-1} > n^2, \tag{2}$$

but if all the a's are equal, the two sides of the above are equal.

If the interval from a to b is divided into n equal sub-intervals by $x_1, x_2, x_3, \cdots, x_n = b$; and, if we set $a_i = \varphi(x_i)$, then

$$\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)} = \lim_{n \doteq \infty} \frac{(b-a)^2}{n^2} \Sigma a_i \Sigma a_j^{-1}, \tag{3}$$

since $\varphi(x)$ is continuous and does not vanish within or at the ends of the interval considered. Hence by (2)

$$\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)} \geqq (b-a)^2. \tag{4}$$

If $\varphi(x)$ is a constant, *i.e.*, all the a's are equal, the equality sign holds. Thus the minimum of the left side of (4) is $(b-a)^2$ and this minimum is reached when $\varphi(x)$ is any constant not zero.

Note on the above Solution by Otto Dunkel, Washington University.

The proof above shows that the minimum of the left side of (4) is reached when φ is a constant, but it does not show that this is true *only* in that case. For we cannot conclude by the above reasoning that, if φ is not a constant, the inequality above holds true in (4). To conclude this we should show that, when the a's are not all equal, the left side of (2) is greater than pn^2, where p is a fixed number greater than unity. The proof may be completed as follows: The right side of (1) may be written

$$n^2 + \Sigma \frac{(a_i - a_j)^2}{a_ia_j}.$$

Hence

$$\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)} = (b-a)^2 + \int_a^b \int_a^b \frac{[\varphi(x) - \varphi(y)]^2}{\varphi(x)\varphi(y)} dxdy. \tag{6}$$

If $\varphi(x)$ is not a constant in the interval considered, then for some pair of values, α and β $(a < \alpha < b,\ a < \beta < b)$ $\varphi(\alpha) \neq \varphi(\beta)$. Since $\varphi(x)$ is continuous we can determine in the region $a \leqq x \leqq b$, $a \leqq y \leqq b$, a small region containing (α, β) throughout which the integrand on the right in (6) is greater than some fixed number $m > 0$. If the area of this region is A, then the left side of (6) is greater than $(b-a)^2 + Am$. It then follows that the left side of (6) is greater than $(b-a)^2$ if $\varphi(x)$ is not a constant and equal to $(b-a)^2$ if $\varphi(x)$ is a constant.

The important rôle of the double integral in (6) in the above proof suggests the following simplification:

Consider the integral

$$\begin{aligned} \int_a^b \int_a^b \frac{[\varphi(x) - \varphi(y)]^2}{\varphi(x)\varphi(y)} dxdy &= \int_a^b \int_a^b \frac{\varphi(x)}{\varphi(y)} dxdy + \int_a^b \frac{\varphi(y)}{\varphi(x)} dxdy - 2(b-a)^2, \\ &= 2\int_a^b \int_a^b \frac{\varphi(x)}{\varphi(y)} dxdy - 2(b-a)^2. \end{aligned} \tag{7}$$

We suppose that the same conditions as before are imposed upon $\varphi(x)$. The second form of the right side in (7) follows from an interchange of the letters x and y. By the process of evaluating

a double integral, the last result in (7) may be written

$$2\int_a^b \varphi(x)dx\int_a^b \frac{dx}{\varphi(x)} - 2(b-a)^2. \tag{8}$$

By the proof previously given the left side of (7) is greater than a fixed number which is not zero, if $\varphi(x)$ is not constant: it is obviously zero if $\varphi(x)$ is a constant. Hence (8) is subject to the same conditions, and we have

$$\int_a^b \varphi(x)dx\int_a^b \frac{dx}{\varphi(x)} \geqq (b-a)^2,$$

where the equality sign holds *only* when $\varphi(x)$ is a constant.

Also solved by D. F. BARROW, H. HALPERIN, and A. PELLETIER.

3001 [1923, 41]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

In the plane of a given circle a second circle with a given radius is drawn so that the radical axis of the two circles passes through a given fixed point. Find the locus of the center of the second circle.

SOLUTION BY MABEL M. YOUNG, Wellesley College.

Take the center of the given circle with radius c as origin; (x', y') the coördinates of the fixed point; and (h, k) the coördinates of the center of the second circle of radius r. Then the condition that the radical axis of the two circles $x^2 + y^2 = c^2$ and $(x - h)^2 + (y - k)^2 = r^2$ shall pass through the fixed point is $2x'h - h^2 + 2ky' - k^2 = c^2 - r^2$. But this is the equation of a circle with center (x', y') and radius $\sqrt{r^2 - c^2 + x'^2 + y'^2}$.

This circle, which is the required locus, may always be constructed for all values of r, c, (x', y'), save that when $c > r$, the fixed point may not lie nearer the center of the given circle than $\sqrt{c^2 - r^2}$.

Also solved by S. E. FIELD, L. O. GHORMLEY, MICHAEL GOLDBERG, A. M. HARDING, WILLIAM HOOVER, H. HALPERIN, R. M. MATHEWS, A. PELLETIER, J. B. REYNOLDS, and A. V. RICHARDSON.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

At the fall meeting of the Mathematics Section of the Virginia State Teachers Association, in Richmond, on November 28th, Professor T. McN. SIMPSON, Jr., of Randolph-Macon College, read a paper on the "Social significance of mathematics"; Professor GILLIE A. LAREW, of Randolph-Macon Woman's College, discussed minimum requirements in college mathematics from the standpoint of the actual practice in Virginia and from that of an ideal curriculum; and Dr. H. A. CONVERSE, of the Harrisonburg State Normal School, spoke on the relation of mathematics to other sciences. For the year 1923–1924, Dr. Converse is president and Professor Larew secretary of the Section.

Professor A. E. WHITE, of the Kansas State Agricultural College, spoke before the mathematical round table of the Kansas State Teachers Association at Wichita on October 17th on "Examinations in high school mathematics." On the following day, Mr. W. C. JANES read a paper on "Some geometrical tests" before the corresponding body at Topeka.

At a recent meeting of the Nebraska Academy of Science, Professor R. M. McDill of Hastings College read a paper before the Mathematics Section on the topic, "Are disciplinary values a legitimate aim to be stressed in the teaching of mathematics?"

Dean J. N. Hart, of the University of Maine, who has been head of the department of mathematics for the past thirty years, has been granted a year's leave of absence by the trustees.

Professor W. F. Osgood, of Harvard University, has been elected a member of the Leopoldinisch-Carolinisch Deutsche Akademie der Naturforscher.

Miss Ethel B. Callahan, of the University of Wisconsin, has been appointed head of the department of mathematics at Cedar Crest College, Allentown, Pa.

Mr. W. J. Wagner has been appointed instructor of mathematics at Allegheny College.

Mr. C. C. Wagner, of Allegheny College, has been appointed assistant professor of mathematics at Pennsylvania State College, and is now acting head of the department.

Mrs. Ethel S. Kershner has been appointed assistant professor of mathematics at Lynchburg College.

At the University of North Carolina, Mr. S. B. Smithey and Mr. L. M. Sahag have been appointed instructors of mathematics.

At Elon College (North Carolina), Miss Louise Savage has been appointed instructor of mathematics. Professor T. C. Amick has been elected president of the department of higher education of the North Carolina Education Association for the coming year.

At Queen's College, Charlotte, North Carolina, Miss Edna Berkele has resigned as head of the department of mathematics, and Miss Olive M. Jones has been appointed to that position.

At Chicora College, Columbia, South Carolina, Mr. W. H. Milner, of the George Peabody College for Teachers, has succeeded Miss Julia B. Prosser as head of the department of mathematics.

Miss Annie R. Alford, of the University of Oklahoma, has been appointed instructor of mathematics at Coker College, Hartsville, South Carolina.

At the Florida State College for Women, Assistant Professor Olga Larson has been granted leave of absence, and is at the University of Missouri. Miss Larsen's place is being filled by Miss Myra B. Kearney.

Mr. J. A. Hyden, after a year's leave of absence, has been made assistant in mathematics and professor of physics at Maryville College (Tenn.).

Mr. Garrett Van der Borgh is now a member of the mathematical staff at Hope College, Mich.

Miss Mary Reichelderfer, of the University of Chicago, has been instructor of mathematics at St. Xavier College, Chicago, since September 1922.

At Campion College, Prairie du Chien, Wisconsin, Professor J. H. May has succeeded Professor G. J. Brunner as head of the department of mathematics.

Professor C. W. Strom, of Luther College, Decorah, Iowa, has been granted leave of absence for the current year and is studying at the University of Iowa.

At Fairmount College, Wichita, Kansas, Professor A. J. HOARE has resumed his work as head of the department, having recovered from a serious impairment of vision which he suffered four years ago. Professor H. G. TITT has been appointed professor of mathematics at Huron College, South Dakota. Mr. R. W. ELLIOTT has been appointed business manager and instructor of mathematics.

Mr. C. S. WHITNEY has been appointed professor of mathematics and head of the department at the State School of Mines, Miami, Oklahoma.

At the Panhandle Agricultural and Mechanical College, Goodwell, Oklahoma, a new mathematics department has been appointed, consisting of Mr. F. C. LEMON, of the University of Michigan, Mr. T. J. PRUET, of the N. W. State Teachers College, and Mr. HENRY HOUGHTON, of the University of Oklahoma.

Miss LIDA B. MAY, of the University of Texas, has been appointed to the chair of mathematics at the Kidd-Key College and Conservatory, Sherman, Texas.

Mr. R. Z. NEWSOM, head of the mathematics department at Washington College, Tenn., has succeeded Miss O. KATE CANNON as head of the department of mathematics at Rusk College, Rusk, Texas.

At Simmons College, Abilene, Texas, Professor A. E. CHANDLER has been made bursar, and Associate Professor J. E. BURNAM has been promoted to a full professorship.

Professor EMMA K. WHITON, of the University of Redlands, Redlands, Calif., has been appointed professor of mathematics at Mills College, Oakland, Calif.

NOTABLE PRIZE AWARD.

The committee on the award of the Cincinnati prize of the American Association for the Advancement of Science has adjudged the prize to Dr. L. E. DICKSON, professor of mathematics in the University of Chicago. The contributions for which the prize was awarded were presented before a joint session of Section A of the A. A. A. S., the American Mathematical Society, and the Mathematical Association of America on Friday afternoon, December 28, and before a session of the American Mathematical Society on Saturday afternoon, December 29. The paper presented at the Friday session was entitled "Algebras and their Arithmetics"; the papers presented at the Saturday session were entitled "On the Theory of Numbers and Generalized Quaternions," and "Quadratic Fields in which Factorization is Always Unique." Distinguished mathematicians from all parts of the country, present at these meetings, were unanimous in the opinion that the work presented by Professor Dickson constituted one of those outstanding contributions to the development of science which are made only at rare intervals. The committee on the award of the prize consisted of the following members: Dr. N. M. FENNEMAN, Professor of Geology, University of Cincinnati, Chairman; Dr. HENRY CREW, Professor of Physics, Northwestern University; Dr. C. H. PARKER, Professor of Zoölogy, Harvard University; Dr. E. W. WASHBURN, Professor of Chemistry, University of Illinois; Dr. G. T. MOORE, Director Missouri Botanical Gardens, St. Louis, Mo.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of .... Dollars,

to be known as the ....

.... Fund, and to be used

or[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witnesses: Signature ....

....

....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to D. C. GILLESPIE, Cayuga Heights, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924
NUMBER 4, APRIL

PUBLISHED BY THE ASSOCIATION
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others

Bills Receivable.	
1923 indiv. dues unpaid	$120.00
1923 instit. dues unpaid	21.00
Advertising	100.00
Interest U. S. Treasury Note	11.00
	$252.00

Bills Payable.	
(Either paid in December or estimated)	
Publisher's bills (Nov., Dec., 1923)	$1,525.00
Manager's office	40.00
Editor-in-chief's office	150.00
Other editors' postage	30.00
Committee on Membership	100.00
Secretary-Treasurer's office	200.00
Annals subvention	250.00
Initiation fees due to sections	300.00
Printing annual ballots, programs, etc.	175.00
Life Membership Fund	139.00
	$2,909.00

If to the balance on 1923 business shown in this report, $5.066.41, there be added the bills receivable, $252.00, and there be subtracted the estimated amount of bills payable, $2,909.00, there results an estimated final balance on 1923 business of approximately $2,400.00. The corresponding estimated final balance one year ago on 1922 business was $1,800.00. Thus, aside from the $400.00 which Dr. Chace has generously given for the second year, there is approximately a gain of $200.00 on the year's business. This narrow margin of profit has been gained through the undesirable expedient of combining two months' issues of the Monthly into a single number at various times in 1922 and 1923; this was adopted in order to bring up to date the tardy printing schedule and it brought with it some financial advantage which, while desirable in itself, cannot be repeated. The experience of other scientific journals indicates that printing rates at present are quite unstable and your officers are concerned that the Association's financial basis may be more secure than is as yet the case and that with increasing income further important lines of activity may be undertaken.

W. D. Cairns, *Secretary-Treasurer.*

RESEARCH FELLOWSHIPS IN MATHEMATICS.

The National Research Council in 1919 was entrusted by the Rockefeller Foundation with the expenditure of an appropriation of half a million dollars within a period of six years for promoting fundamental research in physics and chemistry, primarily in educational institutions in the United States. Recently this grant has been renewed for a second period of five years, and has been increased to $625,000, to take effect July 1, 1925. At the same time these fellowships were extended to include mathematics, this provision to take effect at once, so that applications for mathematical fellowships will be considered at the April meeting of the Fellowship Board. This Board, hereafter, is to contain two mathematicians.

The primary feature of the plan is the initiation and maintenance of a system of National Research Fellowships which are to be awarded by the National

Research Council to persons who have demonstrated a high order of ability in research, thus enabling them to conduct investigations at educational institutions which make adequate provision for effective prosecution of research in physics, chemistry and mathematics. The results which are expected to follow are: (*a*) the benefit of a scientific career opened to a larger number of able investigators, (*b*) the opportunity for more thorough training of young men and women in research, (*c*) resultant extension of knowledge of the fundamental principles of physics, chemistry and mathematics, and (*d*) the creation of more favorable conditions for research in the educational institutions of this country.

These Fellowships are administered by a Board representative of the different sciences involved and are awarded, after careful consideration, to such as may be found entitled to them. The fundamental requirements of an applicant include a Ph.D. degree or its equivalent, evidence from the past of high ability and good character, promise for the future backed by the opinions of men closely associated with him in scientific work, and finally the general nature of the problem submitted for investigation.

Applicants are permitted to select, with the approval of the Board, the institutions at which they may choose to work. Fellowships are awarded for a period of twelve months with a minimum stipend of $1,800 for men without dependents, and $2,300 for men with dependents. Fellows may be re-appointed for a second or a third year with an advance in stipend as determined by the Board in each particular case. Fellows are expected to devote their entire time to research except that during the college year they may, at their option, give a portion of their time—in general not more than one-fifth—to teaching deemed to be of educational value to themselves, or to attendance on advanced courses of study. The results of their investigation must be made available to the public without restriction.

THE PRESENT STATE OF THE DIFFERENCE CALCULUS AND THE PROSPECT FOR THE FUTURE.[1]

By R. D. CARMICHAEL, University of Illinois.

1. Remarks concerning the present state of the difference calculus. It is probably correct to say that the modern development of the difference calculus began with a memoir by Poincaré published in 1885 in the seventh volume of *The American Journal of Mathematics*. He considered a difference equation of the form

$$f(n+k) + p_{k-1}(n)f(n+k-1) + \cdots + p_1(n)f(n+1) + p_0(n)f(n) = 0,$$

where the independent variable n runs over the set of positive integers and where the coefficients $p_i(n)$ are functions of n such that

$$\lim_{n=\infty} p_i(n) = A_i, \qquad i = 1, 2, \cdots, n;\ A_0 \neq 0.$$

[1] Retiring presidential address before The Mathematical Association of America. Cincinnati, Dec. 27, 1923.

In case the characteristic algebraic equation

$$\rho^k + A_{k-1}\rho^{k-1} + \cdots + A_1\rho + A_0 = 0$$

has k roots ρ_i no two of which are equal in absolute value, then, if $f(n)$ denotes any particular solution of the given difference equation, the quotient $f(n+1)/f(n)$ approaches a limit as n approaches infinity and this limit is one of the roots ρ_i of the characteristic algebraic equation. This is an important theorem established by Poincaré. Investigations more or less closely related to this pathbreaking memoir by Poincaré have been carried out by Pincherle, Ford, Nörlund, Perron, Horn, Erb, Van Vleck, and others.

These researches have yielded important applications in the study of convergent series and continued fractions and in the integration of linear differential equations. In these researches, so far as difference equations are involved, one is concerned principally with the asymptotic character of the solutions for large *integral* values of the independent variable. A much more important and characteristic problem of the difference calculus first came into view in the study of the analytic properties of the solutions for general real or complex values of the independent variable; this takes its most interesting form in the case when functions of a complex variable are treated. This type of research was initiated almost simultaneously by Carmichael, Birkhoff, Galbrun, and Nörlund about the end of the first decade of our century. The investigations carried out in this connection have brought into notice a large and important class of functions having properties in many respects analogous to those of the classic gamma function, the latter indeed affording an example of this class of functions. It is a historical fact of some interest that a function of such importance as the gamma function was frequently employed in analysis for several generations before it came to be recognized as but the simplest instance of a large class of useful functions lying rather close to hand but as yet undiscovered.

The genesis of scientific investigation is often a matter of considerable interest. Perhaps I may be pardoned for saying a word here about the origin of the American contributions to the theory of the difference calculus, especially since my dissertation holds a place in the early part of this history. The first impulse in America, and the first in the world so far as I know, toward the development of the difference calculus from the point of view of general function theory, was given at the University of Wisconsin in 1909 in the lectures of E. B. Van Vleck. This fact has been put on record by G. D. Birkhoff [1] in the following words: "To the best of my knowledge, the importance of the functional standpoint in the field of difference equations was emphasized first by Van Vleck in an inspiring series of lectures given at the University of Wisconsin in the spring of 1909, in which he conjectured the existence of sets of solutions analytic on either the left or the right side of the complex plane."

In 1907 Birkhoff received his degree at the University of Chicago. In the two years 1907–1909 he was an instructor at the University of Wisconsin; he

[1] *Transactions of the American Mathematical Society*, vol. 12 (1911), p. 243.

took advantage of the opportunity afforded him to attend the lectures given there by Van Vleck and in this way became thoroughly conversant with the point of view upon which Van Vleck was insisting relative to the most promising direction in which to undertake the development of the difference calculus. In the fall of 1909 he went to Princeton University as a preceptor. At the same time I became a student at Princeton University, attending the lectures of Birkhoff on differential equations. In discussing with him possible subjects for my dissertation I learned of the nature of some outstanding problems in the difference calculus and especially of those whose importance had been insisted upon by Van Vleck in his Wisconsin lectures and I determined to undertake the establishment of general existence theorems for linear homogeneous difference equations for the case of functions of a complex variable.

That the power series did not afford a suitable tool for this investigation was evident from certain general considerations; it was therefore natural to try some form of the method of successive approximations. The formal aspects of this method were fairly obvious; but a certain obstacle in its application was early met with owing to the very difficult character of the problem of finite integration, a problem which had not been solved in a form suitable for use in the investigation in hand. The chief novelty of method in my investigation was in the development of a means of finding the finite integrals of certain classes of functions which had to be dealt with in the course of the argument. To this general problem of finite integration I shall return later on in the lecture.

A similar difficulty with finite integration had been met with a few years earlier by J. Horn and the investigation which he carried out was restricted in range by the lack of a suitable tool for finite integration, as he himself later stated;[1] the method of finite integration employed in my dissertation would have sufficed to overcome the main difficulty encountered by Horn. In the investigations of Birkhoff, published soon after my own, the theory of the solutions of linear homogeneous difference equations reached a state of development distinctly in advance of what had previously been attained. The essential improvement in method consisted in a certain fairly obvious extension of the means of finite integration introduced by me, the extension having the effect of giving the method a considerable increase in flexibility, an advantage entirely unforeseen by me at the time of writing my dissertation. Birkhoff also employed the method of successive approximations, giving it a different (and very convenient) form by the use of the matrix notation.

Two other basic investigations arriving at general existence theorems for linear homogeneous difference equations were carried out almost simultaneously with those due to Birkhoff and myself. Galbrun treated the problem by the aid of the Laplace transformation and the consequent reduction of the investigation to a dependence on known results concerning differential equations. Nörlund treated the problem by aid of factorial series; in some respects the method employed by him is more satisfactory than any other yet developed. I have never

[1] Crelle's *Journal*, 141 (1912), p. 183.

been certain that his method is the best possible. It has seemed to me to be probable that a generalization of the method of Nörlund, effected by means of the series treated in section 5 of this address, may afford a still more satisfactory method than that of Nörlund, especially for the important case in which the coefficients in the difference equation are rational functions.

A considerable development of the theory of linear difference equations has followed the basic investigations by the four mathematicians whose work we have just outlined, this having been carried out both by them and by others. Up to the present time but little progress has been made in the development of the theory of non-linear equations. Furthermore, even in the case of linear equations, we have so far had developed in detail the theory of only a few of the particular equations whose solutions may be of especial interest. The outstanding example is that of the equation $g(x+1) = xg(x)$ and the gamma functions which satisfy it. The theory of these functions was already well known before the development of general existence theorems for linear difference equations. There is much still to be done in the way of basic investigations of particular equations and of the development of a general theory of non-linear equations. Hardly a start has been made toward the development of a theory of partial difference equations.

An article by Nörlund, finished in April, 1922, and entitled "Neuere Untersuchungen über Differenzengleichungen," has been published in the *Encyklopädie der Mathematischen Wissenschaften.* This gives an adequate account of the present state of the difference calculus with precise references to the literature of the subject. A paper on the present state of the theory of difference equations had already been published by Nörlund in *Bulletin des Sciences Mathématiques* in 1921. The existence of these papers justifies me in giving but few explicit references in this address.

2. The rôle of the arbitrary elements. We have already indicated incidentally two of the reasons why the modern development of the difference calculus was so long delayed. It could not proceed far without the development of a fairly adequate theory of finite integration; and this is a problem of no little difficulty. The tool which was so successful in the development of the theory of differential equations, namely, the power series, is altogether inadequate for the difficulties encountered in the case of difference equations; power series are lacking in a certain flexibility requisite in any tool serving this purpose. But it seems certain that these difficulties—great as they are—would have been overcome if they had been resolutely faced. A certain important fact about difference equations probably operated to prevent for a long time a serious attack upon the problem afforded by them. This lies in the nature of the arbitrary elements.

To make the matter of arbitrary elements explicit let us consider the equation

$$f(x+k) + c_{k-1}(x)f(x+k-1) + \cdots + c_1(x)f(x+1) + c_0(x)f(x) = 0,$$

where the coefficients $c_i(x)$ are analytic at infinity. In terms of a suitably independent set $f_1(x), f_2(x), \cdots, f_k(x)$ of solutions of the given equation the general

solution is expressible in the form

$$f(x) = p_1(x)f_1(x) + p_2(x)f_2(x) + \cdots + p_k(x)f_k(x),$$

where $p_1(x)$, $p_2(x)$, $\cdots$, $p_k(x)$ are arbitrary periodic functions of period unity. In the case of the q-difference equation

$$f(q^kx) + c_{k-1}(x)f(q^{k-1}x) + \cdots + c_1(x)f(qx) + c_0(x)f(x) = 0, \qquad |q| \neq 1,$$

the general solution takes the form

$$f(x) = P_1(x)f_1(x) + P_2(x)f_2(x) + \cdots + P_k(x)f_k(x),$$

where $P_1(x)$, $P_2(x)$, $\cdots$, $P_k(x)$ are arbitrary q-periodic functions, that is, functions which satisfy the relation $P(qx) = P(x)$ and are otherwise arbitrary.

The difficulty which these arbitrary elements bring to light is of the following sort. In each case there is a significant region of the complex plane throughout which the solution is quite arbitrary, owing to the nature and place of the arbitrary elements in the solution. In a preliminary view of the situation one might well be led to conclude that little of interest is likely to arise from the theory of an equation whose general solution has so much of the arbitrary in it. It might not be easy to realize in advance that there is a ready means at hand for effectively reducing the rôle of the arbitrary elements; this is the more likely to be true in the case of the ordinary difference equation since the solution in this case is arbitrary in an infinite strip. It has seemed to me that the presence of these arbitrary elements in the general solution of difference equations probably operated to prevent the early development of an interest in them from the point of view of general function theory.

This could hardly have been true if the q-difference equation had been contemplated from the beginning as the ordinary difference equation had; for it is a relatively easy matter to see how to reduce the arbitrariness in the solution by confining attention to a certain class only of the solutions of the equation. Let us consider a q-periodic function $P(x)$, $|q| \neq 1$. Then $P(qx) = P(x)$. Let us suppose now that we restrict this function to be analytic at the point $x = 0$. It is easy to see that $P(x)$ must then be a constant. In view of this fact one may readily define a large class of linear homogeneous q-difference equations having a fundamental system of solutions each function of which is analytic at the point $x = 0$, while the most general solution *which is analytic at the point* $x = 0$ is expressible linearly in terms of the fundamental system, the multipliers being *constants*. The uncomfortable excess of arbitrariness is thus removed by means of a *descriptive condition* by which the admissible solutions are to be restricted.

In the case of the difference equation the matter is not so simple. Here the arbitrary elements are to be restricted by means of the asymptotic character of admissible solutions for certain defined methods of approach of the independent variable to infinity. This can not be made explicitly clear in short space; consequently the matter will be passed over with the general statement that it is possible to obtain the desired restriction of arbitrary elements by means of sets of limiting conditions at infinity.

Some important applications of the difference calculus, I believe, are yet to be made in connection with the theory of elliptic functions. As is well known, these functions are subject to a great variety of conditions arising from the addition theorems and results of a similar character. These may be particularized in many ways so as to give rise to difference equations, or q-difference equations, or to mixed equations of many different forms. The arbitrary elements in the general solution of these functional equations are often of such sort as to render apparently uninteresting the study of the general solution. But if one restricts the class of solutions, say by requiring analyticity at a certain point or in some other suitable way, one will often find that the solution is then restricted to be essentially the elliptic function from which the equation arose. We have had no systematic examination of the theory of elliptic functions from this point of view. I believe that such an investigation would serve a rather useful purpose in simplifying the exposition of certain topics in the theory of elliptic functions; this would be especially true in connection with the solutions of various types of functional equations, such, for instance, as the three-term functional equation for the Weierstrass σ-function.

3. The problem of finite integration. Perhaps the most important and the most difficult problem in the difference calculus is that of the determination of the solution of the equation

$$F(x+1) - F(x) = \varphi(x).$$

This is the problem of the finite integration of the function $\varphi(x)$. The existence of a solution is immediately obvious. The general solution is obtained by adding to a particular solution an arbitrary periodic function of period unity. The main problem of finite integration is to select from the infinite totality of particular solutions that one or those solutions which are characterized by the possession of suitably chosen fundamental properties. The difficulty of the problem consists in seeing how to select those properties and how to construct the solution or solutions possessing them. This fundamental problem has been extensively investigated by Nörlund. Besides the foregoing equation he studies also the following related equation:

$$G(x+1) + G(x) = \varphi(x). \tag{2}$$

He shows in particular the existence of principal solutions $F(x)$ and $G(x)$ of these respective equations defined by use of a method of summation analogous to that by means of which one obtains the solution of the equation

$$\frac{dH(x)}{dx} = \varphi(x)$$

in the form of a definite integral; and he denotes these solutions by the symbols

$$F(x) = S\varphi(x)\Delta x \qquad \text{and} \qquad G(x) = S\varphi(x)\Delta x$$

respectively. For these investigations we must refer the reader to Nörlund's

Encyclopædia article already mentioned and to the papers there cited; one should also examine the very recent paper by Nörlund in *The Transactions of the American Mathematical Society*, vol. 25.

A less important aspect of the problem of finite integration may be referred to here as setting certain additional problems which it seems desirable to have investigated. All these are connected with the so-called formal solutions of equations (1) and (2). To a considerable extent, but not completely, these have already been treated in the case of equation (1). It seems that much less has been done directly with equation (2), although the latter is in some respects rather simpler than equation (1).

Two formal solutions of equation (1) are given by the well-known direct sums to the right and to the left, respectively, namely, the solutions:

$$F(x) = -\varphi(x) - \varphi(x+1) - \varphi(x+2) - \cdots,$$
$$F(x) = \varphi(x-1) + \varphi(x-2) + \varphi(x-3) + \cdots.$$

The corresponding formal solutions of (2) are:

$$G(x) = \varphi(x) - \varphi(x+1) + \varphi(x+2) - \varphi(x+3) + \cdots,$$
$$G(x) = \varphi(x-1) - \varphi(x-2) + \varphi(x-3) - \varphi(x-4) + \cdots.$$

The formal series for $F(x)$ converge, and hence define an actual solution of (1), in case $\varphi(x)$ is analytic at infinity and vanishes there to at least the second order. The series for $G(x)$ converge if $\varphi(x)$ is analytic at infinity and vanishes there to at least the first order. Various modifications of the series for $F(x)$ have been employed, the modifications having been made with the purpose of extending the range of validity of the resulting series. Corresponding modifications of the series for $G(x)$ may also be made and with a like extension of the range of the validity of the resulting series. It appears to me that results of usefulness are likely to emerge from a systematic examination of the possible modifications of the foregoing formal solutions, those modifications being sought which will have the effect of giving a widened range of functions $\varphi(x)$ for which the resulting series shall actually afford a finite integral. In this connection one will probably need to examine summable divergent series as well as convergent series.

The well-known Euler-Maclaurin summation formula is obtained by writing (1) in the symbolic form

$$(e^D - 1)F(x) = \varphi(x)$$

by means of the Taylor expansion of $F(x+h)$ in powers of h (taken for $h = 1$), whence

$$F(x) = (e^D - 1)^{-1}\varphi(x),$$

where D denotes differentiation with respect to x, the explicit formula for $F(x)$ being gotten by expanding $(e^D - 1)^{-1}$ in powers of D and then operating with it term by term upon $\varphi(x)$. A similar formal solution of (2) may be obtained from the formula

$$G(x) = (e^D + 1)^{-1}\varphi(x).$$

Wedderburn[1] has obtained another formal solution of (1) by first expanding $(e^D - 1)^{-1}$ in partial fractions and then operating upon $\varphi(x)$ with the result. In a similar way a formal solution of (2) could be obtained by expanding $(e^D+1)^{-1}$ in partial fractions and then operating upon $\varphi(x)$ term by term with the result. I do not remember to have seen these formal solutions of (2) treated anywhere in the literature, but their formation is obvious in view of the known solutions of (1). It appears to me to be a matter of some interest to have a systematic analysis made of the possible modifications of these formal solutions and an investigation of the range of validity of the resulting series. It appears that the literature contains little (if anything) looking in the direction of these modifications or extensions of such formal solutions.

In case $\varphi(x)$ is an entire function and we write

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 +$$

we have a formal solution of (1) in the form

$$F(x) = \frac{a_0}{1} B_1(x) + \frac{a_1}{2} B_2(x) + \cdots + \frac{a_n}{n+1} B_{n+1}(x) + \cdots, \tag{3}$$

where $B_1(x)$, $B_2(x)$, $\cdots$ are the Bernoulli polynomials. This series is in general divergent. Appell (Liouville's *Journal,* (4) 7 (1891): 157–176) subtracts from $B_n(x)$ the first n terms of its trigonometric series, denoting the result by $\Psi_n(x)$. Then he shows that the series

$$F(x) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} \Psi_n(x) \tag{4}$$

converges uniformly in a given bounded region and defines a solution $F(x)$ of (1) which is an entire function. A similar method can be applied to the determination of a solution of equation (2).

In case $\varphi(x)$ is formally or actually represented by an expansion in factorial series or by an expansion in binomial coefficients, a term by term finite integration will yield a formal or an actual solution of the problem of the finite integration of $\varphi(x)$. For these expansions a similar problem may be proposed.

The way in which the divergent series in (3) is transformed into the convergent series in (4) may be suggestive of suitable methods of dealing with the other problems characterized in this section.

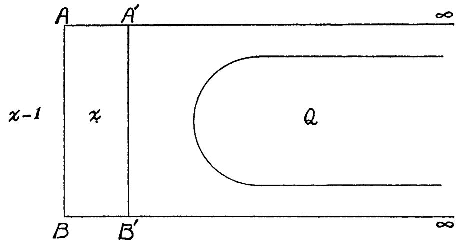
Two other related formal solutions of ( ) will be mentioned briefly; they are the ones first employed in my dissertation and afterwards extended in Birkhoff's memoir. [Corresponding formal solutions of (2) exist, but they have probably not yet made their appearance in the literature.] They are expressed by means of contour integrals. Let $\varphi(x)$ be analytic outside of a region Q extending to the right and lying within a finite distance of the positive half of the axis of reals, as in the figure: then if the contour L, $L = \infty AB \infty$ is drawn so as to enclose Q,

[1] *Annals of Mathematics,* vol. 16 (1914): 82–85.

and passes between $x - 1$ and x in the manner shown in the figure, the function

$$F(x) = \int_L \frac{p(x, z)\varphi(z)dz}{e^{2\pi\sqrt{-1}(x-z)} - 1}$$

formally satisfies (1) as may be seen by means of the theory of residues, provided that the function $p(x, z)$ is an analytic function of x and of z for all finite values of these variables while $p(x + 1, z) = p(x, z)$ and $p(x, x) = 1$. In a similar manner a contour integral to the left may be used to define another formal solution of (1), the corresponding function $p(x, z)$ being such that $p(x, x) = - 1$; it is the latter which is handled explicitly in my dissertation, but only for the case $p(x, z) \equiv - 1$. The more general form involves the extension made by Birkhoff. No general investigation has been made of the range of validity of these formal solutions. They are shown to be effective in the investigations in which they have been employed. It appears to me to be distinctly desirable to have a systematic analysis made of the range of validity of these formal solutions as actual solutions of the problem of finite integration. In this connection one would investigate also the various possible modifications and the related solutions, such (for instance) as those earlier employed by Guichard and Weber.[1]



4. **Numbers defined by recurrence relations.** An application of the difference calculus in the theory of numbers, the importance of which has not been generally recognized, is attached to equations in which the independent variable runs over the set of positive integers alone. Many of the classic theorems in the theory of the divisibility and primality of integers appear as special cases of theorems relating to sequences of integers defined by means of recurrence relations; and not a few rather novel results also emerge in this connection. The importance of this method of investigation was emphasized by Lucas in a series of investigations the principal results of which were brought together in his memoir in the first volume of *The American Journal of Mathematics*. The more general features of this method I have set forth in some fulness in a recent memoir.[2] For some years it has appeared to me that there is good reason to think that Fermat made frequent use of sequences of integers defined by recurrence relations and that he was in this way led to some of his most remarkable results.

For details and for references to the literature the reader may consult the papers mentioned and especially the full analysis and bibliography given in the first volume of Dickson's monumental history of the theory of numbers. We shall mention here only the one or two facts which we need as affording the point of departure for some additional remarks. Let us consider the infinite sequence of integers $u_0, u_1, u_2, \cdots$ defined uniquely in terms of the initial numbers u_0,

[1] See Nörlund's Encyclopædia article.

[2] *Quarterly Journal of Mathematics*, vol. 48, No. 192, 1920.

$u_1, \cdots, u_{k-1}$ by the recurrence relation

$$u_{x+k} + \alpha_1 u_{x+k-1} + \cdots + \alpha_k u_x = \alpha$$

in which $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_k$ are given integers. Let $r_0, r_1, r_2, \cdots$ be in order the least non-negative residues of the integers $u_0, u_1, u_2, \cdots$ with respect to the modulus m, where m is a given integer. A fundamental property of the sequence $r_0, r_1, r_2, \cdots$ is the following: The elements in the sequence, after a finite number at the beginning, repeat themselves in cycles of μ elements, where μ is a certain characteristic integer associated with the sequence. In view of this fact the sequence $u_0, u_1, u_2, \cdots$ is said to be periodic modulo m. Many important properties of the sequence are intimately connected with this property of periodicity. Accordingly a systematic study of the latter property leads to a considerable number of results of genuine interest. For these the reader is referred to the original memoirs.

In connection with these facts I shall now undertake to characterize some problems of considerable interest. It appears natural to look upon the periodicity property just mentioned as the analogue of simple periodicity in the theory of functions of a complex variable. If one conceives it so he can not fail to raise the question as to what corresponds, among recurrent sequences of integers, to the property of double periodicity in the theory of elliptic functions. That this question was in the thought of Lucas is evident from some remarks which he made in the article mentioned; but, so far as I know, no one has ever brought to explicit notice the analogue in consideration. If there is any property of sequences of integers (ordinary or algebraic) which truly corresponds to double periodicity in the theory of elliptic functions, an investigation of it has a large chance of proving of considerable interest.

More general sequences of integers than those already mentioned explicitly could be easily brought into consideration. If, in the last foregoing equation, the quantities $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_k$ are functions of x each of which is integral-valued for every integral value of x, then the equation defines an infinite sequence of integers in terms of k initial integers. [This might also be extended to the case where the integral values are replaced by algebraic integers; it seems also to be possible to extend the range of x by means of equations and systems of equations involving various algebraic units (as well as 1) among the differences of the independent variable x.] For this more general class of sequences—as well as for those mentioned in the sentence in brackets—one may consider such problems as those which have already been treated for the simpler case. So far, the problem has hardly been touched in the literature.

In connection with the linear recurrence relations which have been treated certain non-linear recurrence relations have come into consideration. Some of these appear in certain remarkable theorems giving necessary and sufficient conditions for the primality of certain classes of integers; they also come up in other connections. The purposes which they serve, where they have arisen in this incidental way, afford good grounds for believing that fruitful investigations

may be attached to certain classes of these non-linear equations. It may well be that the problem of double periodicity of sequences of integers is intimately bound up with certain classes of these non-linear equations.

Corresponding questions arise for linear q-difference equations where the multiplier q is an integer. So far as I know these have not been treated. I have not considered the problem sufficiently to have a clear judgment as to whether it may be a promising or a useful one.

5. Expansions generalizing the factorial series. In the solution of differential and difference equations functions have frequently arisen which have usually been represented in a two-fold way: the asymptotic character of one of these functions near infinity is set forth by means of a diverging power series; another form of expansion is found for it by means of which to put in evidence the analytic character of the function. Obviously, it is desirable, if possible, to have a single expansion of such character that it is capable of exhibiting the asymptotic properties of the function near infinity and of yielding at the same time a convenient and workable representation of it in the finite plane or in a significant portion of the finite plane. It is hardly to be expected that any single class of series will afford a form of such a tool best suited to all situations; but it may very well be true that certain classes of functions of great importance and very frequently recurring in practice are capable of representation in one or another sort of series all of which belong to a single type and possess a unitary theory. A few years ago it became apparent that factorial series and certain immediate generalizations of them serve just such a purpose to a remarkable degree.

But there appears to be a certain lack of flexibility of these series which restricts their range of application in such a way as to render them suited to meet only a part of the indicated need. In meditating on this matter a few years ago I was led to observe that a certain important class of functions, frequently in evidence in investigations in the difference calculus, were such as to afford the basis of definition for a large class of series of which the factorial series is but an instance, while many properties of the whole class of series are fundamentally quite as simple as the corresponding properties of the factorial series. These series are of the form

$$\Omega(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)},$$

where the coefficients c_n are independent of x and where $g(x)$ is a given function of the complex variable whose most frequently used properties are expressed by means of the asymptotic relation

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \right),$$

in which $P(x)$ and $Q(x)$ are polynomials in x, $P(x)$ not being a constant in the special case in which $Q(x)$ is of degree less than 2. The general theory of the convergence of these series is very simple in its main aspects. Numerous important

special cases of them have appeared in the literature.[1] I believe that other special cases of these series will have a considerable use in the development of the theory of differential and difference equations. The fundamental theorems concerning the class of functions which are representable by means of them are yet largely to be discovered: the problem has been partially treated in several papers of my own.

Intimately related to this is the problem of the representation of fuctions by means of integrals of the form

$$J(x) = \int_0^\infty \varphi(t) g(x + t) dt,$$

where the function $g(x)$ has essentially the same properties as in the foregoing case. The convergence theory for these integrals is particularly simple and elegant, having many points of contact with that of the series $\Omega(x)$. I have given the main basic theorems in a paper in *The Transactions of the American Mathematical Society*, vol. 20 (1919): 313–322. I believe that special cases of these integrals will be found useful in much the same way as corresponding cases of the series $\Omega(x)$. Several important cases of the integrals $J(x)$ have appeared in the literature.

It seems to me that a promising field for investigation lies in certain subclasses of the series $\Omega(x)$ and integrals $J(x)$; they have an unusual flexibility for the representation of certain classes of functions— and precisely of some of those frequently arising in connection with differential and difference equations. This conviction has lain in my mind for a number of years; but pressure of other work has interfered, and promises still to interfere, with the work of following up the conviction to see explicitly in how far it is justified. Perhaps there are others who may take up the problem.

6. Expansions in conjugate functions. That the expansions treated in the foregoing sections are fundamental in the difference calculus (and elsewhere) has been shown by the effective use already made of special cases of them, particularly the factorial series and certain immediate generalizations of them. For a time I thought that they almost surely afforded the principal expansion problem of the difference calculus. So far the evidence still indicates that this is probably true. The series still appear to me to have all the importance which I attached to them from the beginning. But there is another class of series in the difference calculus which appear to me now to be of greater absolute, and hence of greater relative, importance than I had at first supposed. These are the series analogous to Fourier series and other expansions in orthogonal and biorthogonal functions in the infinitesimal calculus. Concerning them we know, up to the present, nothing but a few of their formal properties. We shall bring the address to a close by setting forth the nature of these series. Since the ordinary difference equation has been more in evidence in the address than the q-difference equation, we shall describe this expansion problem with reference to q-difference equations; but the

[1] See *Bulletin of the American Mathematical Society*, (2) 23 (1917): 407–425.

reader will remember that a similar expansion problem exists for the ordinary difference equation.

Let us consider the adjoint systems of q-difference equations

$$u_i(qx) - u_i(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda\psi_{ij})u_j(x), \qquad i = 1, 2, \cdots, n, \tag{5}$$

$$v_i(x) - v_i(qx) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda\psi_{ji})v_j(qx), \qquad i = 1, 2, \cdots, n, \tag{6}$$

where q is a constant whose absolute value is different from unity and where the φ_{ij} and ψ_{ij} are functions of x which are analytic at infinity and have a zero there. These systems of equations possess fundamental systems

$$u_{1j}(x),\ u_{2j}(x),\ \cdots,\ u_{nj}(x);\quad v_{1j}(x),\ v_{2j}(x),\ \cdots,\ v_{nj}(x)$$

of solutions each function of which is analytic at infinity, say, analytic for $|x| \geqq R$, R being an appropriately chosen positive constant; moreover, the constant term in the solution $u_{ij}(x)$, and that in the solution $v_{ij}(x)$ as well, is δ_{ij}, where δ_{ij} denotes unity or zero according as j is or is not equal to i. Any solution which is analytic at infinity is made up from the foregoing solutions by taking linear combinations of them with constant coefficients. We confine attention to those solutions of both systems which are analytic for $|x| \geqq R$.

If we multiply (5) member by member by $v_i(qx)$ and (6) by $-u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$\sum_{i=1}^{n} \delta\{u_i(x)v_i(x)\} = 0, \tag{7}$$

where δ denotes the operation defined by the relation $\delta f(x) \equiv f(qx) - f(x)$. Let a be a number such that $|a| \geqq R$. In (7) sum as to x from a to ∞, where x runs over the values $a, qa, q^2a, q^3a, \cdots$ or the values $a, q^{-1}a, q^{-2}a, \cdots$ according as $|q|$ is greater than or less than unity; thus we have

$$\sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} = 0. \tag{8}$$

The first member of this equation is a non-singular bilinear form in the two sets of n variables each

$$u_1(\infty),\ u_2(\infty),\ \cdots,\ u_n(\infty),\ u_1(a),\cdots,\ u_n(a);$$
$$v_1(\infty),\ v_2(\infty),\cdots,\ v_n(\infty),\ v_1(a),\ \cdots,\ v_n(a).$$

This bilinear form may be written in an infinite number of ways in the normal form

$$\sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} \equiv \sum_{i=1}^{2n} U_i(u)V_i(v),$$

where the $U_i(u)[V_i(v)]$ are homogeneous linear expressions (with constant coefficients) in the variables $u[v]$. Then with our systems of q-difference equations

we associate the boundary conditions

$$U_i(u) = 0, \qquad i = 1, 2, \cdots, n; \tag{9}$$

$$V_i(v) = 0, \qquad i = n+1, \cdots, 2n. \tag{10}$$

Then relation (8) is satisfied in virtue of the boundary conditions alone.

Systems (5) and (9) define what may be called the u-problem; similarly the v-problem is defined by (6) and (10). The characteristic values λ are the same for the two problems. If $u_i^{(k)}(x)$ and $v_i^{(l)}(x)$ are solutions of the u-problem and the v-problem, respectively, the first for $\lambda = \lambda_k$ and the second for $\lambda = \lambda_l$, and if $\lambda_k \neq \lambda_l$, then we have the fundamental condition of conjugacy, namely,

$$\sum_{s=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ji}(at^s) u_i^{(k)}(at^s) v_j^{(l)}(at^{s+1}) = 0, \tag{11}$$

where $t = q$ or q^{-1} according as $|q|$ is greater than or less than unity.

Let us now suppose that the problem is set up so that we have the infinitude of characteristic values $\lambda_1, \lambda_2, \lambda_3, \cdots$ and corresponding solutions of the u-problem and the v-problem; and let us suppose that the first member of (11) is different from zero when l and k are equal. Then if given functions $f_i(x)$, $i = 1, 2, \cdots, n$ have suitable expansions in the form

$$f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \qquad i = 1, 2, \cdots, n,$$

the same coefficients c_k being employed for each of the functions, these coefficients c_k have the following values

$$c_k = \frac{\displaystyle\sum_{s=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ji}(at^s) f_i(at^s) v_j^{(k)}(at^{s+1})}{\displaystyle\sum_{s=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ji}(at^s) u_i^{(k)}(at^s) v_j^{(k)}(at^{s+1})}, \qquad k = 1, 2, 3, \cdots.$$

This is sufficient to put in evidence the formal aspects of an expansion problem in the q-difference calculus analogous to expansions in orthogonal and biorthogonal functions in the infinitesimal calculus. Two modifications of it may be suggested. In one of these the rôle of the point infinity may be played by the point zero; that is, the summations may be from a to 0 instead of from a to ∞. This gives rise to a problem altogether analogous to the one already described; in fact, one of these problems goes essentially into the other by changing x into $1/x$. The second modification consists of a sort of combination of the two just mentioned, the summation being from 0 to infinity in the same sense in which a Laurent series affords a summation from $-\infty$ to $+\infty$. The latter form of expansion promises certain advantages over the other two.

Corresponding problems arise in connection with ordinary difference equations. Moreover, if we apply certain limiting processes which have become classic through the investigations of Volterra, we are led through to integro-difference equations and integro-q-difference equations. For these functional equations

there also exist expansion problems analogous to that which we have just treated. In each case the first formal aspects of the problem lie close to hand. It is of considerable importance to have a detailed analysis of the character of the functions which are representable in the form of certain of these expansions. No investigation of this type has yet appeared in the literature.

ON DANIEL BERNOULLI'S "MORAL EXPECTATION" AND ON A NEW CONCEPTION OF EXPECTATION.

By CHARLES JORDAN, University of Budapest.

1. Mathematical Expectation. Even those events whose accomplishment is not certain, but only more or less probable, exercise before their happening some influence over us. Therefore we attribute in advance to these events more or less utility and value.[1] From the beginning, the philosophers have agreed that this value may be considered as being proportional to the probability p of the event; moreover that this valuation depends upon the nature of the event itself. For instance, if a certain gain is expected, the greater it is, the greater the value of the expectation will be. But the philosophers soon disagreed about the degree of appreciation.

First, mathematicians and philosophers thought that the value of the expected gain might also be taken as proportional to its magnitude. If we denote the probability of the event by p, and the expected capital by x, then from this point of view the value of the expectation would be equal to $\lambda \cdot px$, where λ is an individual factor of proportionality. Since in the calculus of probability the quantity $p \cdot x$ is called mathematical expectation, the mentioned hypothesis may be thus enunciated: The effect, or utility, and hence the valuation of a coming event is proportional to its mathematical expectation; therefore our actions are regulated by the principle of mathematical expectation.

If we expect one of several events, which may respectively produce the gains $x_1, x_2, \cdots$ or x_m, then the mathematical expectation of these eventualities is the sum of the partial mathematical expectations; *i.e.*, $\Sigma p_i x_i$.

Mathematical expectation is closely connected with arithmetical mean; indeed, if we suppose that the aggregate of the n possible cases contains the quantity x_1 n_1 times, and the quantity x_2 n_2 times, and so on, and in the end the quantity x_m n_m times, then the mean value of the quantities contained in the aggregate, is $\Sigma(n_i/n)x_i$. Taking into account the fact that, according to the frequency definition of mathematical probability, n_i/n is equal to the probability p_i of the quantity x_i, we conclude that the mathematical expectation of the quantities $x_1, x_2, \cdots x_m$ is equal to the arithmetical mean of the quantities constituting the aggregate of the possible cases.

---

[1] The terms "utility," "value" and kindred expressions are used here without attempt at an analysis of their various possible significations in much the same manner as certain undefined terms are used in geometry. For an extended analysis of "value" see "*The Measurement of General Exchange-Value*" by C. M. Walsh, Macmillan, 1901.—EDITOR.

Accepting the principle of mathematical expectation, it follows, if p_i is the probability that our capital x increases by the quantity Δx_i, that the effect or utility ΔU_i of this change will be expressed by $\Delta U_i = \lambda p_i \Delta x_i$; and considering all the possible eventualities the total change of the *utility* will be $\Delta U = \lambda \Sigma p_i \Delta x_i$, where $\Sigma p_i = 1$.

According to this principle, we may class the events or operations under three categories: (1) favorable operations, corresponding to $\Delta U > 0$; (2) indifferent operations, corresponding to $\Delta U = 0$; (3) unfavorable operations, corresponding to $\Delta U < 0$.

If our actions were really directed by the principle of mathematical expectation, we should only participate in operations belonging to the first group. Our readiness to participate is an increasing function of ΔU.

Experience shows that this is not the case. Actually, every day we see people take part in lotteries, insurances and other transactions in which $\Delta U < 0$. On the other hand, as Nicolas Bernoulli has shown in the Petersburg paradox, there are cases in which the mathematical expectation ΔU is unlimited, and nevertheless nobody would engage in the corresponding transaction. Moreover, in every operation at least two persons are opposed to each other. If the mathematical expectation of the first is greater than zero, then that of the second must necessarily be less than zero; so that for at least one of the participants the operation must be unfavorable. According to the principle, this person should refuse to take part. No operations would be possible, except those in which $\Delta U = 0$, and they would be superfluous.

2. Moral Expectation. The principle of mathematical expectation being unacceptable, there arose the need to look for other principles. Daniel Bernoulli[1] was the first to give the following new hypothesis: If our fortune is x and p is the probability that it will increase by Δx, then the value of the expectation is, as in the former hypothesis, directly proportional to the probability p and to the expected gain Δx, but is moreover inversely proportional to the fortune x; that is

$$\Delta U = \lambda p(\Delta x)/x.$$

It results from integration that, if p is the probability that our fortune will become equal to x, the value of its utility is

$$U = \lambda p \log (x/x_0),$$

where x_0 is the initial sum or that sum which has no effect at all on us ("threshold of consciousness").

If there is a probability p_1 that our fortune will become x_1, and p_2 that it will become x_2, and so on,[2] then the value of these eventualities is, according to this principle, the sum of the values of the different expectations; *i.e.*,

$$\overline{U} = \lambda \Sigma p_i \log x_i - \lambda \log x_0 = \lambda \log [x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m}] - \lambda \log x_0.$$

[1] *Specimen Theoriæ novæ de Mensura Sortis*, 1738.

[2] We will call the aggregate of these conditions (A).

If our fortune is x, its utility or value to us (setting $p = 1$ before the risk is taken) is

$$U = \lambda \log x - \lambda \log x_0.$$

Therefore we conclude that the value of the above expectations is the same as if our fortune were actually equal to $x_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}$. This last quantity is called, after D. Bernoulli, "moral fortune."

Let us return to the aggregate of the possible cases mentioned before; the geometrical mean of the quantities contained in it is

$$G(x_i) = \sqrt[n]{x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}} = x_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}.$$

Hence the moral fortune (or capital) is equal to the geometrical mean of the different possible quantities.

If our capital before the transaction was x, and if, after it, we are placed in the conditions (A), then the change in the value will be

$$\overline{U} - U = \lambda \log [x_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}] - \lambda \log x.$$

The quantity $G = [x_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}]/x$, the relative change of the moral fortune, is called *moral expectation.*

The effect, utility and value of the change is proportional to the logarithm of the moral expectation.

From this second point of view we can also class the events under three categories—favorable, indifferent, and unfavorable events, according as $G > 1$, $G = 1$ or $G < 1$.

One of the most important consequences of D. Bernoulli's hypothesis is the principle called that of decreasing utility; *i.e.*, ΔU decreases with increasing x.

To compare the two hypotheses which we have considered, let us introduce a quantity closely connected with the mathematical expectation. If x was our capital, and if in consequence of an operation the different possibilities are those described under (A), then the mathematical expectation of the gain is $\Sigma p_i x_i - x$; let us denote by $A = \Sigma p_i x_i/x$ the *relative* mathematical expectation of our fortune. It results that if the operation is favorable from the first point of view, then $A > 1$; if it is indifferent, $A = 1$; and if it is unfavorable, $A < 1$. The quantity A is the arithmetical mean of the quantities x_1/x, x_2/x, $\cdots$ x_m/x, contained in the aggregate of possible cases. The corresponding moral expectation G is the geometrical mean of the same quantities, that is

$$G = (x_1/x)^{p_1}(x_2/x)^{p_2}(x_3/x)^{p_3} \cdots (x_m/x)^{p_m}.$$

3. The moral expectation is *always smaller* than the corresponding mathematical expectation. Indeed, the arithmetical mean is greater than the geometrical mean of the same quantities. It results that all the operations unfavorable from the first point of view are still more so from the second, and that the indifferent operations and some of those which are favorable from the first point of view are unfavorable from the second.

4. Let x be our fortune, p the probability of winning a sum equal to b, and q the probability of losing a. If $p + q = 1$, the corresponding moral expectation is:

$$G = \left[\frac{x+b}{x}\right]^p \left[\frac{x-a}{x}\right]^q. \tag{1}$$

Thus, if $a = x$, $G = 0$. For this discussion, it will be assumed that $a < x$,—that the entire fortune is not put at risk. If the operation is indifferent from the point of view of mathematical expectation, we have $bp = aq$, hence:

$$G = \left[1 + \frac{qa}{px}\right]^p \left[1 - \frac{a}{x}\right]^q.$$

To show that if a and x retain their value unchanged, G increases with increasing p, it is sufficient to demonstrate that

$$\frac{d}{dp} \log G > 0.$$

This differential quotient can be written as follows:

$$- \log \left[1 - \frac{a}{px + qa}\right] - \frac{a}{px + qa}.$$

The expansion of this logarithm in power series shows that the quantity above is greater than zero.

Therefore, the capital x and the risked sum a remaining unchanged, an operation indifferent, from the first point of view, becomes more and more unfavorable from the second, as p decreases; *i.e.*, the riskier a game is, the more unfavorable it is. This judgment coincides with everyday experience.

5. The *capital* of a gambler has no effect on the mathematical expectation of a gain, but the moral expectation of a gain increases, in unfavorable operations, if his capital becomes greater; *i.e.*, the influence of a risk is always greater for a poorer man than for a richer, conformable to ordinary opinion.

This can be proved by showing that under these circumstances $d/dx \log G > 0$. From (1) it results that

$$\frac{d}{dx} \log G = \frac{(qa - pb)(1 + x) + ab}{x(x - a)(x + b)}.$$

If $qa \geqq bp$ (unfavorable operations), as x, a, b, p, and q are positive, the above differential quotient is certainly greater than zero.

6. From the first point of view the *loss* of a sum has the same effect, only in the opposite sense, as the gain of the same sum. From the second point of view the influence of the loss is greater than that of the gain. Indeed the effect of the gain a is proportional to $\log (1 + a/x)$, and the effect of the loss to

$\log(1 - a/x)$, but

$$\left|\log\left(1+\frac{a}{x}\right)\right| < \left|\log\left(1-\frac{a}{x}\right)\right|$$

as is easily seen by expansion of the logarithms in powers of a/x.

7. From the first point of view, the *dividing* of a risk between two chances has no effect at all; from the second, it is favorable. Indeed if q is the probability of losing, if x is our capital and if we risk on one chance the sum $2a$; then from the second point of view the effect will be proportional to $\log(1 - 2a/x)^q$. If on the other hand we risk the sum a on each of two chances, the effect will be proportional to $\log(1 - 2a/x)^{q^2}(1 - a/x)^{2pq}$; we have

$$\log\left(1-\frac{2a}{x}\right)^q < \log\left(1-\frac{2a}{x}\right)^{q^2}\left(1-\frac{a}{x}\right)^{2pq},$$

which is equivalent to $0 < a^2/x^2$. This discussion applies to such risks as may bring only loss—such as the risks of transportation, but even in risks which could augment our capital we would be led to the same conclusion.

8. Accepting the second hypothesis the *insurances* must necessarily be favorable, from both points of view, to the insurance companies, otherwise they could not be compelled to accept the risks.

The insured are in a different position; they are involved in risks which diminish their moral fortune; consequently, if they pay less than this diminution to get rid of the risk, the operation is advantageous for them from the second point of view, though unfavorable from the first.

For instance, let x be the capital of the person who insures, and b the value of the object to be insured, q the probability of its loss, a the sum paid for insurance, y the capital of the company, and $q = 1 - p$.

From what we have seen, the moral fortune of the person, if he does not insure, is $x^p(x - b)^q$; if he insures, $x - a$. The operation will be favorable to him from the second point of view, if

$$x^p(x-b)^q < x - a.$$

The moral fortune of the company is, if it does not accept the insurance, equal to y; and if it does, to $(y + a)^p(y + a - b)^q$; the operation will be favorable to it if

$$(y+a)^p(y+a-b)^q > y.$$

It is easy to show that there are values of a which satisfy at the same time both of the inequalities. For these values the tariff will be favorable to both parties, *i.e.*, the insurance will increase the moral fortune of both. This corresponds to the cases in which insurance is considered favorable in practice. Thus, while the hypothesis of mathematical expectation cannot account for insurance, moral expectation gives a good explanation of it.

It is true that an insurance contract does not produce wealth, but it increases moral capital. Its value is measured by this increment.

9. Daniel Bernoulli's hypothesis is greatly supported by Weber and Fechner's celebrated *psycho-physical law,* according to which the sensation y is proportional to the logarithm of the corresponding stimulus x; *i.e.,*

$$y = \lambda \log (x/x_0).$$

Hence this law may be regarded as an extension of Bernoulli's hypothesis to natural phenomena. It has been verified in many instances in which measured quantities are compared with their estimated magnitudes. For instance, if we compare Ptolemy's star-magnitudes, y, to the photometric intensity x of their brightness, we have

$$y_2 - y_1 = - 2.5 \log (x_2/x_1).$$

Analogous formulæ can be found if we compare the estimated pitch of sound with the corresponding number of vibrations per second; or by comparing estimated values with measured values of temperature, weight, distance, height, etc.

The law can moreover be verified by the "difference thresholds," that is, by the least observable differences of the stimuli.

10. Let us now apply D. Bernoulli's law to the problem of *contributions.* Let x be the contributor's capital and a the tax; the effect of the tax will be the same to everybody except for the factor λ, if we have

$$y/\lambda = \log (x - a)/x = \text{constant},$$

that is, if the taxes are proportional to the fortunes of the contributors. On the contrary according to the hypothesis of mathematical expectation the tax, to have the same effect, should be the same for everybody whatever his fortune may be. This would be considered in practice as quite unreasonable.

11. The importance of D. Bernoulli's theory is increased by the fact that the modern *theory of value* is founded upon it. It was the starting point for Gossen's, Jevons', Walras', Edgeworth's and Pareto's works.

If a person possesses the quantity x_i of a commodity C_i, then the *utility* of this quantity of C_i is expressed by

$$U = \int_0^{x_i} f_i(t)dt,$$

where $f(t)$ is a non-increasing function of t; $f(t)$ being the degree of utility of the quantity dt and $f_i(x_i)$ the *final degree of utility, i.e.,* corresponding to the last element dt of x_i. This formula is a generalization of D. Bernoulli's hypothesis.

If a person possesses the quantity x_1 of a commodity C_1, and x_2 of C_2 and so on, then the total utility of his goods will be

$$U = \Sigma \int_0^{x_i} f_i(t)dt.$$

This function plays a rôle in economics analogous to that of entropy[1] in

[1] For a statement of the entropy principle see Clerk Maxwell, *Theory of Heat*, Longmans, 1908, p. 193.

physics. People act in such a manner that the total utility of their possessions increases. Production, exchange and consumption are aptly explained by it. For instance, in exchanges an equilibrium will be reached if the final degrees of utility of every commodity possessed by a person are equal.

There was but one difficulty: the utilities concerning different persons, being subjective, cannot be compared or added, so that it seemed impossible to speak of the total utility relative to a group of persons, though it would be very useful to do so.

This difficulty was removed by Edgeworth in his "*Mathematical Psychics*" (1881) where he gave a method which, by means of determining experimentally the curves and surfaces of "indifference" corresponding to a group of persons, enables us to reach the differential equations expressing the total utility of their goods.

12. New Conception of Expectation; Harmonic Expectation. Among the mathematicians there were several [1] who accepted the principle of mathematical expectation but refused that of moral expectation as being quite arbitrary, not seeing that *a priori* the two principles are equally acceptable, the first being based on the arithmetical mean of the possible cases, the second on their geometrical mean.

As all measures are inevitably affected with errors, the natural laws derived from experience can only approximately interpret physical facts. From time to time they are superseded by more general and more accurate laws. D. Bernoulli's law, therefore, though much nearer the truth than that of mathematical expectation, will nevertheless ultimately share the same fate.

Some of its imperfections can be easily pointed out; for instance, while accounting for the threshold of sensations, it asserts that there is no upper limit for them. The sensations increase, it states, indefinitely with the stimuli. But we know that this is not true; we know that there is an upper limit of sensations, and that when it is reached, by increasing the stimuli there is no change at all.

Consideration of this fact, and of others, has led me to conceive and formulate an analogous law, based on the harmonic mean of the possible quantities.

Let x be the fortune of a person, x_0 the threshold of fortune corresponding to that of consciousness; let a be a certain constant, λ a factor differing from one individual to another; Δx the increment of the fortune, and Δy the corresponding increment of sensation or utility. Let us accept the hypothesis that we have

$$\Delta y = \lambda(x_0 + a)\Delta x/(x + a)^2.$$

By integration we get the utility y produced by the fortune x:

$$y = \lambda\left[1 - \frac{x_0 + a}{x + a}\right]. \tag{2}$$

It results that λ is the upper limit of the utility ($x = \infty$). Moreover the

[1] *e. g.* Bertrand, *Calcul des Probabilités*, p. X, p. 65.

difference of the utility of the quantities x_2 and x_1 is

$$y_2 - y_1 = \lambda(x_0 + a)\left[\frac{1}{x_1 + a} - \frac{1}{x_2 + a}\right]. \tag{3}$$

In the problems considered before, this formula gives results analogous to D. Bernoulli's hypothesis; perhaps even they are more satisfactory. For instance in the problem of contributions, if we want to tax people so that, according to this principle, the produced disutility is the same for everybody, then, if a person possesses a capital x_1, the tax $x_1 - x_2$ is obtained by putting in (3) the quantity $(y_2 - y_1)/\lambda(x_0 + a) = -1/c =$ constant. It results that the tax is

$$(x_1 - x_2) = (x_1 + a)^2/(x_1 + a + c).$$

Thus this third principle leads to progressive taxes. After having paid the tax, the fortune of the contributor will be:

$$x_2 = \frac{(c - a)(x_1 + a) - ac}{x_1 + a + c}.$$

It follows that there is a maximum of x_2; indeed for $x_1 = \infty$, $\lim x_2 = c - a$.

If p_1 is the probability that our fortune will become equal to x_1, and p_2 the probability that it will become x_2, etc., then the value of these eventualities is, from our third point of view, according to (2) and to § 1 (if $\Sigma p_i = 1$) the following

$$y = \lambda\left[1 - (x_0 + a)\Sigma\frac{p_i}{x_i + a}\right] = \lambda\left[1 - (x_0 + a)\frac{1}{H}\right],$$

where H is the harmonic mean of the quantities $x_i + a$; *i.e.*, the utility y is the same as if our fortune were equal to $H - a$. If it was, before the operation, equal to x, the change of the value is

$$\Delta y = \lambda\left[\frac{1}{x + a} - (x_0 + a)\frac{1}{H}\right];$$

this quantity can be termed *harmonic expectation.*

It is hardly necessary to add, in conclusion, that before accepting this principle it would be necessary to prove that it interprets physical facts with more accuracy than Weber's law.

SIMPLE DERIVATIONS OF THE FORMULAS FOR THE DISPERSION OF A STATISTICAL SERIES.

By C. H. FORSYTH, Dartmouth College.

1. Definition of dispersion and summary of certain principles. The purpose of this paper is to offer simple derivations of certain important formulas for dispersion along lines which the writer believes to be either more general or more familiar to the average mathematician or statistician than those employed else-

where.[1] It is believed, however, that a preliminary statement in this section of those elementary principles which lead up to and enter into the derivations would be very desirable.

The dispersion σ of a set of measurements or observations $x_1, x_2, \cdots x_n$ is defined by the relation

$$\sigma = \sqrt{\frac{1}{n} \Sigma(x_r - \bar{x})^2},$$

where $\bar{x}$ denotes the arithmetic average or mean of the observations. The conception of dispersion is of the greatest importance in investigating the relative *consistency* of sets of observations because it constitutes a good measure of the way the values of the observations deviate from the mean.

The formula quoted above is only occasionally employed in practice in the form given because a set of observations is usually given in the form of a frequency distribution; that is, the observations are usually classified according to equal intervals of measurement so that to each representative observation (usually the middle of the interval) there corresponds a frequency. The successive pairs of representative observations and corresponding frequencies constitute a frequency distribution.

The most natural means of handling a frequency distribution has been found to be by the method of moments where the n-th moment of a frequency distribution can be defined by the relation

$$n\text{-th moment} = \Sigma x^n y,$$

where x denotes the values of the representative observations and y the corresponding frequencies. It is important but obvious that the value of any moment of a frequency distribution is but an approximation of the value sought.

It is usually desirable to standardize moments by dividing them by the zero-th moment or the total frequency to obtain *unit* moments (usually referred to simply by the term *moments*) and by employing as a standard vertical axis of reference the axis through the mean. In the latter case the preliminary employment of a trial mean or axis of reference affords a great saving of labor. That is, the labor of computing the moments about an arbitrary axis of reference and transforming or interpreting them in accordance with simple formulas based upon ordinary translation to give moments about the mean is far less than that found necessary in computing the latter moments directly. Several of these formulas may be stated verbally as follows:

(*a*) The first (unit) moment about any axis of reference is the arithmetic average of the deviations from the trial mean and constitutes the correction to be made to this trial mean to give the true mean;

(*b*) the second (unit) moment about a trial mean is corrected to give the second

[1] Arne Fisher, *The Mathematical Theory of Probability*, 1922, pp. 118 ff.

Coolidge, J. L., The Dispersion of Observations, *Bulletin of the American Mathematical Society*, vol. 27 (1921), p. 439.

moment about the true mean by subtracting the square of the first moment about the trial mean.

It is evident that

(*c*) the square root of the second unit moment about the mean is identically the dispersion.

Further considerations show also that

(*d*) the square of the dispersion corresponding to the sum or difference of several independent variables is equal to the sum of the squares of the dispersions of the separate variables.[1]

Let us repeat the statement that the values of the moments obtained through the processes outlined above are necessarily approximations because of the form in which the observations are given although the approximations can be made close enough for most practical purposes.

The formulas stated above are all that are required in the derivations offered in this paper and we can now approach the particular problem to be considered.

2. Two kinds of frequency distributions. Frequency distributions can be arranged into two quite different classes according to whether they can be employed to determine the value of an important ratio or probability or not. Those of the latter type are important and common. An example of that type would be a distribution of the lengths of a large number of ears of corn. However, the most that can be expected from the investigation of such a distribution is the determination of the most representative observation and an idea of the consistency of the observations taken as a whole. We shall concern ourselves here with distributions of the former type because they lead not only to the same kind of conclusions as the other type but also to a determination of an important ratio or probability and an important *diagnosis* of any inconsistencies or apparent lack of reliability.

It will be recalled that there *may* be two ways of arriving at the value of a probability: *a priori* and empirically. Analogously, if the probability of an event is known and constant, the value of the dispersion *may* be determined in each of two ways. The empirical way is the one which has already been outlined in connection with moments. If the probability of the event in a single trial is p $(= 1 - q)$, then the probabilities of a failure every time, of exactly one occurrence, of exactly two occurrences, etc., in n trials are given by the successive terms of the expansion of the binomial

$$(q + p)^n = q^n + nq^{n-1}p + \cdots + p^n. \tag{I}$$

We can think of the terms of this expansion as frequencies of a frequency distribution and the first moment about an axis of reference taken at the "q^n" term is easily found to be np; that is, the mean $= np$. Likewise the second moment about the same axis is found to be $np + n(n - 1)p^2$ and according to formulas (*b*) and (*c*) of the preceding section the square of the dispersion becomes

$$\sigma^2 = npq.$$

[1] For the derivation of this relation see Jones' *First Course in Statistics*, London, 1921, p. 158.

It will be noticed that it is assumed that the probability of the event in a single trial remains constantly equal to p throughout such an investigation. In the next sections we propose to establish formulas for the dispersion when the value of the probability of the occurrence of the event is allowed to vary in important but prescribed ways.

3. Statistical series.[1] **Definitions.** We will suppose that the occurrence or the non-occurrence of a certain event has been noted and tabulated nN times and that these observations are arranged in N sets each of n observations. If we denote the number of times the event occurs in the first set by r_1, in the second set by r_2, etc., then the frequency sequence r_1, r_2, r_3, etc., is called a *statistical series*.

We shall differentiate between three kinds of statistical series: if the probability of the event in question remains the same throughout the investigation, the series is called a *Bernoullian* series; if the probability varies within each set but alike for all sets, the series is called a *Poisson* series; if the probability remains constant throughout each set but varies from set to set, the series is called a *Lexian* series.

We shall let $p\ (= 1 - q)$ denote the arithmetic average of the probabilities of the event in question regardless of whether the probabilities are always the same, vary from observation to observation within a set, or vary from set to set. That is,

$$p = \frac{1}{n}(p_1 + p_2 + \cdots + p_n) \qquad \text{or} \qquad \frac{1}{N}(p_1 + p_2 + \cdots + p_N)$$

according as the series is Poisson or Lexian, where the subscripts follow the order of observations or of sets according to the type of series.

4. Derivations of the formulas for dispersion. We propose to study the most probable value of r_i or the most probable number of occurrences of the event in question in n trials and the nature of the stability of this most probable value by means of dispersion. Hence, if this most probable value differs in several investigations, the best estimate of the most probable value at our disposal in that case might well be taken to be the arithmetic average of the various values. Moreover, in one type of series the value of the dispersion will naturally vary from set to set and as we shall deal almost entirely with the squares of dispersions we shall take the liberty of employing as the most probable value of the square of the dispersion in that case the arithmetic average of the squares of the dispersion of the different sets.

Let us consider first the means of the different types of series:

The Bernoullian series is, of course, the type with which we are familiar and we can write at once that the mean of that type is

$$M_B = np.$$

[1] Lexis, *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft*, Freiburg, 1877. The original development is due to Lexis but the final adaptation and terminology are due to Charlier.

As all the sets of a Poisson series are presumably the same or at least comparable, the mean of such a series is the mean of any set of that series or the arithmetic average of the means corresponding to the individual observations of any set, or

$$M_P = \frac{1}{n}(np_1 + np_2 + \cdots + np_n) = np.$$

As all the observations of any set of a Lexian series are comparable, the mean of a Lexian series is the arithmetic average of the means of all the sets, or

$$M_L = \frac{1}{N}(np_1 + np_2 + \cdots + np_N) = np.$$

We have then that the mean of a Poisson series or of a Lexian series is the same as the mean np of a Bernoullian series whenever p is the arithmetic average of the probabilities of that series.

Let us now consider the dispersions of the different types of series. We have already shown that the square of the dispersion of a Bernoullian series is

$$\sigma_B^2 = npq. \tag{1}$$

Moreover, we can evidently apply this formula to obtain p_iq_i as the square of the dispersion corresponding to the single i-th observation of any set of a Poisson series and if we apply formula (d) of § 1, the square of the dispersion for any and therefore for all sets of a Poisson series is $\sigma_P^2 = \Sigma p_iq_i$, where the summation is to extend from $i = 1$ to $i = n$ inclusive. There is a rather pretty way of performing this summation which will be useful again later: write $p_i = p + (p_i - p)$ and $q_i = q - (p_i - p)$. Hence,

$$p_iq_i = pq - (p - q)(p_i - p) - (p_i - p)^2,$$

whence it is easily verified that $\Sigma p_iq_i = npq - \Sigma(p_i - p)^2$ and we obtain

$$\sigma_P^2 = \sigma_B^2 - \Sigma(p_i - p)^2. \tag{2}$$

As the probability remains constant throughout any, say the i-th, set of a Lexian series, the square of the dispersion of that set of the series is np_iq_i which is about the mean np_i of that set. This is one of the rare occasions, however, when we desire the dispersion about a point other than the mean of the set, namely, the mean np of all sets, and it becomes necessary to reverse the application of formula (b) and *add* $(np_i - np)^2$ which is the square not only of the difference between the means but also, according to formula (a), of the first moment about the trial mean np. The square of the dispersion corresponding to all N sets or the whole Lexian series is then the arithmetic average of the values of $np_iq_i + (np_i - np)^2$ for all N sets, or

$$\sigma_L^2 = \frac{n}{N}\Sigma p_iq_i + \frac{n^2}{N}\Sigma(p_i - p)^2,$$

where the summation is to extend from $i = 1$ to $i = N$ inclusive. As the first summation is exactly the same kind (except for the upper limit) as that

considered in connection with the Poisson series, it is easily found that

$$\sigma_L^2 = \sigma_B^2 + \frac{n^2 - n}{N}\Sigma(p_i - p)^2. \tag{3}$$

Formulas (1), (2) and (3) are the standard formulas[1] which were to be derived and show clearly that the dispersion of a given statistical series yielding an average mean of np or probability p will be greater than $\sqrt{npq}$ if the value of the probability varies only from set to set and less than $\sqrt{npq}$ if the value of the probability varies only within each set.

5. Practical applications and limitations of formulas. The method of applying the formulas derived in this paper was stated concisely at the end of the preceding section. As illustrations, suppose that the following distributions were to give the results obtained in investigating the general death rates of three hypothethical countries each of which is assumed to include ten districts of population 1000. The numbers then are the number of deaths which are assumed to have occurred in one year in the respective districts.

(A)	(B)	(C)	$\lvert x - \bar{x}\rvert$	$\lvert x - \bar{x}\rvert^2$
17	18	24	10	100
16	17	22	8	64
11	10	20	6	36
12	11	19	5	25
13	9	18	4	16
14	12	10	4	16
15	19	9	5	25
16	16	8	6	36
14	10	6	8	64
12	18	4	10	100
140	140	140		10)482
				$\sigma^2 = 48.2$

The average number of deaths in all three countries is evidently 14 or $np = 14$ and hence $p = 0.014$ ($q = 0.986$). Hence $\sigma_B = \sqrt{npq} = \sqrt{14(0.986)} = 3.72$ is the dispersion which we should expect if the probability p remained constant throughout the investigation. If, however, we compute the dispersions directly from the data in accordance with the definition given at the beginning of this paper and as outlined to the right of distribution (C), we obtain

$$(A) \quad \sigma = \sqrt{3.6} = 1.90 \pm 0.03,$$

$$(B) \quad \sigma = \sqrt{14.0} = 3.74 \pm 0.06,$$

$$(C) \quad \sigma = \sqrt{48.2} = 6.94 \pm 0.10,$$

where the probable errors are added to the right to show how much of a deviation

[1] Fisher's derivations (*loc. cit.*) are restricted by the assumption that the distributions of probabilities conform to the expansion (I) and Coolidge's derivations are general but the treatment in both cases is given through the use of expected or mean values.

from the value of the corresponding dispersion might well be expected due to chance alone.

These results would lead us to conclude that greater variation in the death rate occurred within each district of country (A) than from district to district; that the variation was about the same throughout country (B); and that greater variation occurred from district to district in country (C) than within each district.

It has been found from experience that most statistical series obtained in practice are Lexian and it is obvious that this type of series is the one which it is most desirable to avoid, since the value of the basic ratio or probability obtained from any set in such a case is very apt to prove far from representative of all sets.

In the derivations of the formulas considered in this paper the case in which the value of the basic probability varies both within each set and also from set to set was ignored but it will be observed at once that this is the case which occurs most frequently, if not universally, in practice and our problem consists in practice in determining where the greatest variation occurs—within the individual sets or from set to set.

Problems will rarely be found in practice which will involve the exact conditions which are assumed in the derivations of the formulas considered in this paper. The condition which it will be most difficult to find is that of equality of number of observations in each set. It is easy to set up any number of examples based on games of chance whose conditions would be ideal and which would yield satisfactory checks upon the formulas derived here but it would be a practical impossibility to find such examples in practice; for example, it would be practically impossible to find quite a number of districts all of the same size of population as assumed in the illustration given above. Extensions of the theory considered here have been made[1] to cover this probable inequality of sets and many other situations but it is the purpose of this paper to cover merely the fundamental principles of a theory which the writer considers so important and yet so unfamiliar in this country as to merit a simpler treatment than any which he has found in print.

QUESTIONS AND DISCUSSIONS.

Edited by C. F. Gummer, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. Some Limit Proofs in Solid Geometry.

By W. R. Longley, Yale University.

1. Introduction. The proofs of the formulas for the volume of a cone and the volume of a sphere involve limiting processes which have always been difficult

[1] Fisher, *loc. cit.*, pp. 149 ff.

for students of elementary geometry, and usually the teacher dismisses them with an informal explanation. The proofs given in this paper are new in the field of elementary geometry, although they appear to be more direct and natural than the traditional methods. They have been tried out a number of times with college freshmen and are well within the grasp of the majority. It is not unlikely that high school students, also, may find that this treatment is illuminating.

The method is that of the integral calculus and it possesses certain theoretical advantages which are important. In the first place the method is general. The same procedure is followed for any pyramid, for any cone, and for a sphere. It introduces at an early stage of mathematical training the fundamental summation idea of the integral calculus without first developing the machinery of integration. The student who will continue the study of mathematics cannot meet this idea too often or in too many different forms. The pupil who will never study the calculus gets a brief glimpse of a powerful general method which has practical applications in certain approximate formulas for computation.

The underlying concept of dividing a solid by parallel section planes is used in most of the current texts in connection with a triangular pyramid but it is quite worth while to develop this concept a little more because of its fundamental character. As applied to the sphere this concept seems much simpler than the usual one based upon spherical sectors. It is also advantageous to use the same concept in connection with the sphere that is used with the pyramid and cone.

The method involves an algebraic formula which may very properly be taken into geometry without proof. But the proof can be made by mathematical induction and here again is an opportunity for the student to get a glimpse of a powerful general method which may never come to his notice again.

Assuming that the volume of a prism and of a cylinder are known, the only theorems to which direct reference must be made are the following:

THEOREM A. *If two straight lines are cut by three or more parallel planes, the corresponding segments are proportional.*

THEOREM B. *The area of a section of a pyramid, or cone, parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid, or cone.*

THEOREM C. *The sum of the squares of the integers from* 1 *to* n *is given by*

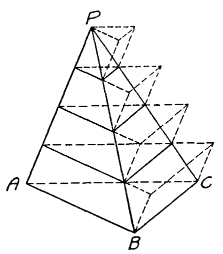
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\cdot$$

2. The volume of a pyramid. The figures represent a triangular pyramid, but the discussion applies to a pyramid having any polygon for a base.

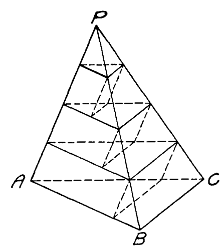
Suppose a lateral edge, AP, of a pyramid (Fig. a) is divided into a certain number (four in the figure) of equal parts, and that planes parallel to the base are passed through each point of division, including the vertex. These parallel planes cut out sections similar to the base and divide the altitude into the same number of equal parts as the edge AP. Suppose that a prism, with lateral edges parallel to AP and altitude equal to the distance between consecutive parallel planes, is

constructed on each section as lower base. The prisms include the pyramid completely and lie partly outside of it. These prisms form a set of circumscribed prisms.

Suppose that a prism (Fig. *b*), with lateral edges parallel to AP and altitude equal to the distance between consecutive parallel planes, is constructed on each section as upper base. These prisms lie entirely within the pyramid and form a set of inscribed prisms.



(*a*)



(*b*)

The total volume of a set of circumscribed prisms is the sum of the volumes of the separate prisms. It is obvious that the total volume of a set of circumscribed prisms is greater than the volume of the pyramid.

Suppose now that AP is divided into twice as many equal parts as before so that new section planes are passed midway between the original ones, and that a new set of circumscribed prisms is formed as before. It appears that the new set of prisms will be entirely within the old set. Hence the total volume of the new set will be less than the total volume of the old set, but will be greater than the volume of the pyramid.

It appears that, as the number of prisms is increased, the total volume of the set of circumscribed prisms will decrease, but will always be greater than the volume of the pyramid.

Similarly, it appears that, as the number of prisms is increased, the total volume of the set of inscribed prisms will increase, but will always be less than the volume of the pyramid.

Referring to the figures and counting down from the top, it is apparent that the first prism of the inscribed set is equal to the first prism of the circumscribed set. This is true also of the second prisms, third prisms, etc., until all the prisms of the inscribed set have been exhausted. There will be left only the lowest prism of the circumscribed set without any counterpart in the inscribed set. Hence for a given number of divisions of the edge AP the total volume of the set of circumscribed prisms exceeds the total volume of the set of inscribed prisms by the volume of the lowest prism of the circumscribed set. As the number of divisions of AP is increased, the altitude of each prism is decreased; hence the volume of the lowest prism of the circumscribed set is decreased, and can be made as small as we wish by making the number of divisions of AP sufficiently large. Hence the difference between the total volume of the set of circumscribed prisms

and the total volume of the set of inscribed prisms can be made as small as we wish. In other words, as the number of divisions of AP is increased, the total volume of each set of prisms approaches the same limiting value.

The preceding discussion justifies the following definition:

Definition. The volume of any pyramid is the limit approached by the total volume of a set of circumscribed (or inscribed) prisms as the number of prisms is increased indefinitely.

Theorem. The volume of a pyramid is equal to one third the product of the area of the base and the altitude.

Given the pyramid P–ABC (Fig. a). Denote the volume by V, the area of the base by b, and the altitude by h.

To prove $V = bh/3$.

Proof. 1. Suppose the edge AP to be divided into n equal parts by passing planes parallel to the base. These section planes divide the altitude into n equal parts. (Th. A.)

2. Suppose a set of circumscribed prisms to be constructed and let the prisms be numbered 1, 2, 3, $\cdots$, n from the vertex downwards. Then the altitude of each prism is h/n.

3. Let b_1 denote the area of the base of the first prism. The distance from the vertex to the base of the first prism is h/n. Hence

$$\frac{b_1}{b} = \frac{(h/n)^2}{h^2}, \qquad \text{or} \qquad b_1 = \frac{b}{n^2}. \qquad \text{(Th. } B.\text{)}$$

4. The volume V_1 of the first prism is

$$V_1 = b_1 \times \frac{h}{n} = \frac{bh}{n^3}.$$

5. The distance from the vertex to the base of the second prism is $2h/n$. Hence, with similar notation,

$$\frac{b_2}{b} = \frac{(2h/n)^2}{h^2}, \qquad \text{or} \qquad b_2 = \frac{4b}{n^2}, \qquad \text{and} \qquad V_2 = \frac{4bh}{n^3}.$$

6. Proceeding in this way,

$$V_3 = \frac{9bh}{n^3}, \qquad V_4 = \frac{16bh}{n^3}, \qquad \cdots \qquad V_n = \frac{n^2bh}{n^3}.$$

7. Let V_c denote the total volume of the set of circumscribed prisms. Then

$$V_c = \frac{bh}{n^3} + \frac{4bh}{n^3} + \frac{9bh}{n^3} + \cdots + \frac{n^2bh}{n^3} = \frac{bh}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2].$$

Hence

$$V_c = bh\left[\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right]. \qquad \text{(Th. } C.\text{)}$$

8. As n is increased, the terms $1/2n$ and $1/6n^2$ are decreased and approach zero as a limit, and V_c approaches the value $bh/3$ as a limit.

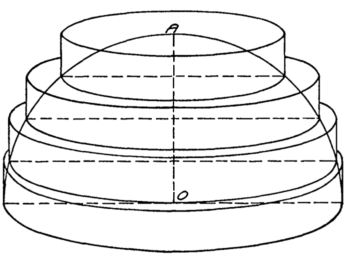
9. Hence $V = bh/3$. (Def.)

The preceding result can be obtained also by using inscribed instead of circumscribed prisms. If V_i denotes the total volume of a set of inscribed prisms obtained by dividing the altitude into n equal parts, it can be shown that

$$V_i = bh\left[\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right].$$

3. The volume of a sphere. To obtain a formula for the volume of a sphere, an argument analogous to that given for a pyramid may be used.

Consider a hemisphere and suppose the radius OA, perpendicular to the base, is divided into a certain number (four in the figure) of equal parts and that planes parallel to the base are passed through each point of division, including the point A. These parallel planes cut out circular sections. Suppose a right circular cylinder with altitude equal to the distance between consecutive parallel planes is constructed on each section as lower base. These cylinders, which include the hemisphere completely and lie partly outside of it, will be called a set of circumscribed cylinders. The total volume of a set of circumscribed cylinders is the sum of the volumes of the separate cylinders. It is obvious that the total volume of a set of circumscribed cylinders is greater than the volume of the hemisphere. Suppose now that OA is divided into twice as many equal parts as before so that new section planes are passed midway between the original ones, and that a set of circumscribed cylinders is formed as before. It appears that the new set of cylinders will lie entirely within the old set. Hence the volume of the new set will be less than the volume of the old set, but will be greater than the volume of the hemisphere. It appears that as the number of cylinders is increased, the volume of the set of circumscribed cylinders will decrease and become more and more nearly equal to the volume of the hemisphere.



As in the discussion of the volume of a pyramid, a set of inscribed cylinders may be formed by constructing cylinders on each section as upper base. The total volume of a set of inscribed cylinders is always less than the volume of the hemisphere and increases as the number of cylinders is increased. The difference between the total volumes of the set of circumscribed cylinders and the set of inscribed cylinders approaches zero as the number of cylinders is increased indefinitely, and the limiting value of the total volume of either set is the volume of the hemisphere.

The preceding discussion justifies the following definition:

Definition. The volume of a hemisphere is the limit approached by the total volume of a set of circumscribed cylinders as the number of cylinders is increased indefinitely.

Theorem. The volume of a hemisphere of radius R is $\frac{2}{3}\pi R^3$.

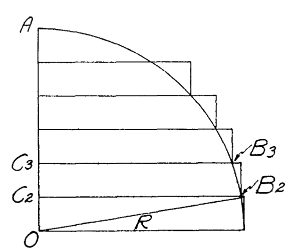
Given a hemisphere of radius R and volume V. To prove $V = \frac{2}{3}\pi R^3$.

Proof. 1. Suppose the radius OA to be divided into n equal parts by passing planes parallel to the base.

2. Suppose a set of circumscribed cylinders to be constructed and let the cylinders be numbered 1, 2, 3, $\cdots$, n from the base upwards. The altitude of each cylinder is R/n.

3. The radius of the first cylinder is R and its volume V_1 is

$$V_1 = \pi R^2 \times \frac{R}{n} = \pi\frac{R^3}{n}.$$



4. Let r_2 ($= C_2B_2$) denote the radius of the second cylinder. From the figure

$$r_2^2 = R^2 - (OC_2)^2 = R^2 - \left(\frac{R}{n}\right)^2 = R^2\left(1 - \frac{1}{n^2}\right).$$

Hence the volume V_2 of the second cylinder is

$$V_2 = \pi r_2^2 \times \frac{R}{n} = \pi\frac{R^3}{n}\left(1 - \frac{1}{n^2}\right).$$

5. The radius r_3 ($= C_3B_3$) of the third cylinder is given by

$$r_3^2 = R^2 - (OC_3)^2 = R^2 - \left(\frac{2R}{n}\right)^2 = R^2\left(1 - \frac{4}{n^2}\right).$$

Hence the volume V_3 of the third cylinder is

$$V_3 = \pi r_3^2 \times \frac{R}{n} = \pi\frac{R^3}{n}\left(1 - \frac{4}{n^2}\right).$$

6. In a similar way the volume of each of the n circumscribed cylinders may be calculated.

7. Let V_c denote the total volume of the set of n circumscribed cylinders. Then

$$V_c = \pi\frac{R^3}{n} + \pi\frac{R^3}{n}\left(1 - \frac{1}{n^2}\right) + \pi\frac{R^3}{n}\left(1 - \frac{4}{n^2}\right) + \cdots + \pi\frac{R^3}{n}\left(1 - \frac{(n-1)^2}{n^2}\right).$$

Hence, by collecting terms,

$$V_c = \pi\frac{R^3}{n}\left[n - \frac{1^2 + 2^2 + \cdots + (n-1)^2}{n^2}\right].$$

8. The sum of the squares of the integers from 1 to $n - 1$ is

$$1^2 + 2^2 + \cdots + (n-1)^2 = \frac{(n-1)(n)(2n-1)}{6} = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}. \quad \text{(Th. } C.)$$

Hence

$$V_c = \pi R^3\left[1 - \frac{1}{3} + \frac{1}{2n} - \frac{1}{6n^2}\right] = \pi R^3\left[\frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}\right].$$

9. As n is increased, the terms $1/2n$ and $1/6n^2$ are decreased and approach zero as a limit, and V_c approaches $\frac{2}{3}\pi R^3$ as a limit. Hence the volume of the hemisphere is $\frac{2}{3}\pi R^3$. (Def.)

Corollary. The volume of a sphere of radius R is given by

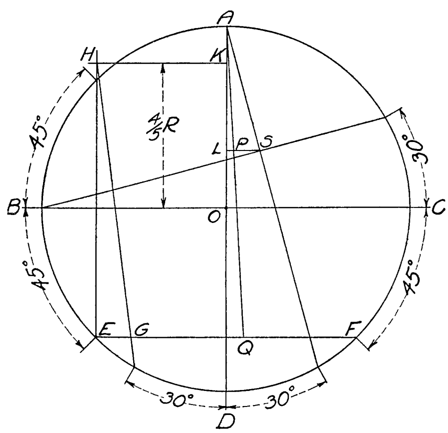
$$V = \frac{4}{3}\pi R^3.$$

II. An Approximate Construction of the Side of a Regular Inscribed Nonagon.

By T. R. Running, University of Michigan.

In the figure AP is a remarkably close approximation to the side of a regular inscribed nonagon. Taking the radius of the circle as unity, and using nine decimal places,

$$AP = .684040255,$$
$$\text{Correct length of side} = .684040286.$$



In the figure R represents the radius of the circle. The two diameters AD and BC are perpendicular. LS is drawn parallel to BC. Q bisects the line GF. AQ intersects LS in P. AP is the approximate length of a regular inscribed polygon of nine sides.

If the construction could be carried out for a circle the size of the earth's equator and AP laid off as a chord nine times, the terminal point would fall short of the initial point by about 6.3 ft., or approximately an arc which measures at the center an angle of $0''.062$.

RECENT PUBLICATIONS.

Edited by D. C. Gillespie, Cornell University, to whom communications should be sent.

REVIEWS.

Einführung in die Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage. By L. Schlesinger. Third Edition. Berlin, Vereinigung Wissenschaftlicher Verleger, Walter de Gruyter and Co., 1922.

Among the applications of the theory of functions of a complex variable in analysis, no doubt that to the theory of differential equations is one of the most interesting. The subject was at its height about a quarter or a half of a century ago. Since then, as the author remarks in the preface to the book under review,

there has been practically a cessation of effort in the development of this theory. In the hopes of inspiring new interest in the subject and indicating some lines of research, this revision of the first edition, which appeared in 1900, has been made.

The first chapter takes up the existence theorems for differential equations in the field of real variables. The example of the motion of a simple pendulum leads to the mention of the desirability of treating differential equations in the field of the complex variable, and the remainder of the book is limited to this field. The second chapter contains a consideration of the existence of analytic solutions of differential equations of the first order, leading up to the question of singularities. Interest centers upon differential equations with fixed singularities, with consequent discussion of the Riccati equation and the linear differential equation of the first order. The third chapter treats of differential equations of the first order in which the derivative is given as an implicit algebraic function of the independent variable and again the objective is the consideration of differential equations with fixed singularities, which is carried out in detail in the succeeding chapter. There is much of interest in this latter chapter, especially in the closing sections which give a brief historical resumé of the material considered, and some indications of possible future developments.

The remainder of the book is devoted to differential equations of higher order, and is limited largely to linear differential equations and their singularities. Existence theorems are proved on the basis of the matricial idea, which is of importance in the question of generalizations, and this idea is brought in wherever feasible. The question of singularities and regular solutions leads naturally to the Fuchsian equations, and their discussion, including the Gaussian differential equation and the hypergeometric functions. There is some discussion of the nature of the solutions of differential equations in the neighborhood of non-regular points of the solutions. The concluding chapter hints at generalizations—for instance in the field of integro-differential equations—and points out some of the directions in which further research in the field considered in the book might be made.

The judging of a book ought, no doubt, to be done on the basis of objects specified in the preface and title. As an "introduction" to the subject, the book seems to the reviewer to be a little advanced. To be sure there are examples illustrating the theory as the preface would lead one to expect, but they do not appear in as great numbers as they might for an elementary understanding of the subject matter. Indications for futher research are made at two points. In view of the avowed purpose of the book, indications and suggestions for further investigation might have been given more frequently.

On the other hand, the method of approach underlying the volume, that of making the fixed singularities of the differential equations a guide for the development, seems a very happy one. Throughout the book one feels that the author has lent interest by the variety of attack on the problems considered. There is much of value to him who wants assistance in gaining a better working knowledge of the theory of analytic functions and much that will repay careful study.

T. H. HILDEBRANDT.

The Outline of Radio. By J. V. L. HOGAN. Boston, Little, Brown and Co., 1923. 256 pages. Price $2.00.

The striking achievements in the field of radio transmission of intelligence and the simplicity of the apparatus which will enable one to receive any message which may be passing have resulted in an enormous general interest in the nature of the processes which are involved. Mr. Hogan's little book aims to bring an understanding of the nature of radio processes to the average intelligent reader, whether or not he has been trained to the understanding of the physical laws of natural phenomena. For those who have no scientific training it will have the appeal of an easy style and a beginning, under each topic, with facts and laws which are matters of daily experience. Nor will they find it difficult to follow the author's explanations of the less familiar natural phenomena involved in the somewhat extended chain of transformations going to make up the complete radio message. Each of the links in this chain is taken up in its turn, and its nature explained by analogy and comparison with other more familiar phenomena. So far back does the author go in some cases in his purpose to enable the uninitiated to pick up the thread, that he occasionally approaches the metaphysical, as for example, in his comments on the nature of intelligence and that of the ether, with a result, however, altogether helpful to his purpose.

To the scientist in other fields the book also makes its appeal by reason of its complete assembly of all the elements of the radio chain, and particularly by its presentation of the numerical limits of the various quantities involved, and their peculiarities within the different ranges. The discussion of frequency and wave length is especially happy, taking the reader over the entire range of the values of each to be met in nature, from ultra-violet light at one end to the shortest sound waves on the other.

The easy style enlivens the historical review, as well as all that follows it, and a well-chosen series of diagrams also does much to help the unaccustomed reader on his way to an understanding of the nature of radio. Mr. Hogan reveals himself a good physicist and a good teacher. His book should prove a popular response to the present widespread demand for general knowledge of the nature and possibilities of radio.

J. B. WHITEHEAD.

Ausgleichsrechnung nach der Methode der kleinsten Quadrate. By V. HAPPACH. Leipzig and Berlin, B. G. Teubner, 1923. 8vo. 76 pages. Price $0.38.

This short treatise, Band 18 of Teubner's "*Technische Leitfäden*" written by "Ingenieur Vollrat Happach," gives the formulas needed for the practical use of the method of least squares. No attempt is made to explain the theory of probability, upon which they are grounded; comparatively few proofs are given. It is shown, however, that the least square hypothesis leads to the arithmetic mean and to the weighted mean of the measurements.

Most of the errors or misprints noticed appear in Sections 7 and 8; but some of these, like the occurrence of 2 as a subscript instead of as an exponent, in line 5 of page 23, would cause no trouble.

Problems are not assigned. However, sixty-three illustrative examples are worked out in considerable detail, showing forms for computation, and this is, indeed, one of the most attractive features of the book. The reader should get a good working knowledge of least squares, with its many applications.

E. L. Dodd.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Annals of Mathematics, second series, volume 24, no. 2, December, 1922: "Spherical representation of conjugate systems and asymptotic lines" by W. C. Graustein, 89–98; "On a short method of least squares" by B. H. Camp, 99–108; "On the convergence of the Sturm-Liouville series" by J. L. Walsh, 109–120; "The functional equation $g(x^2) = 2ax + [g(x)]^2$" by J. H. M. Wedderburn, 121–140; "On convergence factors in triple series and the triple Fourier's series" by B. M. Eversull, 141–166; "On the minimizing of a class of definite integrals" by P. R. Rider, 167–174; "A Pythagorean functional equation" by E. Hille, 175–180—No. 3, March, 1923: "On the potential of a homogeneous spherical cap, of a magnetic shell, and of a steady circular current" by C. De Jans, 181–208; "On cyclic harmonic curves" by H. Hilton, 209–212; "Multiple integrals in n-space" by P. Franklin, 213–226; "On symmetric forms in n variables" by A. Dresden, 227–236; "Algebraic fields" by J. H. M. Wedderburn, 237–264; "A theorem concerning certain unit matrices with integer elements" by H. R. Brahana, 265–270.

Journal of Washington Academy of Science, volume 13, no. 10, May 19, 1923: 'The reduction of all physical dimensions to those of space and time" by A. P. Mathews, 195–210.

Mathematics Teacher, volume 16, no. 5, May, 1923: "Empirical theorems in Diophantine analysis" by R. D. Carmichael, 257–265; "Pennsylvania State course of study in mathematics" by J. A. Foberg, 266–273; "Fate and Freedom" by A. Korzybski, 274–290; "Mechanics" by G. R. Mirick, 291–294; "Varieties of minus signs" by F. Cajori, 295–301; "Correlation of mathematical subjects develops mathematical power" by C. A. Stone, 302–310.

Philosophical Magazine, volume 46, no. 276, December, 1923: "Electronic conduction in metals" by A. Bramley, 1053–1073.

School Science and Mathematics, volume 24, no. 1, January, 1924: "Stereoscopic harmonic curves" by W. F. Rigge, 29–35; "A thread of mathematical history" by Z. Ferguson, 37–44.

Science, volume 59, no. 1514, January 4, 1924: "American mathematics during three quarters of a century" by G. A. Miller, 1–6. January 11, 1924: "Mathematics and geophysics" by W. D. Lambert, 30–31.

Scientific American, volume 130, no. 2, February, 1924: "The Unity of Mathematics," editorial, 84.

Transactions of the American Mathematical Society, voume 25, no. 2, April, 1923: "Developments associated with a boundary problem not linear in the parameter" by R. E. Langer, 155–172; "Invariant points of a surface transformation of given class" by J. W. Alexander, 173–184; "Applications of analysis to the arithmetic of higher forms" by E. T. Bell, 185–189; "On the second derivatives of an extremal-integral with an application to a problem with variable end points" (Supplementary paper) by A. Dresden, 190–192; "Abstract group definitions and applications" by W. E. Edington, 193–210; "On the integrals of elementary functions" by J. F. Ritt, 211–222; "Invariants of the linear group modulo P^k" by M. M. Feldstein, 223–239; "On certain polar curves with their application to the location of the roots of the derivatives of a rational function" by B. Z. Linfield, 239–258; "Orthogonal systems of hypersurfaces in a general Riemann space" by L. P. Eisenhart, 259–280; "Ruled surfaces with generators in one-to-one correspondence" by E. P. Lane, 281–296; "Symmetric tensors of the second order whose first covariant derivatives are zero" by L. P. Eisenhart, 297–301.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed in the MONTHLY. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3070. Proposed by J. L. RILEY, Tarleton Station, Texas.

A triangle circumscribes the circle $x^2 + y^2 = r^2$ and two of its vertices lie on the circle $(x - a)^2 + y^2 = R^2$; find the condition necessary that the third vertex may also lie on this circle.

3071. Proposed by E. L. POST, Cornell University.

Prove that, when $|h| < 1/\epsilon$,

$$1 + \frac{(x+h)}{1!} + \frac{(x+2h)^2}{2!} + \cdots + \frac{(x+nh)^n}{n!} + \cdots = c\epsilon^{mx},$$

where both c and m are independent of x, and the latter satisfies the equation $\epsilon^{mh} = m$. Note: Under the given conditions both members of the first equation considered as functions of x directly satisfy the mixed difference equation $f'(x) = f(x + h)$. The identification of these two equations is therefore of some interest.

3072. Proposed by W. H. RASCHE, Virginia Polytechnic Institute.

If the four vertices, A, B, C, and D (taken in cyclic order), of a simple quadrangle are four points in a plane rigid body undergoing complanar motion, and if two opposite sides, as AB and DC, represent vectorially the accelerations of A and D respectively, then Clifford's point of the quadrangle is the center of acceleration of the rigid body.

Note: The Clifford point associated with four complanar straight lines is the point common to the circumcircles of the four triangles formed by omitting the lines in turn.

3073. Proposed by P. E. HEMKE, U. S. Naval Academy.

Evaluate the integral:

$$\int_{\omega}^{\omega'} \log\left[\frac{(pu - pu_1)}{(pu - pv)}\right]\frac{(pu - pu_1)(pu - pu_2)du}{(pu - pv)^2},$$

where pu is here the Weierstrassian function, ω, ω' and $\omega'' = \omega + \omega'$ (ω real and ω' pure imaginary) are its half periods in the usual notation, u_1, u_2, and v are known and are represented in the Argand diagram by points on the segments $\omega\omega''$, $O\omega'$, $O\omega$, respectively.

3074. Proposed by J. H. MURPHY, Pittsburgh, Pa.

Two circles of radii, R_1, R_2, intersect. A third circle passes through their points of intersection. Find the radius of this third circle, when the sum of the areas of the two crescents cut from it by the other two circles is a maximum.

3075. Proposed by E. T. BELL, University of Washington.

What becomes of Achilles and the tortoise in a time and space which are both discrete (quantised space-time)?

SOLUTIONS.

3000 [1923, 41]. Proposed by J. ROSENBAUM, Milford, Conn.

With use of compasses alone, to construct a circle whose area shall be n times the area of a given circle, where n is any positive integer.

SOLUTION BY H. HALPERIN, Agri. & Mech. College of Texas.

Let r and R represent the radii of the given and the required circles, respectively. Then $\pi R^2 = n\pi r^2$, and $R = \sqrt{n}r$. First of all, we know how to construct, with compasses alone, the vertices of an equilateral triangle of side r. This being the case, we construct the vertices of such a triangle $A_1A_2B_1$, then the third vertex B_2 of a second equilateral triangle $A_2B_1B_2$, then the third vertex A_3 of a third equilateral triangle $A_2B_2A_3$, and so on. In this way we obtain as many points $A_1, A_2, A_3, \cdots, A_n$ as we please, which are easily seen to lie on a straight line such that $A_1A_n = (n-1)r$.

If now n is even we take $n-1$ of the A points $A_1, A_2, \cdots, A_{n-1}$; and with A_1 and A_{n-1} as centers and radii $A_1A_{(n/2)+1} = A_{n-1}A_{(n/2)-1}$ we describe arcs of circles intersecting at M, which must be on the perpendicular to A_1A_{n-1} at $A_{n/2}$. Then from the similarity of the triangles $A_1A_{(n/2)+1}M$ and $MA_{(n/2)-1}A_{(n/2)+1}$ we have

$$MA_{(n/2)+1} = \sqrt{A_1A_{(n/2)+1} \times A_{(n/2)-1}A_{(n/2)+1}} = \sqrt{(n/2)r \times 2r} = \sqrt{n}r;$$

and hence $MA_{(n/2)+1}$ is the required radius.

If n is odd we take n points, and from A_1 and A_n as centers we describe arcs with radii $A_1A_{(n+3)/2} = A_nA_{(n-1)/2}$, meeting at M which lies on the perpendicular to A_1A_n at $A_{(n+1)/2}$. We have then $MA_{(n+1)/2} = \sqrt{n}r$, the radius of the required circle.

Also solved by MICHAEL GOLDBERG, ABIGAIL E. JOHNSON, L. E. LUNN, A. PELLETIER, and the PROPOSER.

3002 [1923, 41]. Proposed by C. N. MILLS, State Normal, Aberdeen, S. Dak.

The diagonals of any quadrilateral are in length a and b respectively and are inclined at an angle A. Show that the greatest rectangle which can be drawn with its four sides passing through the four corners of the quadrilateral is $\frac{1}{2}ab(1 + \sin A)$.

SOLUTION BY R. M. MATHEWS, Wesleyan University.

Through the ends of a draw parallels inclined at an angle B to it. Then parallels through the ends of b perpendicular to the first pair make a rectangle which we may say is circumscribed, in a general sense, to the quadrilateral. The altitude of the rectangle is $a \sin B$, and the base is $b \cos (B - A)$, by a proper choice of B. Therefore, the area is

$$\Delta = ab \sin B \cos (B - A) = \frac{ab}{2}[\sin (2B - A) + \sin A].$$

The maximum Δ is obviously given by setting $B = (90° + A)/2$; we thus obtain for this maximum

$$\Delta = \tfrac{1}{2}ab(1 + \sin A).$$

It is interesting to remark that the positions of the vertices of the quadrilateral on the diagonals have no effect on the directions of the sides of the maximum rectangle. These directions are found from the angle A by bisecting the angle found by adding 90° to A; this bisector and its perpendicular are the required directions.

Also solved by MICHAEL GOLDBERG, A. M. HARDING, H. HALPERIN, ABIGAIL E. JOHNSON, E. J. OGLESBY, A. PELLETIER, A. V. RICHARDSON, J. B. REYNOLDS, and HAZEL E. SCHOONMAKER.

3003 [1923, 41]. Proposed by R. M. MATHEWS, Wesleyan University.

The angle PAM rotates around A and meets a line rotating around B in P and M. When M moves along the perpendicular bisector of AB, the locus of P is an equilateral hyperbola of which the mid-point of AB is the center. Generalize.

SOLUTION BY THE PROPOSER.

The pencil at B described by BM is perspective with the pencil at A described by AM, and the latter is congruent to the pencil described by AP. Thus (BM) and (AP) are projective pencils and the locus of P is a conic.

When BA bisects angle MAP, then BM is parallel to AP; and this happens again when BM is advanced a right angle from that position. Thus the conic has two points at infinity in mutually orthogonal directions and must be an equilateral hyperbola.

When AM passes through B, AP is the corresponding ray to BA and hence is the tangent to the conic at A. When AP passes through B, its corresponding ray is the tangent to the conic at B, and it is easily seen that it must be parallel to the tangent at A. Hence AB is a diameter and its mid-point is the center.

A generalization may be made by replacing the straight line upon which M moves by a conic through A and B. We obtain in this way four conics through A and B, the loci of M, N, P, Q, respectively. See [*1923*, 78] III. *Discussion of 2903* where a similar proof of the first part is given.

This construction is a special case of Newton's method of generating conics: two constant angles A and B in the same plane rotate around A and B while the intersection M of two sides traverses a fixed line; then the intersection P of the other two sides will in general describe a conic through A and B.

It does not seem to have been remarked that the loci of N and Q, the other two intersections of the sides of the angles, are conics through A and B. The relations of the three conics P, N, and Q may be of interest.

Also solved by H. Halperin, William Hoover, A. Pelletier, J. B. Reynolds, A. V. Richardson, and M. Young.

3010 [1923, 76]. Proposed by F. D. MURNAGHAN, John Hopkins University.

Find an expression for the volume of the pedal tetrahedron, with respect to the tetrahedron of reference, of a point whose perpendicular distances from the sides are (x_1, x_2, x_3, x_4); from this expression show that the locus of points the feet of whose perpendiculars on the faces of the tetrahedron of reference are coplanar is Steiner's cubic surface

$$\frac{A_1}{x_1} + \frac{A_2}{x_2} + \frac{A_3}{x_3} + \frac{A_4}{x_4} = 0,$$

where the A's are the areas of the faces of the tetrahedron of reference.

Solution by Theodore Bennett, University of Illinois.

In Cartesian coördinates, let the faces of a tetrahedron be

$$p_i = a_ix + b_iy + c_iz + d_i = 0, \qquad i = 1, 2, 3, 4.$$

Let Δ be the determinant of this system of equations, and let α_i be the cofactor of a_i, β_i that of b_i, etc. If P_i be the vertex opposite the face p_i, we find that $P_i = \left(\frac{\alpha_i}{\delta_i}, \frac{\beta_i}{\delta_i}, \frac{\gamma_i}{\delta_i}\right)$; hence the volume of the tetrahedron is

$$V = \frac{\Delta^3}{6\delta_1\delta_2\delta_3\delta_4}\,.$$

If h_i be the altitude from the vertex P_i, we find that $h_i = \Delta/\delta_ir_i$, where $r_i = \sqrt{a_i^2 + b_i^2 + c_i^2}$. Knowing the volume, and the altitude from each vertex, we find the area of the face opposite P_i to be

$$A_i = \frac{\delta_ir_i\Delta^2}{2\delta_1\delta_2\delta_3\delta_4}\,.$$

In order to avoid any ambiguity of sign, we observe that the equations p_i can be written in such a manner that Δ is positive, and P_i on the positive side of p_i. Then h_i is positive, from the method of its formation, and hence δ_i is positive. Therefore V and A_i are also positive.

Now let $Q = (x', y', z')$ be a point whose distance from p_i is x_i, and let Q_i be the projection of Q upon p_i. The coördinates of Q_i are the values of x, y, z, obtained by solving the equations

$$\frac{x - x'}{\frac{a_i}{r_i}} = \frac{y - y'}{\frac{b_i}{r_i}} = \frac{z - z'}{\frac{c_i}{r_i}} = -x_i.$$

Hence,

$$Q_i = \left(x' - \frac{a_i x_i}{r_i},\quad y' - \frac{b_i x_i}{r_i},\quad z' - \frac{c_i x_i}{r_i} \right).$$

The volume of the tetrahedron formed by the points Q_i may be written as a determinant, from which we can eliminate x', y', z' at once, and write the result in the form

$$V' = -\frac{1}{6}\frac{x_1x_2x_3x_4}{r_1r_2r_3r_4}\begin{vmatrix} a_1 & b_1 & c_1 & r_1/x_1 \\ a_2 & b_2 & c_2 & r_2/x_2 \\ a_3 & b_3 & c_3 & r_3/x_3 \\ a_4 & b_4 & c_4 & r_4/x_4 \end{vmatrix}.$$

Expanding on the fourth column, and replacing $\delta_i r_i$ by $2A_i\delta_1\delta_2\delta_3\delta_4 \div \Delta^2$, this becomes

$$V' = -\frac{1}{3}\frac{\delta_1\delta_2\delta_3\delta_4 x_1x_2x_3x_4}{r_1r_2r_3r_4\Delta^2}\left(\frac{A_1}{x_1} + \frac{A_2}{x_2} + \frac{A_3}{x_3} + \frac{A_4}{x_4}\right).$$

Using the values of h_i and V as given before, we can eliminate r_i and δ_i from V', with the result that

$$V' = -\frac{h_1h_2h_3h_4x_1x_2x_3x_4}{108V^2}\left(\frac{A_1}{x_1} + \frac{A_2}{x_2} + \frac{A_3}{x_3} + \frac{A_4}{x_4}\right),$$

which gives V' in terms of the various dimensions of the tetrahedron of reference.

If the points Q_i are coplanar, $V' = 0$, whence Q must be on the surface

$$\frac{A_1}{x_1} + \frac{A_2}{x_2} + \frac{A_3}{x_3} + \frac{A_4}{x_4} = 0.$$

Also solved by W. J. MARTIN, and A. PELLETIER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

The program of the winter meeting of the Association of Teachers of Mathematics in Southern Massachusetts, held February 9, 1924, in Fall River, included the following papers: "The imaginary in geometry," by Professor W. C. GRAUSTEIN, of Harvard University; "Standardized tests in plane geometry," by Miss VERA SANFORD, of the Lincoln School of Columbia University, and "The course in freshman mathematics at Brown," by Professor R. W. BURGESS of Brown University.

At the request of the Commission on New Types of Examination of the College Entrance Examination Board, Professor L. P. EISENHART of Princeton University has formed a committee of mathematicians to examine critically certain statistical methods used in the investigations of the Commission. The other members of the committee are Professors R. W. BURGESS, W. L. CRUM, E. V. HUNTINGTON, H. H. MITCHELL, H. L. RIETZ, and J. H. M. WEDDERBURN.

Mr. P. L. EVANS, formerly instructor of mathematics in the Manhattan High School, has been appointed instructor of mathematics and engineering drawing at Baker University.

At the University of Chicago, Associate Professor A. C. LUNN has been promoted to a full professorship of mathematics. Dr. MAYME I. LOGSDON, instructor of mathematics, has recently been appointed to a deanship in the College of Science.

Dr. Pauline Sperry, instructor in mathematics at the University of California, has been promoted to an assistant professorship.

Dr. Ganesh Prasad, dean of the Faculty of Science in the Benares University, has been appointed Hardinge professor of higher mathematics in the Calcutta University, in succession to Professor C. E. Cullis.

At Cornell University, Assistant Professor Arthur Ranum has been promoted to a full professorship. Professor F. W. Owens has been granted leave of absence for the second semester of the year 1923–24, and will spend the time in Europe. Dr. E. L. Post, formerly of Columbia University, has been appointed instructor of mathematics for the second semester of the current year.

Mr. H. A. Simmons, instructor in mathematics in the University of Michigan, has been appointed assistant professor of mathematics at the University of Pittsburgh.

Associate Professor H. H. Dalaker, of the University of Minnesota, has been promoted to a full professorship of mathematics.

Professors G. D. Birkhoff, of Harvard University, and R. C. Archibald, of Brown University, will lecture at the University of California in the summer of 1924.

Professor H. F. Blichfeldt, of Stanford University, will lecture at Columbia University during the summer session.

Professor Earl Church, of the Pennsylvania Military College, has resigned to take charge of the computation and least squares adjustment of the geodetic survey of the Hawaiian Islands.

Professor Joseph Lipka, of the Massachusetts Institute of Technology, died January 15, 1924, at the age of forty. Professor Lipka, who secured his doctorate from Columbia in 1912, had been on the mathematical staff of the institute since 1908, with the rank of assistant professor since 1917.

Professor James Harkness, of McGill University, acting dean of the faculty of arts of that university, died December 7, 1923, at the age of fifty-nine years. Professor Harkness had been a member of the American Mathematical Society since 1891, and had held office as vice-president.

At Harvard University, the following have been appointed to part time instructorships of mathematics: Messrs. H. L. Garabedian, E. B. Ham, E. R. C. Miles, F. W. Perkins, T. L. Smith, and L. E. Ward. Professor W. P. Russell, of Pomona College, is at Harvard as visiting lecturer for the second semester of the current year.

Dr. Philip Franklin, now Benjamin Peirce teaching fellow at Harvard University, has been appointed instructor of mathematics at the Massachusetts Institute of Technology, 1924–1925.

Professor R. E. Wilson of Northwestern University died January 30, 1923, at the age of fifty-one. He had recently been appointed Dean of Men. Professor Wilson was a charter member of the Mathematical Association of America.

Correction—The International Congress of Mathematicians is to meet in Toronto, Canada, August 11–16, 1924, and not on the dates previously announced (*1923*, 458).

The following reports of Summer Sessions to be held in 1924 have been received.

University of California, Intersession, May 12 to June 21, and Summer Session, June 23 to Aug. 2. In addition to the usual courses in Algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses will be offered: *Intersession*: By Professor T. M. PUTNAM: Mathematical theory of investment; Solid analytic geometry; Theory of infinite series. By Professor F. IRWIN: Selected topics in the theory of equations. By Professor M. H. McDONALD: Differential equations. *Summer Session:* By Professor G. D. BIRKHOFF: Fourier's series and their application; Mechanics. By Professor R. C. ARCHIBALD: Elementary geometry for advanced students; History of elementary mathematics. By Professor M. W. HASKELL: Higher geometry; Functions of the complex variable. By Professor D. N. LEHMER: Advanced analytic geometry; Solid analytic geometry; Theory of infinite series.

University of Chicago, first term, June 16 to July 23; second term, July 24 to August 29. In addition to the usual courses in College algebra, Plane analytic geometry, and Calculus, the following advanced courses are announced: By Professor G. A. BLISS: Functions of a real variable; Thesis work in analysis. By Professor L. E. DICKSON: Theory of Numbers, I; Thesis work in number theory. By Professor H. E. SLAUGHT: Elliptic integrals; Differential equations. By Professor M. FRÉCHET: Theory of abstract sets; Theory of probability. By Professor E. T. BELL: General theory of numbers; Theory of equations. By Professor F. R. MOULTON: Functions of infinitely many variables; Analytic mechanics, II. By Professor E. P. LANE: Synthetic projective geometry. By Doctor MAYME I. LOGSDON, Introduction to higher algebra.

Cornell University, July 5 to August 15. By Professor W. L. G. WILLIAMS: Analysis. By Professor C. F. CRAIG: Projective geometry. The following reading and research courses are also offered: By Professor J. I. HUTCHINSON and Professor CRAIG: Functions of a complex variable. By Professor VIRGIL SNYDER: Algebraic geometry. By Professor F. R. SHARPE: Hydrodynamics and elasticity. By Professors D. C. GILLESPIE and W. H. HURWITZ: Analysis. By Professors W. B. CARVER and F. W. OWENS: Projective geometry. By Professor WILLIAMS: Algebraic invariants.

Harvard University, July 7 to August 16. Elementary courses are offered in Trigonometry, Analytic geometry, and Calculus.

University of Illinois, June 16–August 9. In addition to the usual courses in College algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses are offered: By Doctor H. A. BENDER: Advanced algebra. By Dr. C. C. CAMP: Differential equations. By Dr. E. E. LIBMAN: Vector calculus. By Professor E. J. TOWNSEND: Theory of integration. By Professor A. B. COBLE: Differential geometry.

University of Iowa, first term, June 5–July 18. In addition to courses in Algebra, Trigonometry, Analytic geometry, and Calculus, the following courses are offered: By Dr. ROSCOE WOODS: Theory of equations; Advanced coör-

dinate geometry. By Dr. J. O. OSBORN: Differential equations. By Dr. F. M. WEIDA: Elements of statistics. By Professor E. W. CHITTENDEN: Theory of functions. Second term, July 21–August 22. By Professor J. F. REILLY: Differential equations; Green's theorem with applications. By Dr. J. O. OSBORN: Projective geometry.

University of Michigan, June 23–August 15. By Professor W. B. FORD: Theory of functions of a complex variable; Advanced calculus. By Professor L. C. KARPINSKI: Teaching of algebra; History of mathematics. By Professor PETER FIELD: Analytic mechanics. By Professor T. R. RUNNING: Graphical methods. By Professor T. H. HILDEBRANDT: Theory of functions of a real variable. By Professor H. C. CARVER: Advanced mathematical theory of statistics; Finite differences. By Professor C. J. COE: Differential equations. By Professor NORMAN ANNING: Solid analytic geometry. By Mr. S. E. FIELD: Projective geometry. By Mr. W. A. JENKINS: Theory of probability.

University of Minnesota, first term, June 21–July 31; second term, August 1–September 5. The department of mathematics will offer an intensive course entitled: Selected topics in advanced mathematics. The topics for 1924 are: First term: By Professor DUNHAM JACKSON: Vector analysis. By Professor A. L. UNDERHILL: Differential equations. By Professor R. W. BRINK: Interpolation. Second term: By Professor W. L. HART (topic to be announced later).

University of Oklahoma, June 4 to July 29. By Professor S. W. REAVES: Differential geometry; Theory of equations. By Professor J. O. HASSLER: Analytic mechanics; Teachers course in mathematics.

University of Pennsylvania, July 7–August 16. In addition to the usual courses in Solid geometry, Trigonometry, College algebra, Analytic geometry, and Calculus, the following courses are offered: By Professor G. H. HAZLETT: Introduction to functions of a complex variable. By Professor H. H. MITCHELL: Theory of probability. By Professor J. R. KLINE: Elementary statistics; Point set theory.

Columbia University, July 7 to August 15. In addition to courses in Logarithms and Trigonometry, Solid geometry, College algebra, Analytic geometry, and Calculus, and a series of courses for teachers of secondary mathematics, the following advanced courses are offered: By Professor H. F. BLICHFELDT: Elementary exposition of selected topics in modern mathematics; Theory of groups of finite order. By Professor W. B. FITE: Theory of functions of a complex variable. By Professor J. F. RITT: Differential equations. By Professor G. A. PFEIFFER: Projective geometry.

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to D. C. GILLESPIE, Cayuga Heights, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

The following are dates of Section meetings of the Association in 1923 (unless otherwise specified):

ILLINOIS, Elgin, May 2–3, 1924

IOWA, Des Moines, November; Ames, April, 1924

KANSAS, Topeka, January 20

KENTUCKY, Center College, April, 1924

MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Annapolis, December 8, 1924

MINNESOTA, Northfield, May 19

MISSOURI, University of Missouri, Columbia, November 30–December 1, 1924

OHIO, Ohio State University, Columbus, March 30–31

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 1924

SOUTHEASTERN, University of Georgia, Athens, Ga., March 7–8, 1924

TEXAS, Fort Worth, November 30–December 1, 1924

MICHIGAN, Ann Arbor, April 3, 1924

Missing Numbers of the Monthly

Cash will be paid for certain single numbers as follows, up to a limited number of copies:
February, March, May or September, 1913; September, 1914; February, March, April or June, 1915; February or September, 1918—fifty cents; September, 1915— seventy-five cents; May, 1915—one dollar (See MONTHLY, March, 1921, p. 152)

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924
NUMBER 5, MAY

PUBLISHED BY THE ASSOCIATION,
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The fourteenth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held on Saturday, December 8, 1923, in McDowell Hall, St. John's College, Annapolis, Maryland. Members of the Association were guests at lunch of the faculties of the U. S. Naval Academy, the Naval Post Graduate School, and St. John's College.

There were forty-seven (47) present including the following members of the Association:

O. S. Adams, R. N. Ashmun, G. A. Bingley, C. C. Bramble, J. A. Bullard, P. Capron, G. R. Clements, A. Cohen, L. S. Dederick, A. Dillingham, H. English, J. B. Eppes, H. W. Ficken, D. M. Garrison, W. M. Hamilton, P. E. Hemke, W. D. Lambert, E. S. Mayer, L. T. Moore, F. Morley, F. D. Murnaghan, C. A. Nelson, E. C. Phillips, C. H. Rawlins, L. J. Reed, H. M. Robert, R. E. Root, W. F. Shenton, J. Tyler, E. W. Woolard.

The following papers were presented:

(1) "Mathematics and geophysics" by Mr. W. D. LAMBERT, U. S. Coast and Geodetic Survey;

(2) "Mathematics of meteorology" by Mr. E. W. WOOLARD, U. S. Weather Bureau;

(3) "The use of mathematics in naval construction" by Commander A. J. CHANTRY, Jr., U. S. N., Head of the Department of Mathematics, U. S. Naval Academy (by invitation);

(4) "The growth function" by Dr. L. J. REED, Johns Hopkins School of Hygiene;

(5) "Mathematics of a warped airplane wing" by Professor J. B. SCARBOROUGH, U. S. Naval Academy;

(6) "Variation on an old theme" by Professor FRANK MORLEY, Johns Hopkins University;

(7) "The differential operator" by Professor JOHN TYLER, U. S. Naval Academy;

(8) "A question in the theory of numbers" by Dr. RAINICH, Johns Hopkins University (by invitation);

(9) "The minimum distance problem for four points in space" by Dr. F. D. MURNAGHAN, Johns Hopkins University.

The following are the abstracts of most of the papers: No. 1 by Mr. Lambert appeared in *Science* for Jan. 11, 1924. An abstract of it has already appeared in the MONTHLY (*1923*, 410–411). No. 9 by Dr. Murnaghan was read by title only and will appear in full in the MONTHLY.

2. The singularly fundamental rôle played by mathematics in all domains of exact scientific thought, by virtue of which any science as it becomes increasingly

mature and perfect becomes correspondingly more and more mathematical in character, was illustrated in Mr. Woolard's paper by the history and present status of the science of meteorology. Meteorology proper is the study of the mechanics and thermodynamics of the earth's atmosphere. The problem before us is: Given complete observations describing the state of an extensive three-dimensional portion of the atmosphere, to determine from the laws of mathematical physics the conditions that will result at some given time in the future. The practical difficulties in the way of obtaining adequate observational data, and the mathematical difficulties involved in the theoretical treatment of the data, are such that weather forecasting is now, and must for many generations remain, largely empirical; it is hopeless to attempt the mathematical calculation of the coming weather in the way that the movements of the heavenly bodies are forecast. Nevertheless, any insight into the mechanism of atmospheric phenomena which we may obtain through advances in theoretical meteorology will sooner or later be of practical use in improving weather predictions, and it is not unreasonable to hope that meteorology may ultimately become a truly exact science. The supreme importance of theoretical investigations into meteorological phenomena was emphasized; the past history of meteorology was compared with that of other sciences; and brief reference was made to each of the more important contributions to mathematical meteorology, including the early work of Ferrel, Helmholtz, Guldberg and Mohn, Hertz, von Bezold, Oberbeck, Sprung, Margules, *et al.*, and the modern and contemporary work of Hildebrandsson, Teisserenc de Bort, Kobayasi, Ryd, Bjerknes, Taylor, Jeffreys, Shaw, Richardson, Fujiwara, etc.

The paper is to appear in full in a future number of the *Monthly Weather Review.*

3. Commander Chantry said in substance: "In no other physical science is there found a closer relationship with pure mathematics, nor a greater and immediate direct use of it, than in naval architecture. Men have been going upon the water in ships since early history, but only in recent years have they been able to claim at least comparative mastery of the waves. The greatest advance in this mastery dates from the time that the hull of a ship was conceived as a geometric body, and was accordingly submitted to mathematical analysis.

"One of the principal steps in the design is the determination of the *lines* of the ship, which are the intersections with the hull of the ship by three series of mutually perpendicular planes. These are shown by orthographic projection on three views. This furnishes the data for finding by approximate integration the volume of the displacement of the ship up to any water line, and the center of gravity of this volume. This, known as the center of buoyancy, is of extreme importance in stability calculations, and must be found in all three dimensions. A further beautiful application of mathematics arises in the determination of the metacentric radius. In these considerations of stability, and in weight calculations, much approximate calculation of areas, volumes, centers of gravity, and moments of inertia must be made.

"In strength calculations the device of the 'equivalent girder' is employed. This brings in the whole theory of the strength of beams, and involves a great deal of detailed computation. Thus the determination at each point to be tested of I, the moment of inertia of cross-sectional area of the equivalent girder, involves the moment of inertia of members cut by a section about their own axes, and then the determination of the moment of inertia of the whole section about the neutral axis of the ship as a whole, by means of formulæ involving transfer of axes.

"In the resistance and powering of ships we find extensive and complex use of mathematics. Such a study enables us to predetermine with great accuracy the power required for a certain ship, based on the resistance of a model towed in a tank.

"A combination of experiment and mathematical investigation has given us Joessel's formula, which gives the turning moment ensuing when the rudder is put over, in terms essentially of the angle of inclination of the rudder as independent variable. The usual methods of the calculus give us the angle of inclination for maximum turning moment, and hence we need only provide for throw of rudder slightly beyond that angle.

"The behavior of a ship in a seaway is of extreme importance, especially as regards her periods of roll and pitch. These periods of roll and pitch are connected with the metacentric height of the ship by an equation derived from consideration of the ship as a pendulum, suspended from the metacenter. This equation takes the form $T = C/\sqrt{GM}$, where T is the period of roll, C is a constant, and GM is the metacentric height. From this the effect on the period of roll due to an increment in metacentric height is easily computed. Such an increment in metacentric height is most easily given by weight changes on the vessel. Thus the ship operator who knows the science of his profession may at any time alter the behavior of his ship in this respect, within certain limits.

"The anti-rolling device possessing the most promise of practicability is that involving the gyroscopic principle. Certainly in this field, no progress is possible without exhaustive and complex mathematical treatment.

"The launching of a ship involves a neat problem on the inclined plane, including determination of velocity, acceleration and coefficient of friction.

"It has been possible to touch only a few of the high spots, and such as are of basic nature. Detailed ship calculations reveal a multitude of mathematical problems. Naval architecture is indeed applied mathematics. It is hoped that two impressions may remain from this presentation. The first is a realization of the enormous contribution that mathematicians are making in the progress of naval architecture. The second is that we of our calling realize this full well and tender you our deepest appreciation of your help to us, and wish you unstinted success in further developments."

5. In 1922, Dr. Max Munk of Washington published a brief outline of a powerful and elegant method for investigating the aerodynamical behavior of a warped airplane wing of elliptic plan form. Professor Scarborough gave a

somewhat detailed exposition of the mathematical theory underlying the method.

By applying complex function theory to the fluid motion in a plane perpendicular to the direction of flight, it was shown that for a wing of elliptic form the induced and effective angles of attack have the same ratio at all points along the wing span. This gives a simple linear relation between the geometric and effective angles of attack. To get the most general warping, the geometric angle of attack is expressed as a Fourier series. Then the effective angle of attack and the density of lift along the wing span can likewise be expressed in terms of the same series.

6. The curve of pursuit for a circle leads to a differential equation which is in the ordinary sense not integrable. The point of Professor Morley's note was that, when the man M describes a circle, and the path of the dog D is always at right angles to DM, the equation is integrable. The relative path is a Cartesian oval, and the actual path can be mapped on a line explicitly.

8. Dr. Rainich gave Filippov's proof of the proposition that if the numbers $x^2 + x + m < (4m - 1)/3$ are primes all the numbers of this form which do not exceed m^2 are primes. Then with the aid of the identities

$$a^2 + ab + mb^2 = b^2(x^2 + x + m) + (a - bx)(a + bx + b)$$

for $x = 0, 1, \cdots n$ it was shown that the numbers $a^2 + ab + mb^2$, with a and b relative primes, have no prime divisors $< 2n - 1$ which do not divide at least one of the numbers $x^2 + x + m \leq n^2 + n + m$. Putting $n = m - 1$ and supposing that the numbers $x^2 + x + m < m^2$ are primes, Frobenius's theorem results that under these conditions the numbers $a^2 + ab + mb^2 < m^2$ are primes.

HARRY ENGLISH, *Secretary-Treasurer.*

DETERMINANTS WHOSE ARRAYS ARE MAGIC SQUARES.

By J. E. TREVOR, Cornell University.

1. The General Magic Array. For the present purpose, a magic square shall be understood to be an array of n^2 numbers such that the sum of the elements of each row, of each column, and of each principal diagonal is the same number s. When no further conditions are imposed, the $n^2 - 2n$ arbitrary elements may conveniently be taken to be the elements remaining when a corner element and the opposite row and column are deleted, as for example the elements x_{ij} in the array:

$$\begin{array}{lllll}
a_{11} & x_{12} & \cdots & x_{1,\,n-1} & a_{1n} \\
x_{21} & x_{22} & \cdots & x_{2,\,n-1} & a_{2n} \\
\cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot \\
x_{n-1,\,1} & x_{n-1,\,2} & \cdots & x_{n-1,\,n-1} & a_{n-1,\,n} \\
a_{n1} & a_{n2} & \cdots & a_{n,\,n-1} & a_{nn}.
\end{array}$$

Let s_b be the sum of the elements in the array B obtained by deleting the first and last rows and columns of the square; let s_d' and s_d'' be the sums of the elements of the first and second principal diagonals, respectively, of B; and let c be the sum of the four corner elements of the square. Then

$$s_b = (n-2)s - (2s - c), \qquad s_d' + s_d'' = 2s - c.$$

Hence

$$(n-2)s = s_b + s_d' + s_d'', \tag{1}$$

which expresses s in terms of the $(n-2)^2$ elements of the array B.

To complete a magic square from $n^2 - 2n$ arbitrary elements arranged as stated, it is necessary to determine a corner element. The corner elements a_{11}, a_{1n}, a_{nn} are connected by the relations

$$\begin{aligned} a_{11} + a_{1n} &= s - \sum_{j=2}^{n-1} x_{1j} \\ a_{11} + a_{nn} &= s - s_d' \quad = s - (n-2)s + s_b + s_d'' && \text{By (1).} \\ a_{1n} + a_{nn} &= s - \sum_{i=2}^{n-1} a_{in} = s - \left[(n-2)s - \sum_{i=2}^{n-1} x_{i1} - s_b\right]. \end{aligned}$$

On subtracting the members of the last equation from the sum of those of the first two, we find

$$2a_{11} = s + s_d'' - \sum_{i=j=2}^{n-1} (x_{1j} + x_{i1}).$$

For the summation, which is the sum of the elements in the "fringe," let us write s_f. Then

$$2a_{11} = s + s_d'' - s_f. \tag{2}$$

We observe that a_{11}, unlike s, is a function of all the arbitrary elements. The magic square determined by the $n^2 - 2n$ arbitrary elements of the above array is readily completed by employment of (1) and (2).

To the last column of the determinant D whose array is a magic square let us add all the other columns, and then to the last row add all the other rows. Hereupon, factoring s out of the last row and column, we obtain

$$D = s^2 \begin{vmatrix} a_{11} & x_{12} & \cdots & x_{1,\,n-1} & 1 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ x_{n-1,\,1} & x_{n-1,\,2} & \cdots & x_{n-1,\,n-1} & 1 \\ 1 & 1 & \cdots & 1 & n/s \end{vmatrix}.$$

Let us write A for the determinant obtained by deleting the last row and column of D, and A_i' for the determinant obtained from A by replacing each element of its ith row by unity. Then, on expanding the above determinant with reference to the elements of its last column, the equation becomes

$$D = s^2\left(\frac{n}{s}A - \sum_{i=1}^{n-1} A_i'\right). \tag{3}$$

Let $\Delta^{(k)}$ denote the determinant formed from any given determinant Δ by adding an arbitrary constant k to each element of Δ. Then

$$A^{(k)} = A + k\sum_{i=1}^{n-1} A_i', \qquad \partial A^{(k)}/\partial k = \sum_{i=1}^{n-1} A_i'.$$

Hence (3) becomes

$$\frac{D}{s} = nA - s\frac{\partial A^{(k)}}{\partial k}\,. \tag{4}$$

In seeking to determine the effect of the addition of the arbitrary constant k to each independent variable in D, we observe, by (1) and (2), that such addition increases s by nk, and increases each dependent element of the array of D by k. And we note that the value $\sum A_i'$ of $\partial A^{(k)}/\partial k$ is unaffected. So, for the determinant $D^{(k)}$, the equation (4) is

$$\frac{D^{(k)}}{s+nk} = nA^{(k)} - (s+nk)\frac{\partial A^{(k)}}{\partial k}\,; \tag{5}$$

whence, differentiating,

$$\frac{\partial}{\partial k}\frac{D^{(k)}}{s+nk} = 0; \tag{5a}$$

i.e., the ratio differentiated is independent of k. This result establishes the following theorem.

THEOREM. *When D is a determinant, of order n, whose array is a magic square having the row-sum s, the ratio D/s is a function of the differences of the $n^2 - 2n$ independent variables x_{ij} that determine D.*

2. Odd-rowed Concentric Magic Arrays. Consider any odd-rowed magic square from which successive magic squares are obtained by successive removal of the bounding rows and columns. Let x_1 be the central element. Then the arbitrary fringe-elements X_{31}, x_{31} determine the corner element c_3 and the other dependent elements of the next square, of order three. Continued repetition of the process yields the following array, in which x_1 and the X_{ij} and x_{ij} are the $(n^2 - 2n + 3)$ arbitrary elements.

$$\begin{array}{ccccccc}
c_n & x_{n1} & x_{n2} & \cdot & \cdot & x_{n,\,n-2} & c_{n1} \\
X_{n,\,n-2} & \cdot & \cdot & \cdot & \cdot & \cdot & A_{n,\,n-2} \\
\cdot & \cdot & c_3 & x_{31} & c_{31} & \cdot & \cdot \\
\cdot & \cdot & X_{31} & x_1 & A_{31} & \cdot & \cdot \\
X_{n2} & \cdot & c_{32} & a_{31} & c_{33} & \cdot & A_{n2} \\
X_{n1} & \cdot & \cdot & \cdot & \cdot & \cdot & A_{n1} \\
c_{n2} & a_{n1} & a_{n2} & \cdot & \cdot & a_{n,\,n-2} & c_{n3}
\end{array}$$

By (1), $(n-2)s_n = s_b + s_d' + s_d''$, the row-sum s_n for the array of the nth order is given by $(n-2)s_n = (n-2)s_{n-2} + 2s_{n-2}$. Hence we have, successively,

$$s_n = \frac{n}{n-2}s_{n-2}, \qquad s_{n-2} = \frac{n-2}{n-4}s_{n-4}, \qquad \cdots, \qquad s_5 = \frac{5}{3}s_3, \qquad s_3 = \frac{3}{1}x_1;$$

and the above equation becomes

$$D/s_n = nA_n^{(-x_1)}. \tag{7}$$

We now proceed to replace the dependent variables in A_n by their values from (6a), (6b), (6c), and to form the determinant $A_n^{(-x_1)}$ by adding $-x_1$ to each element of A_n. To reduce the determinant so obtained we first add to the last row all the other rows except the first, whereby the last row becomes

$$\sum_{j=1}^{n-2} (X_{nj} - x_1) \quad 0 \quad 0 \quad \cdots \quad 0.$$

Hence, if C is the cofactor of the non-vanishing element of this row, the determinant is equal to $-\sum (X_{nj} - x_1)\cdot C$, the product being negative because $n-1$ is even. Next, in C, we add to the last column all the other columns, whereby the last column becomes

$$\sum_{j=1}^{n-2} (x_{nj} - x_1) \quad 0 \quad 0 \quad \cdots \quad 0.$$

So C is equal to $+\sum (x_{nj} - x_1)\cdot A_{n-2}^{(-x_1)}$, the product being positive because $n-2$ is odd. Hence

$$A_n^{(-x_1)} = -\sum_{j=1}^{n-2} (X_{nj} - x_1)\cdot \sum_{j=1}^{n-2} (x_{nj} - x_1)\cdot A_{n-2}^{(-x_1)}.$$

Thus each application of the foregoing process reverses the sign of the expression obtained. Continued repetition of the process yields a succession of similar

---

illustrated by the case $n = 5$, which may be written as follows:

$$\Delta_5 = \begin{vmatrix} Y_5 & x_{51} & x_{52} & x_{53} & y_5 \\ X_{53} & Y_3 & x_{31} & y_3 & -X_{53} \\ X_{52} & X_{31} & y_1 & -X_{31} & -X_{52} \\ X_{51} & -y_3 & -x_{31} & -Y_3 & -X_{51} \\ -y_5 & -x_{51} & -x_{52} & -x_{53} & -Y_5 \end{vmatrix}.$$

In reducing this determinant, to the last row add the first row, and to the last column add the first column. Expansion then gives

$$\Delta_5 = -(Y_5^2 - y_5^2)\Delta_3.$$

On beginning with the determinant Δ_n of order n, continued repetition of this operation yields

$$\Delta_n = (-1)^{(n-1)/2}\cdot y_1 \cdot \prod_{n=3}^{n} (Y_n^2 - y_n^2), \quad (n \text{ odd}),$$

which is the expansion in question.

When n is even, the two-rowed kernel of the array is

$$\begin{matrix} Y_2 & y_2 \\ -y_2 & -Y_2 \end{matrix}$$

and the expansion is

$$\Delta_n = (-1)^{n/2}\cdot \prod_{n=2}^{n} (Y_n^2 - y_n^2), \quad (n \text{ even}).$$

factors,

$$A_n^{(-x_1)} = (-1)^{(n-1)/2} \prod_{n=3}^{n} \left(\sum_{j=1}^{n-2} (X_{nj} - x_1) \cdot \sum_{j=1}^{n-2} (x_{nj} - x_1) \right).$$

On substituting this value in (7), which is $D = ns_n \cdot A_n^{(-x_1)} = n^2 x_1 A_n^{(-x_1)}$), we are enabled to state the following theorem.

THEOREM. *When D is a determinant, of order n, whose array is an odd-rowed concentric magic square having the row-sum s_n, the ratio D/ns_n or D/n^2x_1 is a product of linear factors such that*

$$D = (-1)^{(n-1)/2} \cdot n^2 x_1 \cdot \prod_{n=3}^{n} \left(\sum_{j=1}^{n-2} (X_{nj} - x_1) \cdot \sum_{j=1}^{n-2} (x_{nj} - x_1) \right), \quad (n \text{ odd}).$$

It should be noted that every magic square of order three is a concentric magic square, and hence that every determinant whose array is a three-rowed magic has the form

$$-3^2 x_1 (X_{31} - x_1)(x_{31} - x_1).$$

3. Even-rowed Concentric Magic Arrays. We shall consider three cases of even-rowed magic squares from which successive magic squares are obtained by successive removal of the bounding rows and columns. If the two-rowed kernel of the square is to be magic, it will be an array of identical elements; if the kernel is freed from the summation-condition on its diagonals, each diagonal will consist of identical elements; if the kernel is freed from all conditions, it will be an array of arbitrary elements. In these successive cases the kernels are

$$\begin{matrix} x_1x_1 & \quad x_1x_2 & \quad x_1x_2 \\ x_1x_1 & \quad x_2x_1 & \quad x_3x_4 \end{matrix}$$

In the first case, let the first and second rows of the kernel be x_2, a_2 and a_2, a_2 respectively.[1] The arbitrary fringe-elements X_{41}, X_{42}, x_{41}, x_{42} determine the corner element c_4 and the other dependent elements of the next square, of order four. Continued repetition of the process yields an array similar to that displayed in the preceding section, and in which x_2 and the X_{ij} and x_{ij} are the $(n^2 - 2n + 2)/2$ arbitrary elements. The process of deducing (6), $\cdots$, (6c) yields here the same equations, save that x_1 is everywhere replaced by x_2; and we again find $D/s_n = nA_n^{(-x_2)}$. On evaluating $A_n^{(-x_2)}$ as before, we find it to reduce to a product as in the preceding case, save that the last factor is .

$$\begin{vmatrix} c_4 - x_2 & x_{41} - x_2 & x_{42} - x_2 \\ X_{42} - x_2 & 0 & 0 \\ X_{41} - x_2 & 0 & 0 \end{vmatrix} = 0.$$

In this case, then, D vanishes identically.

In the second case we build up the concentric magics as before about the kernel $|x_1\, x_2,\, x_2\, x_1|$, obtaining an array in which x_1, x_2 and the X_{ij} and the x_{ij}

[1] Here, of course, it is meant that $a_2 = x_2$.

are the $(n^2 - 2n + 4)/2$ arbitrary elements. The process of deducing (6), $\cdots$, (6c) yields here the same equations for the dependent elements, save that x_1 is everywhere replaced by $(x_1 + x_2)/2$, and that the corner elements of the four-rowed square are

$$\begin{array}{l|l} c_4 = x_1 + 2x_2 - \frac{1}{2}(F_4 + f_4) & c_{42} = x_1 - \frac{1}{2}(F_4 - f_4) \\ c_{41} = x_1 + \frac{1}{2}(F_4 - f_4) & c_{43} = -x_1 + \frac{1}{2}(F_4 + f_4). \end{array}$$

In deducing the equation analogous to (6), we choose $k = -(x_1 + x_2)/2 = -\xi$, and so obtain

$$D/s_n = nA_n^{(-\xi)}.$$

On evaluating $A_n^{(-\xi)}$ in the same way as before, we find

$$A_n^{(-\xi)} = (-1)^{(n-2)/2}\cdot\frac{x_2 - x_1}{2}\cdot\prod_{n=4}^{n}\left(\sum_{j=1}^{n-2}(X_{nj} - \xi)\cdot\sum_{j=1}^{n-2}(x_{nj} - \xi)\right)\cdot$$

On substituting this value in the above equation,

$$D = ns_nA_n^{(-\xi)} = n^2\frac{x_1 + x_2}{2}A_n^{(-\xi)},$$

we are enabled to state the following theorem.

THEOREM. *When D is a determinant, of order n, whose array is an even-rowed concentric magic square having a two-rowed kernel of identical elements and the row-sum s, the value of D is zero. But when the kernel has the form $|x_1\,x_2, x_2\,x_1|$, the ratio D/ns_n or $2D/n^2(x_1 + x_2)$ is a product of linear factors such that*

$$D = (-1)^{(n-2)/2}\cdot n^2\frac{x_2^2 - x_1^2}{4}\cdot\prod_{n=4}^{n}\left(\sum_{j=1}^{n-2}(X_{nj} - \xi)\cdot\sum_{j=1}^{n-2}(x_{nj} - \xi)\right), \qquad (n \text{ even}),$$

where $\xi = (x_1 + x_2)/2$.

In the third case cited, in which the array of the determinant is a concentric magic square having a kernel of arbitrary elements, the procedure employed here leads to no simple formulation.

THE COCHLIOID.[1]

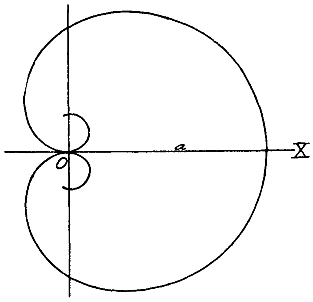
By ROSCOE WOODS, University of Iowa.

1. Introduction. The term "cochlioid" as applied to a curve is now used in a different sense from that employed by the ancients. It was the name given by Pappus to Nicomedes' conchoid which was used as a means of solving the famous problems of trisecting an angle and duplicating a cube. Also Apollonius of Perga called the quadratix of Dionostratus the sister of the cochlioid.[2] But in more recent times Benthem and Falkenburg[3] associated the name "Cochleoïde"

[1] Read before the Iowa Section of the Mathematical Association of America, April 28, 1923.
[2] P. Tannery, *Bulletin des Sciences Mathématiques*, 1883, p. 283.
[3] *Nieuw Archief voor Wiskunde*, t. X, 1883, p. 76.

with the curve whose equation in polar coördinates is $r\theta = a \sin \theta$, and Benthem notes that Catalan gave this equation[1] in 1857. In the *Philosophical Transactions of the Royal Society of London,* 1706, there is an anonymous discussion[2] of the equation $r(\pi - 2\omega) = 4a \cos \omega$. E. Wölffing has shown that this article was by J. Perks.

In the present article, the curve whose equation in polar coördinates is $r\theta = a \sin \theta$ will be designated as the cochlioid.[3] My interest in the curve began when I obtained it as the trace of a helical beam of electrons on a plane perpendicular to its axis. The data, described later in the paper, was furnished by Dr. C. J. Lapp. In what follows detailed discussion is given only in case of special interest. The graph and some of the properties of the curve are first exhibited. Then follows five methods of defining the curve. Finally there are two paragraphs on the tangents and normals.



2. **Graph.** The graph of the cochlioid is easily obtained from its equation. The curve is symmetrical to the polar axis. In each half of the plane we find an infinite number of non-intersecting ovals, each of which is tangent to the polar axis at the pole, O. The intercept on the polar axis is a. At this point the radius of curvature is $3a/4$. The intercepts (taken without regard to sign) on any radius vector except the polar axis form a harmonic divergent series. If the radius vector makes an angle φ with the polar axis, the intercepts are

$$\frac{a \sin \varphi}{\varphi}, \quad -\frac{a \sin \varphi}{\varphi + \pi}, \quad \frac{a \sin \varphi}{\varphi + 2\pi}, \quad \cdots.$$

The area swept over by the radius vector is

$$\frac{1}{2}\int_0^\theta r^2 d\theta = \frac{a^2}{2}\int_0^\theta \frac{\sin^2 \theta}{\theta^2}\, d\theta.$$

This integral cannot be expressed in terms of the elementary functions, but if θ varies from 0 to ∞, the limiting value of the area is $\pi a^2/4$,[4] or one fourth the area of a circle of radius a.

3. **Methods of defining the cochlioid.** There are several ways of defining the cochlioid. Some of the most important are as follows:

(a) The cochlioid is the locus of the center of gravity of a variable arc measured from a fixed point X on a circle of radius $OX = a$.[5]

---

[1] E. Catalan, *Manuel des Candidats à l'Ecole Polytechnique,* vol. 1, 1857, p. 331.

[2] Gino Loria, *Spezielle Algebraische und Transcendente Ebene Kurven,* 1902, pp. 418–424.

[3] It may be of interest to point out that this equation enjoys the distinction of having been given by the Oui-ja board. See Oliver Lodge, *The Survival of Man,* pp. 130-134.

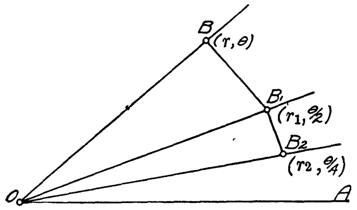
[4] Byerly, *Integral Calculus,* p. 105.

[5] E. Egger, *Ann. di Matem.* (1864), p. 21. Also L. Stoeckley, *Archiv der Math. u. Physik,* first series, 1868, p. 110. See also Haton de la Goupillière, *Comptes Rendus de l'Académie des Sciences,* 1906, p. 1130.

(*b*) If upon all the circles tangent to the x-axis at the origin O an arc of constant length is measured from O, the locus of the end point of this arc is the cochlioid.[1] Or, as Falkenburg has stated it:[2] Consider a variable isosceles triangle OVP whose legs are OV and PV. The vertex V moves on the Y-axis. O is the origin so that the leg OV coincides with the Y-axis. If PV makes an angle with the Y-axis that varies inversely as the distance OV, then the vertex P describes the cochlioid.

(*c*) In 1874, Gregor Fontana[3] proposed and solved this problem. Draw a line OB making an angle θ with a fixed line OA. At any point B on OB erect a perpendicular and let it cut the bisector of the angle θ in B_1. At B_1 erect a perpendicular and let it cut the bisector of $\theta/2$ in B_2, and so on. Continuing this process indefinitely, what is the limiting position of B_i on OA and upon what curve will the points B_i lie?

Let the coördinates of B be (r, θ) and those of B_i be $(r_i, \theta/2^i)$ for $i = 1, 2, 3, \cdots$. From the figure we then have $r_1 = r/\cos(\theta/2), \cdots, r_n = r_{n-1}/\cos(\theta/2^n)$. Hence $r_n = r/(\prod_1^n \cos\theta/2^i)$ and since we can write $\sin\theta = 2^n \sin(\theta/2^n)\prod_1^n \cos(\theta/2^i)$ we have, after rearranging the terms and substituting in the above, $r_\infty \sin\theta = r\theta$ for n approaching infinity. If the curve is required to pass through the point (a, φ), it then has the form $r\theta \sin\varphi = a\varphi \sin\theta$. If we let φ approach zero, we have the cochlioid in its simple form.



(*d*) The cochlioid appears as the solution of the following differential equation, $(x^2 + y^2)(xdy - ydx) - a(x^2 - y^2)dy + 2axydx = 0$. This equation is a special type of a more general differential equation studied by G. Fouret.[4] If a point on a right helicoid surface is luminous, Fouret showed that the projection, on a plane perpendicular to the axis of the helicoid, of the boundary of the shadow is a cochlioid. From this Brocard deduced that the cochlioid is the projection of a circular helix, from a point on the same, on a plane perpendicular to its axis.

(*e*) The trace of a helical beam of electrons on a plane perpendicular to its axis is the cochlioid.

If a fine stream of electrons is caught in a magnetic field, the paths of the electrons become helices on cylinders of radius r, where $r = mV_v/eH$, in which m is the mass of the electron, V_v its radial velocity, e its charge and H the strength of the magnetic field. If the velocity of the stream is uniform and equal to V and if the beam is projected at an angle φ with the horizontal, we have the following relations, $V_v = V \sin\varphi$ and $V_h = V\cos\varphi$. If a plate p, at a distance S

[1] A problem considered by Bernoulli and Goldbach in 1726.

[2] *Archiv der Math. u. Physik*, first series, 70, p. 257.

[3] *Mem. de la Soc. della Scienze*, 11 (1874).

[4] *Bulletin de la Société Mathématique*, t. 7, 1879, p. 199. See also H. Brocard, *Mathesis*, 1901, p. 109.

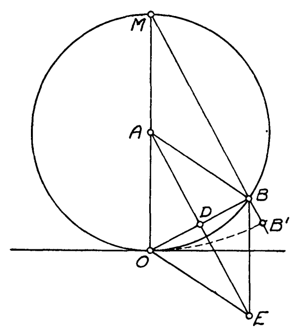
from the source, be inserted across the path of the helix, perpendicular to its axis, the coördinates of the position of the electron in p are (with the proper choice of the X-axis)

$$X = S, \qquad Y = r \sin \theta, \qquad Z = r(1 - \cos \theta), \tag{1}$$

where θ is the angle measured about the axis of the cylinder about which the electron rotates during its flight. Further we know that

$$T = S/V_h = (n + \lambda)t, \tag{2}$$

where T is the time required for the electron to traverse the distance S, t is the time of one convolution (being constant for electrons of any velocity in a given field) and $n + \lambda$ is the number of convolutions of the electron during its flight in which n is an integer and λ is a fraction. Since $\theta = (n + \lambda)2\pi$, θ and r can be eliminated from equations (1). If this result be thrown into polar coördinates, the equation of the trace of the electrons in p has the form $r = k \sin \pi\lambda/(n + \lambda)\pi$, where k is a constant.[1]



In this connection, it might be well to point out that the cochlioid is the inverse of the quadratix, $y = x \cot \pi x/2a$. It can therefore be used to square the circle. By means of Falkenburg's interpretation of Goldbach's problem, it can be used to divide an angle into any given number of equal parts. In fact Falkenburg discovered the cochlioid in connection with the design of the starting gear of a steam engine. In the adjacent figure, let $OA = r$, angle $OAB = \pi a/r$, E be the fourth vertex of the rhombus $OEBA$, and let D be the intersection of its diagonals. Then it can be proved that B and B' describe the same cochlioid if $MB' = MO$. The point E describes a hyperbolic spiral and D a second cochlioid. The variable line AB envelops a curve E' whose equation is[2]

$$\sin\left(\frac{ay + \sqrt{ax(x^2 - y^2 - ax)}}{x^2 + y^2}\right) = \frac{y\sqrt{ax} - x\sqrt{x^2 - y^2 - ax}}{x^2 + y^2}. \tag{3}$$

Furthermore the cochlioid is the pedal of E' with respect to O.

4. **Tangents.** The equation of the tangent at the point (r_1, θ_1) is

$$ar \sin(\theta - 2\theta_1) + ar_1 \sin \theta_1 - rr_1 \sin(\theta - \theta_1) = 0. \tag{4}$$

This equation is always satisfied by the point $(a, 2\theta_1)$. Therefore all the tangents to the cochlioid at its intersections with a given radius vector from the pole, O, pass through the same point, the locus of which is a circle with center O and radius

[1] The experimental work of this problem was carried on by Dr. C. J. Lapp while at the University of Illinois. Dr. Lapp is now connected with the Physics Department of the State University of Iowa. He has photographs of the traces which serve as a partial check on the above equation. Since I found the above equation in this connection, I was led to investigate the curve. See *Transactions of Illinois Academy of Science*, vol. 15, p. 331.

[2] Neuberg, *Wiskundige Opgaven*, vol. 2, 1884, p. 249; also *Mathesis*, 1885, p. 91.

a. This theorem is due to Cesàro who also noted that the tangent to the curve generated by the center of gravity of a variable arc always passes through the moving point on the fixed circle.[1]

The points of contact of a system of parallel tangents lie on

$$r = a \sin (2\theta - \varphi)/ \sin (\theta - \varphi), \tag{5}$$

where φ is the angle between the tangents and the polar axis. This curve is an oblique strophoid. If $\varphi = \pi/2$, it becomes the right strophoid, but it degenerates into a circle if φ becomes zero.

5. Normals. For a discussion of the normals it is advisable to throw the equation of the curve into rectangular coördinates. It is

$$ay = (x^2 + y^2) \text{ arc tan } y/x. \tag{6}$$

The equation of the normal at the point (x, y) is

$$(x^2 + y^2)^2 - (x^2 + y^2)(xX + yY + ax) + aX(x^2 - y^2) + 2axyY = 0, \tag{7}$$

where X, Y are running coördinates. But if we think of the point (X, Y) as fixed, the feet of the normals from the point (X, Y) to the cochlioid lie on the bicircular quartic (7).

If the equation (7) is required to be the product of the equations of two circles, the point (X, Y) must lie on the strophoid whose equation is

$$Y^2 = X(X - a)^2/(2a - X). \tag{8}$$

The vertex of this curve is at the origin, its double point is $(a, 0)$ and its asymptote is the line $X = 2a$. This curve can be represented parametrically by the equations

$$X = 2at^2/(1 + t^2), \qquad Y = at(t^2 - 1)/(1 + t^2). \tag{9}$$

If we substitute these values in (7) and factor, the equations of the two circles are found to be

$$x^2 + y^2 - ax - aty = 0 \quad \text{and} \quad (1 + t^2)(x^2 + y^2) - 2at^2x + 2aty = 0, \tag{10}$$

where t can have all real values. The centers of these two circles generate two curves when t takes all its values. These curves are

$$x^2 + y^2 - ax = 0 \quad \text{and} \quad x = a/2. \tag{11}$$

If we eliminate t from equations (10), the equation of the variable intersection of these two circles is found to be

$$(x^2 + y^2 - ax)^2 + 2a(a - x)(x^2 + y^2 - ax) + a^2y^2 = 0. \tag{12}$$

The x-intercepts of this curve are 0, a, and $2a$. The point $(a, 0)$ is a double point of this curve which in shape is very similar to the figure eight.

[1] *Nouvelle Correspondance Mathématique*, t. IV (1878), p. 283.

Keeping X, Y constant, the quartic (7) is the inverse of the conic K, where

$$K \equiv aXx^2 - aXy^2 + 2axyY - ax - xX - yY + 1 = 0. \tag{13}$$

The condition that K be two straight lines is (8). If X, Y runs over the strophoid (8), the intersection of the two lines that form K generates the strophoid whose equation is

$$a^2Y^2 = (1 - aX)^2(2aX - 1)/(3 - 2aX). \tag{14}$$

Finally if in the quartic (7) we set $y = \pm ix - u$, where $i = \sqrt{-1}$ and let x approach infinity in the result, the following quadratic equation is obtained,

$$2u^2 + (Y \mp iX \mp ia)u - a(X \pm iY) = 0. \tag{15}$$

If the roots of this quadratic are unequal, the quartic (7) has four distinct asymptotes, two passing through each circular point. Hence the quartic (7) has a node at each circular point. If the roots are equal, the asymptotes coincide and the quartic (7) has a cusp at each circular point. The condition that (15) has equal roots is

$$(Y \mp iX \mp ia)^2 + 8a(X \pm iY) = 0 \tag{16}$$

or

$$Y^2 - X^2 + 6aX - a^2 = 0 \qquad \text{and} \qquad Y(X + a \pm 4a) = 0. \tag{17}$$

For the plus sign the points of intersection are $[a(3 \pm 2\sqrt{2}),\ 0]$ and $(-5a \pm 2a\sqrt{14})$. For the minus sign, they are $[a(3 \pm 2\sqrt{2}),\ 0]$ and $(3a,\ \pm 2ai\sqrt{2})$. If X, Y are restricted to real quantities, there are only four positions in the plane for the point (X, Y) which make the quartic (7) have two cusps on the line at infinity. When the condition (16) is satisfied, the quartic (7) is known as a "cartesian."

THE CORRELATION BETWEEN TWO VARIATES ONE OF WHICH IS NORMALLY DISTRIBUTED.[1]

By P. R. RIDER, Washington University.

It is the purpose of this paper to discover the correlation between two variates x and y ($y = kx^n$, $k > 0$), where x is distributed according to the so-called normal law of error. A problem of this type would arise if, for instance, one wished to find the correlation between the diameters and the weights of a set of homogeneous spheres, either the diameters or the weights being normally distributed. A concrete example might be afforded by the apples on a tree. Professor Rietz[2] has given other practical illustrations in discussing the frequency distribution

[1] Presented before the American Mathematical Society, April 19, 1924.

[2] H. L. Rietz, "Frequency distributions obtained by certain transformations of normally distributed variates," *Annals of Math.* (2), vol. 23, pp. 292–300.

of the second variate and has emphasized the importance of considering the properties of the frequency curve of this variate, which is obtained from the variate x by means of the transformation given above, *viz.*, $y = kx^n$, $k > 0$. It would consequently seem worth while to determine the correlation between these two variates, even though they do not vary independently.

Let us assume that the variate x is measured in terms of its standard deviation as a unit, and that its frequency is given by $\varphi(x - \bar{x})$, where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}.$$

The coefficient of correlation, which is independent of k, is then given by the formula[1]

$$r = \frac{\int_0^{2\bar{x}} (x - \bar{x})(x^n - \bar{y})\varphi(x - \bar{x})dx}{\left[\int_0^{2\bar{x}} (x - \bar{x})^2\varphi(x - \bar{x})dx\right]^{1/2}\left[\int_0^{2\bar{x}} (x^n - \bar{y})^2\varphi(x - \bar{x})dx\right]^{1/2}}, \tag{1}$$

in which $\bar{y}$ is the mean of the variate y, *i.e.*,

$$\bar{y} = \frac{\int_0^{2\bar{x}} x^n\varphi(x - \bar{x})dx}{\int_0^{2\bar{x}} \varphi(x - \bar{x})dx}. \tag{2}$$

It will be noted that the limits of the integrals involved in r have been taken as 0 and $2\bar{x}$, it being assumed for convenience that $\bar{x} > 0$. This avoids negative values of x, which lead to imaginary values of certain of the functions dealt with, and simplifies some of the formulas subsequently developed. Moreover, unless limits which are symmetric with respect to $\bar{x}$ are taken, it is not identically true that

$$\bar{x} = \frac{\int_0^{2\bar{x}} x\varphi(x - \bar{x})dx}{\int_0^{2\bar{x}} \varphi(x - \bar{x})dx}. \tag{3}$$

Because of the rapid approach of the function $\varphi(x - \bar{x})$ to zero with increasing absolute value of $x - \bar{x}$, it will not seriously affect the value of r to cut off the integrals at these limits provided $\bar{x}$ is moderately large and n moderately small (say $\bar{x} > 4$, $n < 10$). We shall for the present restrict ourselves to values of n greater than $-\frac{1}{2}$, in which case the integrals contained in r will all be finite, even though some of them may be improper.

[1] Cf. E. V. Huntington, "Mathematics and statistics, with an elementary account of the correlation coefficient and the correlation ratio," this MONTHLY (*1919*, 421–435).

It follows quite readily from (1), if we make use of (2) and (3), that

$$r = \frac{\int_0^{2\bar{x}} (x^{n+1} - \bar{x}\bar{y})\varphi(x-\bar{x})dx}{\left[\int_0^{2\bar{x}} (x^2 - \bar{x}^2)\varphi(x-\bar{x})dx\right]^{1/2}\left[\int_0^{2\bar{x}} (x^{2n} - \bar{y}^2)\varphi(x-\bar{x})dx\right]^{1/2}}$$

$$= \frac{\int_0^{2\bar{x}} x^n(x-\bar{x})\varphi(x-\bar{x})dx}{\left[\int_0^{2\bar{x}} (x^2-\bar{x}^2)\varphi(x-\bar{x})dx\right]^{1/2}\left[\int_0^{2\bar{x}} x^{2n}\varphi(x-\bar{x})dx - \frac{\left\{\int_0^{2\bar{x}} x^n\varphi(x-\bar{x})dx\right\}^2}{\int_0^{2\bar{x}}\varphi(x-\bar{x})dx}\right]^{1/2}} \quad (4)$$

$$= \frac{\int_0^{2\bar{x}} x^n(x-\bar{x})\varphi(x-\bar{x})dx}{[\alpha(\bar{x}) - 2\bar{x}\varphi(\bar{x})]^{1/2}\left[\int_0^{2\bar{x}} x^{2n}\varphi(x-\bar{x})dx - \frac{1}{\alpha(\bar{x})}\left\{\int_0^{2\bar{x}} x^n\varphi(x-\bar{x})dx\right\}^2\right]^{1/2}} \quad (5)$$

where

$$\alpha(x) = \int_{-x}^{x} \varphi(x)dx.$$

The first factor in the denominator of (5) is approximately equal to unity when $\bar{x}$ is moderately large. For example, if $\bar{x} = 4$, we have $\alpha(4) = 0.99994$, $\varphi(4) = 0.00013$, $[\alpha(4) - 2 \times 4\varphi(4)]^{1/2} = 0.9994$.

If in (1) we let $f(x) = (x - \bar{x})\sqrt{\varphi(x-\bar{x})}$, $g(x) = (x^n - \bar{y})\sqrt{\varphi(x-\bar{x})}$, we find that

$$r = \frac{\int_0^{\bar{x}} f(x)g(x)dx}{\left[\int_0^{2\bar{x}} f^2(x)dx\right]^{1/2}\left[\int_0^{2\bar{x}} g^2(x)dx\right]^{1/2}},$$

and it follows from Schwarz's inequality [1] that the numerator of r is not greater in absolute value than the denominator. Thus the maximum absolute value that r can attain is unity. It can be shown directly that $r = 1$ when $n = 1$, as is to be expected, since perfect correlation exists between two variates when and only when each is a linear function of the other.

In the expressions (4) and (5) the numerator is $\int_0^{2\bar{x}} x^n(x-\bar{x})\varphi(x-\bar{x})dx$. Now the function $(x - \bar{x})\varphi(x - \bar{x})$ appearing in the integrand is symmetric with respect to the point $(\bar{x}, 0)$, and since x^n increases or decreases when x increases from 0 to $2\bar{x}$ according as n is positive or negative, it follows that

$$\int_{\bar{x}}^{2\bar{x}} x^n(x-\bar{x})\varphi(x-\bar{x})dx \gtrless \int_0^{\bar{x}} x^n(x-\bar{x})\varphi(x-\bar{x})dx,$$

[1] See G. Vivanti, *Elementi della Teoria delle Equazioni Integrali Lineari*, p. 213.

according as $n \gtrless 0$, and therefore that

$$\int_0^{2\bar{x}} x^n(x - \bar{x})\varphi(x - \bar{x})dx \gtrless 0,$$

according as $n \gtrless 0$. Thus, since the denominator of r is positive, r is positive for positive values of n and negative for negative values of n. It can be shown by a direct evaluation that $r = 0$ when $n = 0$.

Let us now determine the limiting value of r as n approaches infinity. By the law of the mean for definite integrals, we have

$$\int_0^{2\bar{x}} x^m\varphi(x - \bar{x})dx = \varphi(x_m - \bar{x}) \int_0^{2\bar{x}} x^m dx = \varphi(x_m - \bar{x}) \frac{(2\bar{x})^{m+1}}{m+1}, \quad m \neq -1,$$

in which $0 < x_m < 2\bar{x}$. Making use of this relation in (5), we find that

$$r = \frac{\varphi(x_{n+1} - \bar{x}) \dfrac{(2\bar{x})^{n+2}}{n+2} - \varphi(x_n - \bar{x})\bar{x} \dfrac{(2\bar{x})^{n+1}}{n+1}}{[\alpha(\bar{x}) - 2\bar{x}\varphi(\bar{x})]^{1/2} \left[\varphi(x_{2n} - \bar{x}) \dfrac{(2\bar{x})^{2n+1}}{2n+1} - \dfrac{1}{\alpha(\bar{x})} \varphi^2(x_n - \bar{x}) \dfrac{(2\bar{x})^{2n+2}}{(n+1)^2} \right]^{1/2}}, \tag{6}$$

the values x_n, x_{n+1}, x_{2n} being between 0 and $2\bar{x}$. Equation (6) is easily reduced to the form

$$r = \frac{\varphi(x_{n+1} - \bar{x})(2\bar{x}) \dfrac{n+1}{n+2} - \varphi(x_n - \bar{x})\bar{x}}{[\varphi(\bar{x}) - 2\bar{x}\varphi(\bar{x})]^{1/2} \left[\varphi(x_{2n} - \bar{x}) \dfrac{1}{2\bar{x}} \dfrac{(n+1)^2}{2n+1} - \dfrac{1}{\alpha(\bar{x})} \varphi^2(x_n - \bar{x}) \right]^{1/2}}.$$

Since the function φ is finite and different from zero, it can be seen that r approaches zero as n increases without limit.

In order to consider the case $n \leqq -\frac{1}{2}$, we set $n = -m$ and write

$$r = \frac{\displaystyle\int_\epsilon^{2\bar{x}-\epsilon} (x^{-m+1} - \bar{x}x^{-m})\varphi(x - \bar{x})dx}{\left[\displaystyle\int_\epsilon^{2\bar{x}-\epsilon} (x^2 - \bar{x}^2)\varphi(x - \bar{x})dx\right]^{1/2} \times \left[\displaystyle\int_\epsilon^{2\bar{x}-\epsilon} x^{-2m}\varphi(x - \bar{x})dx - \dfrac{\left\{\displaystyle\int_\epsilon^{2\bar{x}-\epsilon} x^{-m}\varphi(x - \bar{x})dx\right\}^2}{\displaystyle\int_\epsilon^{2\bar{x}-\epsilon} \varphi(x - \bar{x})dx} \right]^{1/2}}$$

Applying the law of the mean for definite integrals and reducing, we get

$$r\left[\int_{\epsilon}^{2\bar{x}-\epsilon}(x^2-\bar{x}^2)\varphi(x-\bar{x})dx\right]^{1/2}$$

$$= \frac{\dfrac{\varphi(x_{-m+1}-\bar{x})}{-m+2}\{(2\bar{x}-\epsilon)^{-m+2}-\epsilon^{-m+2}\} - \dfrac{\varphi(x_{-m}-\bar{x})}{-m+1}\bar{x}\{(2\bar{x}-\epsilon)^{-m+1}-\epsilon^{-m+1}\}}{\left[\dfrac{\varphi(x_{-2m}-\bar{x})}{-2m+1}\{(2\bar{x}-\epsilon)^{-2m+1}-\epsilon^{-2m+1}\} - \dfrac{1}{\alpha(\bar{x}-\epsilon)}\dfrac{\varphi^2(x_{-m}-\bar{x})}{(-m+1)^2}\{(2\bar{x}-\epsilon)^{-m+1}-\epsilon^{-m+1}\}^2\right]^{1/2}}$$

$$= \frac{\dfrac{\varphi(x_{-m+1}-\bar{x})}{-m+2}\{\epsilon^{m-(1/2)}(2\bar{x}-\epsilon)^{-m+2}-\epsilon^{3/2}\} - \dfrac{\varphi(x_{-m}-\bar{x})}{-m+1}\bar{x}\{\epsilon^{m-(1/2)}(2\bar{x}-\epsilon)^{-m+1}-\epsilon^{1/2}\}}{\left[\dfrac{\varphi(x_{-2m}-\bar{x})}{-2m+1}\{\epsilon^{2m-1}(2\bar{x}-\epsilon)^{-2m+1}-1\} - \dfrac{1}{\alpha(\bar{x}-\epsilon)}\dfrac{\varphi^2(x_{-m}-\bar{x})}{(-m+1)^2}\{\epsilon^{m-(1/2)}(2\bar{x}-\epsilon)^{-m+1}-\epsilon^{1/2}\}^2\right]^{1/2}},$$

where x_{-m}, x_{-m+1}, x_{-2m} are between ϵ and $2\bar{x}-\epsilon$. It is readily seen that this expression approaches zero with ϵ if $m > \frac{1}{2}$ (*i.e.*, $n < -\frac{1}{2}$), provided m is different from 1 and 2 (*i.e.*, $n \neq -1, -2$). Since $\left[\int_{\epsilon}^{2\bar{x}-\epsilon}(x^2-\bar{x}^2)\varphi(x-\bar{x})dx\right]^{1/2}$, the coefficient of r in the foregoing equation, approaches a limit which is different from zero as ϵ approaches zero, r has the limiting value zero for $n < -\frac{1}{2}$ ($n = -1$ and -2 excepted).

We have now only to consider the special cases $n = -\frac{1}{2}, -1, -2$. For these values of n certain integrals in r become logarithms. It is not difficult to show that for each of these three values r approaches zero with ϵ. The proofs will however be omitted as they can be effected by the methods ordinarily employed in evaluating indeterminate forms.

To summarize: *The correlation coefficient r for the variates x and y ($y = kx^n$, $k > 0$), x being normally distributed, is zero for $n \leqq -\frac{1}{2}$ and for $n = 0$, is negative for $-\frac{1}{2} < n < 0$ and positive for $n > 0$, and approaches the value zero as n approaches ∞; moreover r is equal to unity for $n = 1$ but is less than unity in absolute value for all other values of n.*

COLLEGE GEOMETRY.[1]

By NATHAN ALTSHILLER-COURT, University of Oklahoma.

The ancient Greeks never studied synthetic geometry—at least they never knew they did. Henri Poincaré remarks somewhere in his philosophical writings that of all living creatures inhabiting the earth man alone is mortal—the others do not know that they are to die. The ancient Greeks did not know that the geometry they were studying was synthetic, because they knew of no other kind. The revival of learning in Europe during the Renaissance brought to life the geometry of the Greeks. In the brilliant school of the French mathematicians of the first half of the seventeenth century synthetic geometry had two important exponents, Desargues and de la Hire,—it would be unfair not to mention Pascal also. But the dazzling discovery of analytic geometry made by Descartes and the new avenues opened for it by the invention of calculus entirely absorbed the attention of mathematicians. The contributions of Pascal and Desargues remained unnoticed for a century and a half.

A change occurred at the end of the eighteenth century due to Monge and his pupils: Carnot, Gergonne, Brianchon, Poncelet. Synthetic geometry came again into its own and developed along two distinct lines: projective geometry and modern geometry.

In the last quarter of the nineteenth century modern geometry became enriched by a splendid addition known as the "geometry of the triangle."

Projective geometry is taught widely in the colleges and universities in this country. We can even boast of some very serious contributions to the literature on the subject, as for instance Veblen and Young's two-volume work. But modern geometry has found little or no favor on this side of the Atlantic. For lack of both time and competence I shall not attempt to find a reason for this phenomenon. I shall simply take the liberty to call your attention to the fact, because this state of things seems to me very regrettable.

Here is a body of geometric doctrine, very beautiful in its simplicity, about the existence of which our college students of mathematics have no chance of learning. There are very many things in modern geometry that ought to constitute a part of the mathematical equipment of any college student who takes an interest in mathematics. Take for instance the properties of the radical axis and radical center, Ceva's theorem, the nine-point circle, centers of similitude of two circles, or such more recent things as the Brocard points, the Symmedian lines, and other elementary notions of the geometry of the triangle. With the present day tendency to reduce the high school course in geometry to the very "essentials" (whatever that may mean), the students enter college with a considerably reduced geometric knowledge, as compared with those of former years. They know little of geometric constructions beyond the very elementary ones, and very little or nothing about, say, the escribed circles of a triangle, or the problem of Apollonius.

[1] Read before the Mathematical Association of America at Cincinnati, December 27, 1923.

Aside from its general informational value, the study of modern synthetic geometry in our colleges has yet another very pertinent raison d'être. If we leave out the engineering students, the vast majority of the college students who continue their studies in mathematics beyond their freshman year are prospective teachers of mathematics in our secondary schools. The high school teachers of mathematics of to-day are our students of yesterday, and our students of to-day are the high school teachers of to-morrow. How does their college training prepare them for their task? The traditional course in college mathematics includes algebra, trigonometry, analytic geometry, and the calculus. Aside from the general development of mathematical thinking derived from such a course, the student also gains a great degree of algebraic skill which will help him in his classroom when he teaches algebra. But what specific help does this course give the prospective teacher of high school geometry? A course in projective geometry will widen the geometric outlook of the student. So would a discussion of the foundations of geometry, and far be it from me to belittle the value of these subjects for the prospective teacher. But the methods of projective geometry are totally different from those of Euclidean geometry, and the same is true about the contents of these two subjects. The result is that the prospective high school teacher, upon his graduation from college, knows about Euclidean geometry and its methods of proof exactly as much as he knew when he completed his high school course in this subject. Indeed, he knows much less, because he has had time to forget most of it, since his college studies made but little direct appeal to his knowledge of elementary geometry. When confronted with a problem in plane geometry, he will have only his own resources to fall back upon, since his college professors have done nothing, at least directly, to help him in his task. It is difficult to see how such a state of things can be considered anything short of abnormal. And it appears still more so in view of the fact that there is right at hand a body of doctrine whose study would both extend the field of knowledge of the prospective teacher in the domain of plane geometry and would provide an opportunity for him to review and to give a more mature consideration to his high school geometry; it would teach him to apply the methods of proof and of solving problems which he will use in his classroom with his pupils.

From a course in modern geometry our college students may derive yet other advantages. The traditional college courses, the calculus for instance, are little more than introductions into a vast field, and the student will have to continue to work in such a field considerably more before he can reach the stage where he may find the least opportunity to do something original, something by himself. The vast majority of high school teachers, however, do not go beyond their college course. At the most they may take another year's work. An initiation into modern geometry would suffice to give the student an opportunity to do some little creative work of his own. The results may be modest, but a chance to live through, even on a small scale, the pains of creation and the joy of discovery is much too valuable, much too precious to be given up lightly.

But one may go further. Among all these attempts some real gain for the progress of modern geometry, and thus for the advancement of science, may be

discovered some day. There is no reason why this country should not take the lead in this particular field of mathematical activity. Certainly one may hope that the vast army of high school teachers we have in the United States, a body greater than the corresponding bodies of a good many European countries put together, would be worthily represented in this field of research, if the study of modern geometry should become more widespread among our prospective high school teachers.

Let us now consider the subject from the point of view of the student. However interesting and desirable a change or improvement in the curriculum may be from the point of view of the teacher, the question that we must ask ourselves is: what attitude will the student take toward the innovation? Will the student's appreciation of these things coincide with ours, or will his reactions destroy all the hopes that we had attached to the change? I believe I am in a position to give you some idea of what you may anticipate in this case. For, if I seem to preach a new faith, I am fortunately not in the position of the preacher who does not practise what he preaches.

About six years ago I brought this question to the attention of my colleagues in the University of Oklahoma, and a course in modern synthetic geometry, under the name of "College Geometry," was introduced and has since been conducted under my direction. The course at once found favor in the eyes of the students, and its popularity with those who do their major work in mathematics has been steadily growing. The course is not required, but most students who expect to teach geometry in high school make it a point to enroll in "College Geometry," because they feel the need of it and realize how helpful it will be to them in their high school work.

Still more interesting, perhaps, is the attitude of students in the summer school. In these classes there have been principals of high schools and superintendents of schools, men and women who have been teaching high school mathematics for many years, who would volunteer opinions to the effect that for them as teachers this is one of the most valuable courses they have ever had. Some of them would go so far as to insist that in their opinion no one ought to be permitted to teach high school geometry who has not had a course in "College Geometry." On the intellectual side the subject is nothing short of a revelation to them. They never suspected that right alongside of the geometry they have been teaching all these years there is a direct extension of it, built of the same material, so closely interwoven with the elementary geometry, and yet so interesting, so new, so fascinating. The direct connection between elementary geometry and modern geometry makes most of them feel that there are ample possibilities ahead along these lines, and some of them get the inspiration to try to accomplish something themselves. Last summer one student, a man of middle-age, said to me: "At last I have found something to work upon." I have never had more enthusiastic, more hard-working classes than in "College Geometry."

So far I have spoken of modern geometry as a subject for advanced students in mathematics. I should like to say a few words in favor of it as a freshman sub-

ject. Most of the colleges require the freshmen to take a few hours of mathematics. The courses offered to freshmen are the traditional college algebra and trigonometry. But custom aside, is there any intrinsic reason why a college freshman should continue to study his high school algebra rather than his high school geometry? On narrowly utilitarian grounds neither can be defended, while from a broader cultural and intellectual point of view a continuation of the study of geometry is surely as valuable as an additional course in algebra. One may go still further. In addition to the first year's course in algebra most of the high schools offer the student an opportunity to take another semester's work in the subject. But a year's work in plane geometry seems to be the limit beyond which the inquisitiveness of the student must not be permitted to go. But it would be futile to insist at present on this point. Before we can expect to teach the high school students, we must first prepare the high school teachers, so that we come back to the point where we started.

It is the need of the prospective high school teacher that made me give some thought to the place of modern geometry in the college curriculum. I do not think that the conditions in Oklahoma in this respect are so radically different from those in other states, or in other schools.[1] I was thus prompted to bring the question before the Association in the hope that I may find out what other schools are doing, and I hope that wide discussion will shed upon this subject all the light of which it is worthy.

THE ARITHMETIC CLASSIC OF HSIA-HOU YANG.

By Père LOUIS VANHÉE, S.J., Brussels.

The Chinese mathematician Hsia-hou Yang, a writer of the sixth century, was the author of a brief work, the *Hsia-hou Yang Suan-king,* which has already been mentioned in Professor Smith's *History of Mathematics* (vol. I, p. 150) but which is worthy of a more extended description than such a treatise can be expected to give.

It was one of the standard works demanded for the official examinations in mathematics during the Tang Dynasty (618–907), and has been preserved in the extensive encyclopedia, *Yong-lo,* although the compilers of that work have separated it into parts for the purpose of supplementing the much older treatise, the *K'iu-ch'ang Suan-shu* (*Arithmetic in Nine Sections*). Fortunately the index is so arranged as to allow one to determine the original arrangement of the material, and to this fact we owe the *editio princeps* as prepared in 1776 by the celebrated Peking scholar, Tai Chen. So important was this edition considered that the emperor Khien-Lung wrote the preface and supplied most of the money necessary for its publication.[2]

---

[1] In fact, the success of this course at the University of Oklahoma is corroborated by testimony from several other institutions where such a course has been given. EDITORS.

[2] A copy of this rare work is preserved in the Bibliothèque Nationale at Paris. See Courant's catalogue, No. 4.844.

The work is divided into three parts, spoken of as the higher, middle, and lower sections, and consists of only thirty-six leaves (double pages), of which the several sections have respectively twelve, thirteen, and eleven folios, every page containing nine vertical lines of about twenty-one characters each. Allowing for blank lines and for spaces, there are only about 12,000 words in the entire work. As in all the older Chinese books, no technical rules are given, and the problems are simply followed by the answers, occasionally with very brief explanations.

In the first section the five operations of addition, subtraction, multiplication, division, and square and cube roots are given. The work on division is subdivided into (1) "ordinary division"; (2) "division by ten, hundred, and so on," especially intended for work in mensuration; (3) "division by simplification" (*yo ch'u*). The last problem in the section is as follows:

"There are 1843 *k'o*, 8 *t'ow*, 3 *ho* of coarse rice. A contract requires that this be exchanged for refined rice at the rate of 1 *k'o*, 4 *t'ow* for 3 *k'o*. How much refined rice must be given?" The answer is 860 *k'o*, 534 *ho*. The solution is given as follows: "Multiply the given number by 1 *k'o*, 4 *t'ow* and divide by 3 *k'o* and you will obtain the result." [1]

Fractions are also mentioned, special names being given to the four most common ones, as follows:

$\frac{1}{2}$ is called *chung p'an* (even part);
$\frac{1}{3}$ is called *shaw p'an* (small part);
$\frac{2}{3}$ is called *thai p'an* (large part);
$\frac{1}{4}$ is called *joh p'an* (weak part).

In the second section there are twenty-eight applied problems relating to taxes, commissions, and such questions as concern the division by army officers of loot and food (silk, rice, wine, soy sauce, vinegar, and the like) among their soldiers.

The third section contains forty-two problems each beginning with the word "now," which is here taken as substantially equivalent to the word "if." Five of these problems, translated as literally as possible, are as follows:

Ex. 1. Now for 1 pound of gold one gets 1200 pieces of silk. How many can you get for 1 ounce? Answer: For 1 ounce you get exactly 75 pieces. Solution: Take the given number of pieces, have it divided by 16 ounces, and you will obtain the answer.[2]

Ex. 2. Now you have 192 ounces of silk. How many *choo* have you? Answer: Four thousand six hundred eight.[3]

[1] 1 *k'o* = 10 *t'ow* = 100 *ho*. The *k'o* may be roughly translated as a bushel.

[2] The Chinese pound was, from early times, divided into 16 ounces or *taels*.

[3] The result (4608) is written in words instead of in numeral characters. The zero was not in use in the sixth century; but this did not, of course, prevent the use of the ordinary Chinese numerals of that period. The answer shows that the ounce was at that time divided into 24 *choo*.

At the beginning of most of the early Chinese books on arithmetic there is the following statement from Sun-tzï's *Suan-king*, a work referred to in Smith's *History of Mathematics*, vol. I, p. 141: "Weight begins with one grain of millet; ten grains make one *ts'en*; ten *ts'en* make one *choo*; 24

Ex. 15. Now 2000 packages of cash must be carried to the town at the rate of 10 cash per bundle. How much will be given to the mandarin and how much to the carrier? Answer: 1980 packages and $198\frac{2}{101}$ cash to the mandarin; 19 packages and $801\frac{98}{101}$ to the carrier. Solution: Take the total number as the dividend, and 1 package plus 10 cash as the divisor.[1]

Ex. 24. Out of 3485 ounces of silk how many pieces of satin can be made, 5 ounces being required for each piece? Answer: 697. Solution: Multiply the number of ounces by 2 and go back by one row. Dividing by 5 will also give the answer.[2]

Ex. 42. Now they build a wall, high 3 rods, broad 5 feet at the upper part and 15 feet at the lower part; the length 100 rods. For a 2-foot square a man works 1 day. How many days are required? Answer: 75,000. Solution: Take half the sum of the upper and lower breadths, have it multiplied by the height and length; the product will be the dividend. As the divisor you will use the square of the given 2 feet.[3]

Hsia-hou Yang also uses percentage. He shows considerable ability in finding various areas and volumes. His work is evidently a good type of practical textbook of the time, a fact that is shown by its popularity and the high esteem in which it has always been held. Yüan Yüan,[4] for example, in his well-known biographical work, speaks of it as "an easy text, intended for daily use." We thus have a fair idea of the elementary calculations performed by the Chinese in the sixth century, and a satisfactory basis for comparison of the oriental ability with that of the contemporary and barren period in the West.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. A NOTE ON KNOTS.

By F. V. MORLEY, New College, Oxford, England.

1. The construction of a regular pentagon by tying a simple knot in a strip of paper leads directly to a generalization for the construction of regular polygons of any odd number of sides.

---

choo make one ounce (*tael*); 16 ounces make one pound; 30 pounds make one *kiun*; 4 *kiun* make 1 stone.

[1] A package or string of cash contains 1000 farthings. The two results reduce to periodic fractions. There is no reason given for the division of 2,000,000 by 1010.

[2] This is an early use of our rule for dividing by 5. The expression about going back one row seems to refer to the use of counters. So far as known, the *suan-pan*, in its present form, was not yet invented.

[3] 1 rod = 10 feet = 100 inches. The amount made by one man in a day is evidently intended to be the volume of a rectangular solid of base 2 ft. square and of height 1 rod; that is, 40 cubic feet. The divisor, therefore, is 40.

[4] See Smith, *loc. cit.*, vol. I, p. 535. Yüan Yüan was born in 1764 and died in 1849.

The construction of the pentagon came to me by oral tradition, and I am at a loss for a reference to it.[1] Though no doubt familiar to many, it may be recapitulated here. To perform the operation, take a strip of smooth, pliable paper, sides cut parallel and even, say half an inch wide and ten inches long. Visualization may be aided by Fig. 1. The process is simpler in action than in words.

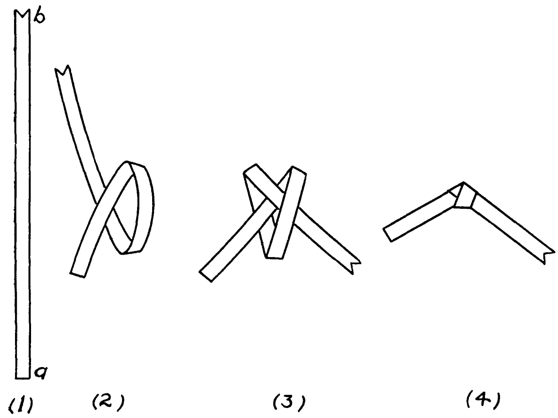


FIG. 1.

1. Hold the strip vertical, the lower end, a, between thumb and forefinger of the left hand; the upper end, b, in the right hand.

2. Carry the end b forward and pass it, from the right, behind a—catching the double thickness between the left thumb and forefinger. We have thus a simple loop, with long end b projecting to the left.

3. Carry b forward again, and pass it from the left axially through the loop.

4. Now pull ends a and b delicately. Folding the paper neatly flat when the knot is tight, we have the pentagon.

If, with a somewhat thinner strip of the same length, say a quarter of an inch wide, we start from stage 3 in the above process, and, instead of pulling the ends, pass b again back and behind the double thickness—so as now to catch three thicknesses between the left thumb and forefinger—we have a double loop, with long end b projecting to the left. This is shown in Fig. 2 (5). Carry b forward and pass it from the left axially through the double loop (Fig. 2 (6)). Pulling both ends of the strip until the knot is tight, and flattening, we have the regular heptagon (Fig. 2 (7)).

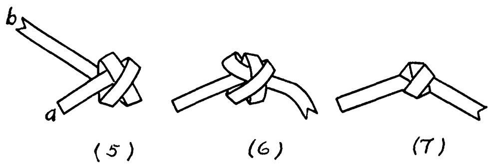


FIG. 2.

This construction applies to any regular polygon of $2n + 3$ sides, where n is the number of loops in the knot. Thus from Fig. 2 (6) we might, instead of pulling the ends, form a triple loop by passing b back and behind. Tying the knot with this triple loop, the result will be a regular nonagon; and so on.

2. The question rises of constructing the even regular polygons by knots. I do not think this can be done with a single strip of paper. Whenever we tie a "four-in-hand" tie, we construct a hexagonal knot; but it does not flatten into a regular hexagon.

[1] Mr. H. W. Richmond tells me that the construction is mentioned in *Scientific Amusements*, by Tom Tit; translated from the French by C. G. Knott, and published by Nelson (no date mentioned).

However, we may take two strips, of equal width, and make a simple loop in each, as in Fig. 1 (2). Call the ends of the strips a, b and α, β respectively. Turn the second loop over, and join it with the first as shown in Fig. 3 (8); that is, with ends α, β through the loop of a, b, and ends a, b through the loop of α, β. Now pulling the ends, and flattening, we get the regular hexagon (Fig. 3 (9)).

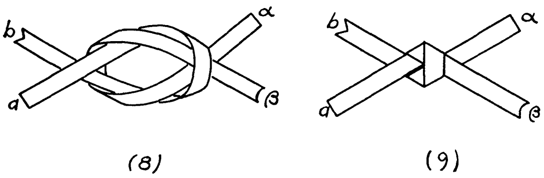


FIG. 3.

If, instead of using two simple loops, we had in this way combined a double loop and a single loop, the result would have been the regular octagon. Combining a triple loop and a single, we get the regular decagon. And so on.

3. The above constructions give any regular polygon of five or more sides. Physically, the strips are not easy to manipulate when the loopage is high. Theoretically the construction may be thought of as that of tying knots in parallel lines, and may so continue *ad infinitum.* If an analysis can be developed to handle such processes of knotting, we shall have, in a treatment of these constructions, a method for solving particular equations of any degree.

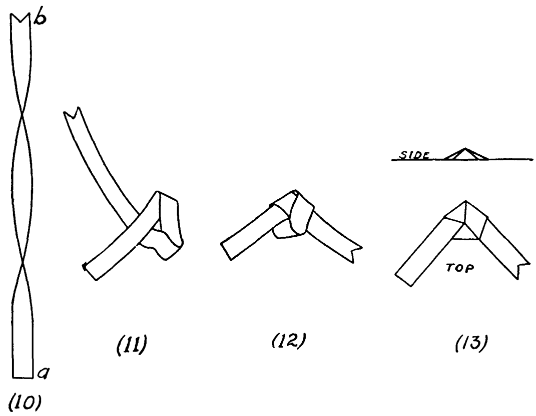


FIG. 4.

There are plenty of possible complications to be studied. One interesting development is to twist the strip. Here it is well to have one face colored, to aid in seeing what happens. If a is held and b twisted through two right angles perpendicularly to the length of the strip—so that b presents the same face as a—we say the strip has one complete twist (Fig. 4 (10)). Tying the pentagonal knot in a strip once twisted, we have in Fig. 4 (12) a knot which will not flatten into a plane polygon; but it "flattens" naturally on a pentagonal pyramid (Fig. 4 (13)). In other words, a simple knot tied in a helical strip may give a pentagonal pyramid. In the strict sense I have no proof, other than having seen it happen.

II. THE DEFINITION OF "VARIABLE."

By A. A. BENNETT, University of Texas.

Among the important tools of mathematicians is to be reckoned the concept of a variable. A search through mathematical treatises brings out the fact that a definition of this term is more often omitted than included. In works entitled

"Theory of Functions of a Real Variable" or "Theory of Functions of a Complex Variable," the term "function" is carefully defined with numerous cautions and illustrations while the idea of a variable is apparently presupposed.[1] This is consistent when the treatise in question also presupposes a familiarity with algebra, analytic geometry and calculus. However not a few laborious tomes start with the first elements of theoretical arithmetic and continue with a supposedly complete set of axioms and definitions, but never define or explain the notion of a variable.

Strictly speaking, algebra as the study of equations and of rules of operation does not require the use of a variable, useful as this notion can be in graphical suggestiveness; but in calculus at least, the idea of variable would appear essential. Yet relatively few books on the calculus and fewer still on analytical geometry suggest a definition of the term. Some careful writers of mathematical treatises explain the term at some length without giving any concise definition. This may be due to a willingness to describe the meaning of the word in an implicit fashion, its properties being implied from significant features of its use in various connections. Such discussions, however grateful, do not meet the logical demand for a definition. One reason justifying this general neglect is that there is little occasion for confusion in the use of the word "variable." The concept is suggested with reasonable vividness by the etymology, and paradoxes based on its use would be hard to find. It might therefore be with some surprise that one notes considerable discrepancies among available definitions, and that most of the familiar formulations are open to objection.

A familiar definition is the following: "A variable is a symbol that denotes more than one quantity in the course of a single discussion." There are two features of this definition that we might criticize. (1) Scholarly writers at least fail to agree that a variable is but a *symbol* of a quantity. To be sure, Keyser in his *Mathematical philosophy* insists at some length that such is the case. Pasch in *Veränderliche und Funktion* refers to "a variable as a number-name. The individual numbers which the number-name may denote are the values of the number-name. It is permissible to understand by the symbol of a variable each arbitrary individual among its values." Hardy in *Pure mathematics* calls a variable "an unspecified element" in a given "field of variation," although he makes no explicit definition. Hobson in *Theory of functions of a real variable* refers to the "essential nature of the variable (as) consisting in its being identifiable with any particular number of the domain," although again there is no concise definition of the word under discussion. Quite irrespective of authority, it is clear that the language used in connection with variables does not suggest any underlying concept of the variable being a symbol other than a quantity of the same sort as the constants entering the problem. The sum of two variables is never regarded as requiring special explanation as would be the case were the variables new symbols, for which in the nature of things there could be no in-

[1] One may note that Caratheodory's *Vorlesungen über Reelle Funktionen* contrives to avoid the word "variable" altogether until page 641, when it slips in undefined.

herent rules of addition. One does not speak of the limit of a variable as the limit of a symbol, and so on through the list of customary locutions. With some care the treatment of a variable as a symbol might be made consistent but it would appear artificial and in fact seems never to have been undertaken. One might well insist that a variable is best left undefined, merely the term "symbol of a variable" being defined, and the use of the term "variable" by itself being justified as a simple and obvious abbreviation in the several phrases in which it appears. (2) One should insist upon the disjunctive application of the term, while the expression "more than one" might well refer to a set taken as a whole, that is, conjunctively. In plane analytical geometry, a point may be represented by a pair of numbers, and therefore by the single symbol (a, b), but not many persons would desire to regard this symbol as a variable simply because it represents more than one number, namely a and b. On the other hand a typical variable, $y = f(x)$, may for some choice of the function f denote but a single number, the constant value of the function. It would seem that any definition of "variable" to be acceptable should cover the special case of a constant. However this seems to be a point about which there is not unanimity. For instance, Fine in his *College algebra* specifically excludes the constant case and refers to a variable as a "letter . . . free to take every possible value and to change from any one value to any other."

A revised definition of a variable as "any one of a set of elements" avoids indeed the conjunctive ambiguity and escapes likewise the awkwardness of regarding a variable as merely a symbol. Much confusion might still result from the ambiguity of the phrase "any one." In one use there is the notion of uncertainty, in another that of free choice. When a variable is used to refer to "any one," it is essential that the reader does not obtain the impression, that might be justified by the words, that a variable is "some one not as yet revealed to the reader but perhaps already chosen by the author, and which the author reserves the privilege of identifying at some later stage of the discussion"; but rather that it is something better expressed as follows, "an unspecified one of a set (the set being mutually agreed upon by author and reader) which the reader is always at liberty to choose in whatever manner he may desire from the given set, but which, for illustrative purposes only, the author may request to be identified in turn with one or more specific elements." The fact that there is no time-factor involved in the notion might seem self-evident. However, Fricke in *Hauptsätze der Differential- und Integral-Rechnung* speaks as follows: "A quantity which in the course of time assumes different values is called a variable quantity, or briefly a variable."

I propose the following definition which presents the notion in an implicit form. Explicit definitions when convenient are of course to be preferred on grounds of elegance. There are however many terms which it seems best to leave not completely defined by explicit statement. Some writers in their effort to make all definitions explicit do violence to current mathematical language. It would be possible to define "point," if algebraic analysis be presupposed, a point being

identified with its set of coördinates. It is possible to define "vector" as a set of point pairs, but since only equality of vectors and the result of combining vectors is essential, it would seem better to leave the term defined only implicitly. Certainly the remark that a given point pair is an element of a vector sounds bizarre. In the same way it does not seem necessary to know just what an irrational number is, whether it is an infinite series, a cut in the number system, an upper bound of a set of rational numbers, etc., so long as we call it a number, and know its properties in connection with other rational and irrational real numbers. The explicit question as to whether a variable is or is not a number may have philosophical interest but does not need to be decided for the investigation of mathematical results. It is in the same position as the question as to whether a rational fraction is a pair of integers, concerning which we need merely insist that a rational fraction is a number and is in one-to-one reciprocal relation to pairs of integers subjected to certain familiar rules and conventions. One might mention incidentally that the phrase "in turn," which is sometimes used in connection with the definition of variable, is obviously inapplicable to the usual case for which the total set is non-enumerable.

Definition: A symbol that denotes any one you please of an assigned set of elements may be called the symbol of a variable element of this set, or more briefly of a variable, when there is no ambiguity as to the set.

It is customary in mathematics to use language that would suggest that the symbol is identified with the object symbolized. The statement "5 is a number" is not to be criticized on the ground that "5" can be nothing other than a mark or token while a number is an abstract concept. A more detailed statement would be "The symbol 5 is the symbol of a number." But such usage is authorized by long and general practice. The language "x is a variable" would be in the same manner consistent with our definition, although a more elaborate statement of the same content would be "The symbol x is the symbol of a variable." We will be content to regard the term "variable" as referring to an abstract, and one might insist undefined notion, so long as one will agree to use it only in connection with the words "symbol of" either expressed or, as is usually the convention, only implied by the context, the entire phrase "symbol of a variable" being defined as above.

RECENT PUBLICATIONS.

EDITED BY D. C. GILLESPIE, Cornell University, Ithaca, N. Y., to whom communications should be sent.

REVIEWS.

Vector Analysis. By C. RUNGE, translated by H. LEVY. New York, E. P. Dutton and Co., 1923. 8vo. viii + 226 pages.

This volume by Professor Runge of the University of Göttingen is a logical development of the subject of vector analysis and forms an excellent introduction

to the application of vectors in many fields. Several pages are devoted to the relation between determinants and vector theory and sufficient of the theory of lattices is given to form a basis for a study of the lattice structure of crystals. Since the author is a mathematical physicist, it is quite natural that he should confine his attention mainly to the realm of physics, the sphere to which vectors seem essentially to belong. Hydromechanics, electromagnetic theory, and elasticity are subjects of frequent reference and in the discussion of tensors there is an admirable introduction to the mathematical theory of relativity.

Chapter I introduces the vector idea and the rules of vector algebra. The author employs a slight variation from the usual method of approach. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors, then $\boldsymbol{ab}$ is called the external product of $\boldsymbol{a}$ and $\boldsymbol{b}$ and is termed a vectorial area of definite sense and of magnitude equal to that of the parallelogram of which $\boldsymbol{a}$ and $\boldsymbol{b}$ are sides, drawn in, or parallel to, the plane determined by $\boldsymbol{a}$ and $\boldsymbol{a} + \boldsymbol{b}$. A vector $\boldsymbol{f}$ perpendicular to $\boldsymbol{ab} \equiv \boldsymbol{F}$ is the representation or complement of $\boldsymbol{F}$, where $\boldsymbol{f}$ is of such magnitude that its absolute value is equivalent to that of $\boldsymbol{F}$. This relationship is written $\boldsymbol{f} = |\boldsymbol{F}$ or $\boldsymbol{F} = |\boldsymbol{f}$. With this beginning, the scalar and vector products of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ may be defined $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}|\boldsymbol{b}$ or $\boldsymbol{b}|\boldsymbol{a}$ and $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{ab} = -|\boldsymbol{ba}$. The external product $\boldsymbol{abc}$ represents in a defined sense the volume of a parallelopiped of edges $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. The external product of two vectorial areas is defined and several pages are devoted to reciprocal vectors and their uses.

In chapter II, the rules for differentiation and integration of vectors are defined and the rules for the use of the operator ∇. Applications are made to space curves, surfaces and volumes. Theorems are developed for the transformation of surface into volume integrals and of line into surface integrals. Green's theorem, rotors and potential as applied to scalars, vectors and vector areas are discussed.

In chapter III, the affine transformation of space is defined as that transformation by which a vector $\boldsymbol{r} = x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c}$ from a fixed point O to a point R transforms into the vector $\boldsymbol{r}' = x\boldsymbol{e} + y\boldsymbol{f} + z\boldsymbol{g}$ from O to the new position R' of R, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{e}$, $\boldsymbol{f}$, $\boldsymbol{g}$ being arbitrary vectors, but the scalars x, y, z being the same in both cases. If $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{c}'$ are the reciprocal vectors of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ respectively, then the operator $\boldsymbol{T} = \boldsymbol{ea}' + \boldsymbol{fb}' + \boldsymbol{gc}'$, which effects this transformation, is termed a tensor. Different types of tensors are then discussed and rules for their use set forth. Cogredience and contragredience end the first volume. The author states in the introduction "This first volume contains the vectorial analysis of three demensions. In the second volume that of four and more dimensions playing an important part in the theory of relativity will be treated."

The reader may question whether all of the notation is necessary but it is sufficiently near to that already recognized so that little difficulty will be experienced in getting accustomed to it. Misprints are few. On page 61, line 17, the second = sign should read − and dr, line 12, page 92, should be $d\tau$. The translation is well done. One encounters a few ambiguous and careless statements, such as the last sentence of paragraph 14, page 38: "The vectorial

product of a vector similarly or oppositely directed is zero since their external product is zero."

This book is a valuable addition to existing texts in English on vector analysis, being comprehensive and meaty yet not too formidable. It is full of suggestions of interesting lines of investigation and should attract the attention of all who desire a modern and authoritative presentation of the subject.

J. B. REYNOLDS.

A History of Magic and Experimental Science during the First Thirteen Centuries of Our Era. By LYNN THORNDIKE. New York, The Macmillan Co., 1923. Vol. I, pp. xl + 835; Vol. II, pp. vi + 1036.

This history, in two massive volumes, is truly a magnum opus. It represents many years of painstaking research among medieval manuscripts in European libraries, as well as the careful weighing of the judgments of older writers on medieval magic and science. Much of the source material examined has not previously been used. This book affords detailed information on writers not readily accessible to most students of history. For example, the reader finds here sixty pages devoted to Pliny's natural philosophy, sixty-five pages to Galen, nineteen to Augustine, thirty-three to "The Pseudo-Aristotle," seventy-six to Albertus Magnus, and another seventy-six pages to Roger Bacon. In other words, the treatment of the leading characters is the equivalent of a small book on each. Another way of describing the contents of the work is to state that the attitude toward magic and science is described for the period of the Roman empire and early Christianity, for the early Middle Ages, for the twelfth and the thirteenth centuries.

Primarily the book is of interest to students of natural philosophy and astronomy, rather than to mathematicians. Yet even the latter will discover points of interest. Since the time of Benjamin Peirce the precise definition of mathematics has repeatedly interested the votaries of that science. Hence the early conceptions of the words mathematics and mathematicians are of interest. In the Roman empire, astrologers were called mathematicians, as is disclosed in the "Recognitions," a book ascribed to Clement of Rome. The western church father Tertullian at the beginning of the third century rejoices that the mathematici or astrologers are forbidden to enter Rome or Italy, the reason being that they are consulted so much in regard to the life of the emperor. Augustine, in the fifth century, refers to the mathematici as astrologers and holds that they enslave human free will by predicting a man's character and life from the stars. Similarly, Isidore of Seville refers to the mathematici who augur the future from the stars, assign the parts of the soul and body to the signs of the zodiac, and try to predict the nativities and characters of men from the course of the stars. An anonymous author of the tenth century refers to the wisdom of the mathematici who think that mundane affairs are carried on under the rule of the constellations. This conception of mathematici was held by Marbod, the Bishop of Rennes, as late as the beginning of the twelfth century. Saint Hildegard of

Bingen called mathematici "deadly instructors and followers of the Gentiles in unbelief."

Gradually a distinction came to be drawn between *mathematica* or mathesis and *matematica* or matesis. The historian Richer, a contemporary of Gerbert, apparently used the word mathesis in the sense of our modern mathematics. Over a century later, Vincent of Beauvais, a contemporary of Abelard, lets mathematica mean sound doctrine and (as with Aristotle) the science of abstract quantity, while matesis signifies that superstitious vanity which places the fall of man under the constellations. Similar distinctions were made later by William of Conches, John of Salisbury, Albertus Magnus and Roger Bacon. Michael Scott of the court of Frederick II draws a very sharp distinction between mathesis or knowledge, and matesis or divination, also between mathematica which may be taught freely and publicly, and matematica which is forbidden to Christians.

Modern mathematicians will be amused by the web of mysticism which ancient thinkers wove around particular numbers. There was a fusion of Pythagorean and Hebrew number mysticism in the writings of Philo Judæus who emphasized the glories of the number 7. He notes that there are 7 planets, 7 circles of heaven, four quarters of the moon of 7 days each, 7 stars in the Pleiades and 7 in the Ursa major, that children born at the end of 7 months live, that the 7th day in disease is critical, that there are 7 ages of man's life, that the lyre has 7 strings, speech has 7 vowels, also that there are 7 divisions of the head (eyes, ears, nostrils, mouth), etc. However 4 and 6 yield little to 7 as mystic symbols. Hippolytus associates the beneficent and masculine with odd numbers, the feminine and malicious with even numbers. Macrobius held the Pythagorean doctrine that the world-soul consists of number, that number rules the harmony of celestial bodies. Puerile reveries occur in Robert Grosseteste: Form is represented by 1, matter by 2, composition by 3, the compound by 4; now $1 + 2 + 3 + 4 = 10$. Wherefore, every perfect thing is 10.

Thorndike utters the philosophical tenet that "magic and experimental science have been connected in their development, that magicians were perhaps the first to experiment." He gives the term "magic" a broad interpretation, "including all occult arts and sciences, superstitions, and folk-lore." He states further that "magic implies a mental state and so may be viewed from the standpoint of the history of thought." Thorndike does not press his philosophic views. It seems to the reviewer that the existence of a causal relation between magic and science is not evident. Did magic appreciably stimulate experimental science? Did number mysticism lead to any real theorems about numbers? Did astrology contribute substantially to the progress of astronomy? Did the horse of Bellerophon which according to legend was suspended in the air by magnets contribute to an exact knowledge of magnetic attraction and mechanical equilibrium? What useful medical practice grew out of the belief of Pliny that a man who has been struck by lightning will speak at once, when turned over on his injured side? What stimulus to science grew out of the claim that sometimes when a crocodile opens its jaw an ichneumon "darts down its throat like a javelin and eats away its intestines"?

Indeed, if we may judge the past by the present, mysticism does not lead to science. On the contrary, such new results of science as appeal to popular imagination are prone to stimulate the undisciplined reason of mystics to fresh absurdities. The mathematician's fourth dimension of space has afforded spirittualists rich fields of speculation. Ethereal waves of light have suggested to telepathists easy explanations of alleged thought-transference. The electron theory assists in certain mystic medical practice. Daily weather-bureau forecasts tempt ambitious prophets to predict the weather weeks and months in advance.

Whether Thorndike's philosophical tenet be valid or not, the rich historical material collected in his book and presented to the reader in attractive diction is of immense value. The publication is an epoch-making contribution to the better understanding of the scientific and pseudo-scientific thought of the first thirteen centuries of our era.

FLORIAN CAJORI.

The Rhind Mathematical Papyrus, British Museum, 10057 and 10058, Introduction, Transcription and Commentary by T. ERIC PEET. The University Press of Liverpool Limited, Hodder & Stoughton, London, 1923, 2 + 136 folio pages + 24 plates (3 of which are folding). Price 63 shillings.

Almost all of our knowledge of Egyptian mathematics is derived from: (*a*) four papyri in hieratic writing, and two wooden tablets the contents of all of which date from the twelfth dynasty [1] which began about 2000 B.C.; (*b*) a Demotic papyrus, Byzantine and Coptic tables of fractions, and the Akhmîm papyrus of much later date. On account of the possibility of contamination from Greek mathematics, these may not be used to prove anything with regard to mathematics of the earlier Egyptian periods.

The best known of the papyri is the one found at Thebes in the ruins of a small building near the Ramesseum. It was purchased in 1858 by A. Henry Rhind and after his death it passed into the hands of a gentleman from whom it was purchased by the trustees of the British Museum in 1864. This Rhind mathematical papyrus is in two parts of the same width, 33 cm., and of length 206 cm. and 319 cm. respectively. These two parts were originally one, about 543 cm. in length. Remarkable to relate, a number of fragments belonging to the 18 cm. missing in the British Museum papyrus became in 1907 the property of the New York Historical Society.[2] They were part of the collection of Edwin Smith, were in his possession in 1862–63, and were found in Egypt with a medical papyrus now well known.

[1] Of other material for the study of mathematics, especially weights and measures, various account papyri are of importance. The most valuable of those already published are Papyrus Bulaq 18, and those discussed by Spiegelberg in his *Rechnungen aus der Zeit Setis I.*

[2] The discovery that the New York fragments were originally part of the British Museum papyrus was only made within the past twenty-two months. When examining the Society's papyri in the autumn of 1922, the British Egyptologist, Percy E. Newberry, took tracings of the fragments, since he suspected connection with the Rhind Papyrus. He brought them to the attention of his colleague of the University of Liverpool, Professor Peet. These facts were kindly furnished to me by Mrs. Caroline R. Williams of the New York Historical Society.

According to Professor Peet, the Rhind papyrus was written between 1756 and 1612 B.C. but it is copied from an older work in the time of a king of the twelfth dynasty who was on the throne from about 1849 to 1801 B.C.

A second papyrus of this dynasty is the one recently acquired by the Museum of Fine Arts in Moscow. It was first described[1] in print in 1917 as containing 19 "problems, some of which give us new types of calculation unknown till now and therefore somewhat difficult to comprehend. Four of these problems are geometrical ones. The first shows how to define the length of the sides of a quadrilateral, when the relation of the sides and area of the quadrilateral is known. The two next give a method of calculating the area of a triangle: a method already known to us." The fourth geometrical problem appears to indicate a familiarity with the formula for the volume of a frustum of a square pyramid, $V = (h/3)(a_1^2 + a_2^2 + a_1a_2)$, where h is the altitude of a frustum, the sides of whose bases are[2] a_1 and a_2. If it is verified that such a result was actually known to Egyptians of the twelfth dynasty, it will lead to an entire revision of our ideas as to the geometrical powers of the Egyptians. Writing in August, 1922, the director of the Moscow Museum informed me that at a congress of Egyptologists, held at Moscow in that month, it was decided that his mathematical papyrus was to be published by the Russian Academy of Sciences, with a translation by Professor W. Strouwé, of Petrograd, and a mathematical commentary by Professor Zienserling.

The third papyrus of the twelfth dynasty consists of fragments found at Kahun in 1889 and published in 1898. Among its contents are: a table of resolutions of all fractions, 2/3 to 2/21, with numerator 2, and denominator an odd number into the sum of two or more fractions with numerator unity; a problem dealing with parallelopipedal containers the sides of whose bases are to one another in a fixed ratio (the solution involves the use of square root); and accounts of a poultry yard.

The fourth papyrus of the twelfth dynasty is Berlin Papyrus 6619, described by Schack-Schackenburg in 1900. One of its problems is to divide 100 square

[1] B. A. Touraeff, *Ancient Egypt*, 1917, pp. 100–102. The attention of mathematicians was first drawn to this by the writer at the Rochester meeting of the Association in 1922.

[2] Brahmagupta (who flourished about 628 A.D.) gave a formula which reduces to this (*Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brahmagupta and Bhâscara*, ed. by Colebrooke, London, 1817, pp. 312–313). So also Mahāvīrācārya (c. 850 A.D.), *The Gaṇita-Sāra-Sangraha* . . . with English translation and notes by M. Raṅgācārya, Madras, 1912, p. 260.

In the Moscow papyrus $a_1 = 4$, $a_2 = 2$, and $h = 6$. In al-Khowârizmî's algebra, the same problem is solved for $a_1 = 4$, $a_2 = 2$, and $h = 10$. (*The Algebra of Mohammed Ben Musa*. Edited and translated by F. Rosen, London, 1831, pp. 83–84); the method of solution is: determine the height (20), and hence the volume, of the completed pyramid, from which is subtracted the volume of the pyramid of height 10 on the upper base. To Democritus (who flourished about 400 B.C.) is due the discovery that the volume of a pyramid is one third that of the prism having the same base and equal height; the first rigorous proof of this result was given by Eudoxus.

The general formula for the volume of a frustum of a pyramid $(h/3)(A_1 + A_2 + \sqrt{A_1A_2})$, A_1 and A_2 being the areas of the bases, seems to have been first given by Leonardo Pisano in his 'Practica Geometriæ' of 1220 (*Scritti di Leonardo Pisano* . . . , vol. 2, Rome, 1862, pp. 177–178).

Heron of Alexandria (Third century A.D.?) discusses the volume of a frustum of a pyramid for which $a_1 = 24$, $a_2 = 16$, $h = 50$. *Heronis Alexandrini Opera quae supersunt omnia*, Leipzig, vol. 3, 1903, pp. 115–116).

cubits into two squares whose sides are in the proportion 1 : 3/4; it involves the use of square root. Another problem in numbers involves the correct determination of the square root of $6\frac{1}{4}$ as $2\frac{1}{2}$.

If such mathematics was known in the twelfth dynasty, it is interesting to speculate how much further we must go back in order to find the beginning of its development. Only a very few facts are known. The decimal system of notation for numbers up to 1,000,000 goes back to the first dynasty (which Breasted in 1906 set as beginning about 3400 B.C.), and there are faint traces of the system's having been originally quinary. In the fourth dynasty (which Breasted in 1906 set as beginning about 2900 B.C.) "land measures of the Rhind Papyrus are," according to Peet, "already in full development in a form which involves correct determination of the area of the rectangle but not of necessity of the triangle or circle. There appears to be no early evidence with regard to the measures of capacity, though one may almost take it for granted that with the measurement of the field on which the corn was grown went that of the containers in which it was stored and sold. That measurement by weighing was practised can hardly be denied."

The first published account of the Rhind mathematical papyrus, which is by far the most elaborate of the hieratic mathematical papyri which we have, was by F. Lenormant [1] in 1867. Then ten years later appeared the valuable German translation and commentary of August Eisenlohr who was assisted in the mathematical work by his brother Friedrich and by Moritz Cantor. In 1872 the British Museum loaned to Eisenlohr a set of plates of the Rhind papyrus. He published tracings of these in a volume accompanying the above-mentioned work. But the splendid facsimile issued by the Museum in 1898 is incomparably superior to Eisenlohr's volume of plates.

During the past 50 years the bibliography of the Rhind papyrus and of Egyptian mathematics has become quite extensive; in 1922 the writer prepared an annotated list of about 100 titles, in nine languages. Whilst some of these titles refer to descriptions, in articles or books, of a trivial nature, there are many others indicating notable advance in the understanding of the papyrus due to vastly extended knowledge of the Egyptian language. Hence for some time Eisenlohr's work has been in need of fundamental revision and modification.

It will be a source of profound satisfaction to many that the new work on the Rhind mathematical papyrus is in the English language. The author is a distinguished scholar, the Brunner professor of Egyptology in the University of Liverpool, director of the Institute of Archæology in the University, and editor of the *Journal of Egyptian Archæology*.

Professor Peet writes in his preface, "Every attempt has been made to render the book intelligible to the mathematician who has no knowledge whatsoever of the Egyptian language. On the other hand, the Egyptologist with little knowledge of mathematics may enter on it without fear. Egyptian mathematics

[1] *Comptes Rendus . . . de l'Académie des Sciences*, Paris, vol. 65, p. 903.

was a simple affair, and the author has tried throughout to deal with it in its own simple terms without clothing it in a modern dress which is totally foreign to it."

Not only is the book intelligible to the mathematician, but from the beginning to the end it is of extraordinary interest for such a reader. The attractive, lucid, and stimulating style of the commentary and the remarkably thorough, judicial and scholarly character of the work as a whole stamp it as a contribution of very high order to our knowledge of Egyptian mathematics.

Of the large folio pages in Professor Peet's work, 1–32 are occupied by introductory matter; 33–131, by a translation of the text and commentary; 131–132, by a general index; 134–135, by an index of Egyptian words discussed; 136, by an explanation of the plates. The plates *A*–D, F–Y (omitting I) contain a hieroglyphic translation,[1] with notes, of the original hieratic writing. Hieratic and hieroglyphic writing are about equally old, but most Egyptologists read the latter more readily. It is also of considerable interest to the mathematician who is not an Egyptologist; the signs for numbers may be learned in a few moments; a pair of legs walking toward the right is the sign for addition,[2] toward the left, subtraction; a bird with its beak at the ground is translated "to find." Plate E contains a slightly reduced copy of the ends of the parts of the Rhind papyrus at the break,[3] with 24 of the New York fragments placed in their proper order with reference to them. At one side of the plate are reproductions of 23 small unplaced fragments.

The sub-headings of the introductory matter are as follows: Previous work on the papyrus, Description of the papyrus, Date of the papyrus, Contents of the papyrus (pages 4–5), Documents available for the study of Egyptian mathematics (6–9), Date of the origin of Egyptian mathematics (9–10), General character of Egyptian mathematics[4] (10–21), Method of setting out the sums (21–24), Egyptian weights and measures (24–26), Comparison of Egyptian mathematics with Babylonian (27–31), The Greeks in Egyptian mathematics (31–32).

In this last section the most interesting thing is with reference to "harpedonaptai" or "rope-stretchers" of Egypt who according to some mathematical historians (for example, Fink and Ball) were acquainted with the fact that a triangle whose sides were in the ratios of the numbers 3, 4, and 5 contained a right angle and that they constructed right angles accordingly, as ancient Indian, and possibly Chinese, geometers did. Concerning this, Peet remarks:

[1] This translation was not made from the British Museum facsimile but from the original papyrus and is therefore, according to Peet, "at variance with the *Facsimile*. The more important of these cases," he continues, "are noted in the critical notes. To note all minor instances would have rendered the notes far too bulky." Peet does not seem anywhere to make clear how a *facsimile* of a papyrus *can* differ from the papyrus itself. Of course one can guess what is meant.

[2] In the Moscow papyrus this sign means to square. Curiously enough Peet fails to comment in Problem 28 on the fact that this same sign enters to mean *subtraction*.

[3] In making this copy five changes had to be made in rectification of the papyrus itself which had been carelessly patched at the break.

[4] Professor Peet does not specifically note that the Egyptian method of finding the product of two numbers is very often similar to that of the so-called Russian peasant system of multiplication. Compare my note in this MONTHLY (*1918*, 139–142).

"Nothing in Egyptian mathematics suggests that the Egyptians were acquainted even with special cases of Pythagoras' theorem concerning the squares on the sides of a right-angled triangle. That the *harpedonaptai* were land-measurers, on the other hand is most probable, indeed we can even see such persons at work in the pictures on the walls of the Egyptian tombs . . . the very unit by which fields were measured was a 'reel of rope ' of 100 cubits in length. . . . This process of land-measuring with a rope, the Egyptian name for which is not known, has been confused by historians of mathematics with the ceremony called . . . 'the stretching of the cord.' This was one of the initial ceremonies at the foundation of a temple. The king, or whoever represented him, took a sighting of the pole-star through a cleft stick, another standing north of him with a plumb-bob attached to a wooden arm. Each then drove a stake in the ground in front of him, and a cord stretched between the two gave a true north and south line and enabled the four corners of the temple to be fixed. Here the stretching of the cord is used not necessarily in measurement, but in the fixing of the orientation. Possibly it was something of this kind which Democritus had in mind when he spoke of the skill of the *harpedonaptai*."

The Rhind mathematical papyrus starts out with a table of resolution of fractions with numerator 2, and denominators the odd numbers 5 to 101, into the sum of two to four other fractions with unity in the numerators. It was one of the New York fragments which extended this table so as to include 2/101 which is given equal to $1/101 + 1/202 + 1/303 + 1/606$. This is the only case in the table that the number in the denominator of a fraction appears again in its resolution. Results of this table are used many times in the problems which follow.

The problems are unnumbered in the original but for convenience were numbered by Eisenlohr. This numbering has been retained by Peet for the same reason, and because they are now current in the literature of the subject.

Numbers 1–40 are problems in arithmetic. Number 3 is: "To divide 7 loaves among ten men": the answer is found to be that each man received $2/3 + 1/30$ of a loaf. (The Egyptian thought of 2/3 as $1/1\frac{1}{2}$.) Number 33 is: "A quantity whose two-thirds, half and seventh are added to it becomes 37; [what is the number?]" The result is found to be $16 + 1/56 + 1/679 + 1/776$. Professor Peet observes (page 20) "that the problems Nos. 24–38 involve the solution of equations of the first degree with one unknown by means of a simple method of trial," but this statement implies more than he seems prepared to grant in his extended comments on page 60 and those which follow. Only arithmetic is required to solve these problems and if we use our modern arithmetic processes we could often abbreviate those of the papyrus. Surely *this* is what should be emphasized in this connection, when attempting to reproduce the thought of the Egyptians. It is a considerable step in advance to assert that the Egyptians of the twelfth dynasty had any such thought as we have when formulating and solving algebraic equations of the first degree.

Problems 41–60 are problems in mensuration, and include: (*a*) volumes of cylindrical and rectangular parallelopipedal containers; (*b*) expressions for the areas of rectangular, circular, triangular pieces of land; (*c*) the angle of slopes of pyramids.[1] In problem 41 we have the area of a circle of diameter d given as

[1] This is the "seq-et" of Heath's discussion in the part of a paragraph at the top of page 128, volume 1, of his *History of Greek Mathematics*. Oxford, 1921. If space permitted, it might be shown that this part of a paragraph is replete with error and false suggestions. This is probably due to Heath accepting Borchardt and Griffith as authoritative in this connection.

$(8/9)^2d^2$; whence we may find $\pi = 256/81 = 3.1605 -$, instead of the correct value, $3.1416 -$.

Problems 61–84 are miscellaneous arithmetic problems, such as: (*a*) Proportionate values of precious metals—62; (*b*) Division of barley into shares in arithmetical progression—64; (*c*) Daily portion of a yearly ration of fat—66; (*d*) Problem 79, which is apparently the following: "Seven houses; in each are 7 cats; each cat kills 7 mice; each mouse would have eaten 7 ears (or grains) of spelt; each ear of spelt will produce 7 *hekat*. What is the total of all of these?" The result, 7×2801, is (as Peet and Cantor have pointed out) exactly what would be obtained by substituting in the modern formula for the sum, S_n, of n terms of a geometric series $(a = r)$ $S_n = a(a^n - 1)/(a - 1)$. On recalling that $S_n = a(S_{n-1} + 1)$ we note that in the problem before us $a = 7$, $S_{n-1} + 1 = 2801$. Was such a formula, even for simple cases, known to the Egyptians?[1] There is no actual evidence that it was.

Number 85 is an unintelligible group of signs; No. 86 is a fragment of accounts; and No. 87 contains calendrical entries. With this the papyrus ends.

These problems of the Rhind papyrus suggest that the mathematics of Egypt was intensely practical, and Peet believes that this is characteristic of all of the sciences of this country.[2] He remarks: "As Plato alone of the Greeks seems to have realized, the Egyptians were essentially a 'nation of shopkeepers,' and interest in or speculation concerning a subject for its own sake was totally foreign to their minds." If, however, it turns out that the frustum problem of the Moscow papyrus is correctly interpreted, we have a result which appears to indicate that the Egyptians studied mathematics for its own sake. Such problems as those in the Rhind papyrus involving numbers in arithmetic and geometric progressions seem also to imply considerations not strictly practical.

Professor Peet's work is admirably printed, with very clear type and generous spacing and margins, and is attractively and substantially bound. It should be in every college and university library of the country, and no teacher of the early history of mathematics should fail carefully to peruse its pages. The work is one which would also prove of great interest to our host of teachers in the secondary schools. To all such, and to students intellectually curious as to ancient cultures, it is most heartily recommended. May the University of Liverpool be rewarded for publishing this outstanding contribution of English scholarship!

R. C. ARCHIBALD.

---

[1] For this suggestion I am indebted to Chancellor A. B. Chace of Brown University.

[2] On the evidence of the Edwin Smith Medical Papyrus alone, J. H. Breasted expressed the belief that the evidence of Egyptian "interest in pure science, as such, is perfectly conclusive." Compare *The New York Historical Society Quarterly Bulletin*, vol. 6, April, 1922, p. 29.

Methodik des mathematischen Unterrichts. By W. LIETZMANN. Second edition, second part. Leipzig, Quelle und Meyer, 1923. Cloth, 8 vo. xii + 367 pages.

It speaks well for the series of handbooks edited by Dr. J. Norrenberg that this work by Dr. Lietzmann, which first appeared in 1916, has met with such success as to warrant a second edition. There is no one in Germany who stands higher than the author as a leader in the reform of the teaching of mathematics. His contributions to the work of the International Commission are well known and the assistance which he rendered to Professor Klein in that great undertaking was very helpful. He is Oberstudiendirektor of the Oberrealschule at Göttingen, a lecturer in the University, and one of the leaders in the pedagogical Seminar. Still a relatively young man, he has the forward look in matters educational, and his editorial duties on the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* have kept him in touch with all the current reform movements in Germany, the most important of which he has recently summarized in an article in *The Mathematics Teacher*.

Such being the equipment of the author, what kind of book has he felt is needed by the German teacher? It would be difficult to find a clearer picture of the difference between the Teutonic point of view and that which prevails in the schools of education in this country. Dr. Lietzmann arranges the 367 pages at his disposal in nine chapters: (1) computation (Der Rechenunterricht), (2) intuitive geometry (Der propädeutische geometrische Unterricht), (3) plane geometry, (4) solid geometry, (5) modern geometry, (6) introduction to algebra (Arithmetik), (7) algebra, and (8) analysis, including (*a*) the function concept, (*b*) analytic geometry, (*c*) the calculus. From this list of topics it is easily seen that the work seeks to present a modern view of the subject matter of instruction, that it attempts no unnatural fusion of topics, and that the author is not one whose mental vision is shut in by the drab walls of educational statistics. This leader in the reform of mathematical teaching has not a word to say about modern tests, leaving this to the trained tester. He would consider as trivial such exercises for teachers as the counting of the problems in factoring in the six "best sellers." What he seeks is to have teachers know the mathematics to be taught and the paths which current and past experience has shown to be the best for their progress. In geometry, for example, he explains and discusses with scientific openness of mind the reforms of Méray and Bourlet and the school which they represented; he considers how much of modern geometry may properly have place in the secondary school; and he takes up such topics as the cinematograph material, the models which have recently appeared, and the modern spirit that is permeating the schools of Europe.

We have nothing of this kind in our American educational literature, and we need it. It is valuable to our teachers to know the nature of the modern tests in mathematics and to be able to use them intelligently, all of which can be taught in a very short time. It is not their business to pose as experts in the making of such tests, however, and it is far more important that they know the subject that they are teaching and the works of real scholarship relating thereto. For

teachers who feel this need and who have some mastery of German this book will be of great help. By others, the book will not be read and will very likely be called reactionary.

D. E. SMITH.

Mensuration and Solid Geometry. By R. M. MILNE. Cambridge University Press, 1923. x + 206 pages, 177 figures.

It is the purpose of this book to develop the formulas most frequently used in mensuration and to afford practice in using them. The author is an instructor in the Royal Naval College at Dartmouth, Eng., and the text is developed with particular regard to the needs of students in that institution but should prove useful to any student who has had training in plane and solid geometry and trigonometry. The style is rapid and condensed. The proofs given are usually brief, and frequently informal. The subject matter is varied and well chosen.

No mention is made of logarithms. Not enough emphasis is laid on the determination of the accuracy that may be expected from computations on given data. Computational methods for finding a result to a prescribed number of significant figures are also neglected. The "mathematical" definitions of a cylinder (page 75) and a cone (page 87) are inaccurate.

The collections of problems constitute one of the most commendable parts of the book. The numerous problems of practical value will serve to arouse and maintain the interest of the student. Teachers of geometry and trigonometry who can devote only passing attention to computation will find this book valuable as a source for problems.

C. H. SISAM.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

JOURNAL OF MATHEMATICS AND PHYSICS, Massachusetts Institute of Technology, volume 3, no. 1, January, 1924: "Note on developable surfaces in hyperspace" by C. L. E. Moore, 1–6; "On Ricci's coefficients of rotation" by J. Lipka, 7–23; "Certain notions in potential theory" by N. Wiener, 24–51.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 10, no. 1, January, 1924: "On the subdivision of 3-space by a polyhedron" by J. W. Alexander, 6–8; "An example of a simply connected surface bounding a region which is not simply connected" by J. W. Alexander, 8–10; "Remarks on a point set constructed by Antoine" by J. W. Alexander, 10–12.

QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS, volume 50, no. 1, October, 1923: "Determination of all the characteristic subgroups of an abelian group" by G. A. Miller, 54–61.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 24, no. 4, December, 1922: "Representations of a complex point by pairs of ordered real points" by W. C. Graustein, 245–254; "Non-loxodromic substitutions and groups in n dimensions" by E. B. Van Vleck, 255–273; "Birational transformations simplifying singularities of algebraic curves" by G. A. Bliss, 274–285; "A symbolic theory of formal modular covariants" by O. C. Hazlett, 286–311; "On the mean-value theorem corresponding to a given linear homogeneous differential equation" by

G. Polya, 312–324.—volume 25, no. 3, July, 1923: "Discontinuous boundary conditions and the Dirichlet problem" by N. Wiener, 307–314; "A type of differential system containing a parameter" by F. H. Murray, 315–324; "On a remarkable class of entire functions" by J. I. Hutchinson, 325–332; "Note on an ambiguous case of approximation" by D. Jackson, 333–337; "Expansions in terms of solutions of partial differential equations, Second paper: Multiple Birkhoff series" by C. C. Camp, 338–342; "Circular plates of constant or variable thickness" by C. A. Garabedian, 343–398; "Permutable rational functions" by J. F. Ritt, 399–448; "On approximation by functions of given continuity" by D. Jackson, 449–458.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3076. Proposed by S. A. COREY, Des Moines, Iowa.

Professor Swift has shown in the MONTHLY (*1920*, 301) that the equation,

$$\frac{1}{n+1} - \frac{1}{6}\cdot\frac{4}{2^n} - \frac{1}{6} = 0,$$

must hold in the development of an analytic function, $\int_0^x f(x)dx$, by Simpson's Formula, and that n has only the positive integral values: 1, 2, and 3. The editor (*1920*, 464) has shown that if n be assumed to be a complex number the function, $\int_0^x x^2 \sin(b\log x - c)dx$, is also developable by Simpson's Formula for certain values of b.

In Weddle's Formula,

$$\int_0^h f(x)dx = \frac{h}{20}\left[f(0) + 5f\left(\frac{h}{6}\right) + f\left(\frac{h}{3}\right) + 6f\left(\frac{h}{2}\right) + f\left(\frac{2h}{3}\right) + 5f\left(\frac{5h}{6}\right) + f(h)\right],$$

the equation for finding n may easily be shown to be

$$(1+n)(2^n + 4^n + 6^n + 5^{n+1} + 6.3^n + 5) = 20.6^n,$$

which holds for all positive integral values of $n \not> 5$.

By taking n to be a complex number find other functions developable by Weddle's Formula, or show that none exist.

3077. Proposed by R. M. MATHEWS, Wesleyan University.

What geometric relations between a curve and its evolute are implied by a point of intersection of the two curves?

3078. Proposed by N. ALTSHILLER-COURT, University of Oklahoma.

Find the curve having the property that the tangent and the normal at any point determine on a fixed line two points conjugate in a given involution.

3079. Proposed by J. J. QUINN, Pittsburgh, Pa.

The line OP is equal to and coincident with the straight line segment AB. As O describes AB uniformly, OP rotates uniformly through 180°. Determine (*a*) the locus of P; (*b*) the length of the curve for a complete revolution; (*c*) the area of a loop.

3080. Proposed by C. N. SCHMALL, New York City.

Given a cyclic quadrilateral $ABCD$ such that its diagonals AC, BD intersect at right angles in P. If O be the center of the circle, and perpendiculars be dropped on the sides of the quadrilateral from O and P, show that the eight feet of these perpendiculars are concyclic.

NOTE: The circle on which these points lie is somewhat analogous to the *nine-point* circle of a triangle. Its center is the middle point of OP.

3081. Proposed by HARRY LANGMAN, New York City.

Show that

$$\begin{vmatrix} \binom{2}{1} & \binom{4}{2} & \binom{6}{3} & \cdots & \binom{2m-2}{m-1} & \binom{2m}{m} \\ 1 & \binom{4}{1} & \binom{6}{2} & \cdots & \binom{2m-2}{m-2} & \binom{2m}{m-1} \\ 0 & 1 & \binom{6}{1} & \cdots & \binom{2m-2}{m-3} & \binom{2m}{m-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \binom{2m}{1} \end{vmatrix} = 2,$$

where the parentheses represent binomial coefficients.

SOLUTIONS.

Solutions of **2980** (*1922*, 271) were received from R. W. GENESE and R. F. DAVIS, Aberystwyth, Wales, and solutions of **2993** (*1922*, 420) and **3002** (*1923*, 41) were received from E. P. BUGDANOFF, Manchuria, China, after solutions of these problems had been sent to the printers.

3005 [1923, 76]. Proposed by F. D. MURNAGHAN, Johns Hopkins University.

Given the recurrence formulæ: $x_{n+1} = x_n(x_n + y_n)$, $y_{n+1} = y_n(2x_n + y_n)$, with the initial values $x_1 = 1$, $y_1 = 1$, it is desired to find a good approximation to $x_n/(2x_n + y_n)$ for large values of n. In particular, for values of n from $n = 20$ to $n = 30$.

Note.—This question arose in connection with a problem in genetics.

SOLUTION BY A. A. BENNETT, University of Texas.

1. The problem may be solved without recourse to any asymptotic formula by a simple transformation. The transformed variable is chosen so as to satisfy a simpler recursion relation. Let $s_n = (y_n + x_n)/x_n$. It is readily verified that the given recursion formulæ for x_n and y_n imply $s_{n+1} = s_n + 1 - 1/s_n$. The desired function is of the form $1/(s_n + 1)$. Using this relation, I have computed s_n on a computing machine for n from 1 to 31 inclusive, and have checked the results. To eight decimal figures, the last digit being possibly erroneous, we have

$s_1 = 2.00000000$	$s_{16} = 14.10441583$
$s_2 = 2.50000000$	$s_{17} = 15.03351606$
$s_3 = 3.10000000$	$s_{18} = 15.96699802$
$s_4 = 3.77741936$	$s_{19} = 16.90436884$
$s_5 = 4.51268836$	$s_{20} = 17.84521254$
$s_6 = 5.29109096$	$s_{21} = 18.78917510$
$s_7 = 6.10209402$	$s_{22} = 19.73595297$
$s_8 = 6.93821585$	$s_{23} = 20.68528402$
$s_9 = 7.79408658$	$s_{24} = 21.63694047$
$s_{10} = 8.66578418$	$s_{25} = 22.59072321$
$s_{11} = 9.55038781$	$s_{26} = 23.54645725$
$s_{12} = 10.44568002$	$s_{27} = 24.50398801$
$s_{13} = 11.34994667$	$s_{28} = 25.46317833$
$s_{14} = 12.26184053$	$s_{29} = 26.42390593$
$s_{15} = 13.18028671$	$s_{30} = 27.38606141$

$$s_{31} = 28.34954648.$$

This suffices for the practical problem raised.

2. To secure an asymptotic formula by the method of undetermined coefficients presents difficulties, since the general form of s_n as a function of n is not obvious.

Using the functional notation $s(n)$ in place of s_n we have $s(n+1) - s(n) = 1 - 1/s(n)$. If the relation had been $\lim_{h\to 0}\left[\frac{s(n+h)-s(n)}{h}\right] = 1 - \frac{1}{s(n)}$, this could have been written $\frac{ds(n)}{dn} = 1 - \frac{1}{s(n)}$, or $\frac{sds}{s-1} = dn$, whence $ds + \frac{ds}{s-1} = dn$ whence $s + \log(s-1) - c = n$. This relation is not of course the actual solution of the proposed problem and was obtained by virtue of an unjustified alteration. However as frequently occurs in the theory of difference equations a hint as to the nature of the solution of a given problem in differences is often obtained by considering the more familiar differential equation which results upon replacing, when possible, $s(n+1) - s(n)$ by $\lim_{h\to 0}\left[\frac{s(n+h)-s(n)}{h}\right]$, that is, by ds/dn. In the present case the analogous differential equation suggests that it might be much more difficult to solve for s as a function of n than to solve for n as a function of s and $\log(s-1)$.

We therefore write, in this case,

$$n + c = s + \log(s-1) + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3} + \frac{a_4}{s^4} + \cdots,$$

where s stands for s_n. Whence, putting $n+1$ for n,

$$n + 1 + c = s_{n+1} + \log(s_{n+1} - 1) + \frac{a_1}{s_{n+1}} + \frac{a_2}{s_{n+1}^2} + \frac{a_3}{s_{n+1}^3} + \frac{a_4}{s_{n+1}^4} + \cdots.$$

Using the relation $s_{n+1} = s_n + 1 - \frac{1}{s_n}$, and writing t for $1/s_n$, we have

$$n + 1 + c = s + 1 - t + \log(s - t) + a_1\frac{t}{1+t-t^2} + a_2\left(\frac{t}{1+t-t^2}\right)^2 + \cdots.$$

By subtracting the initial expansion from this, and noting that

$$\log\left(s - \frac{1}{s}\right) - \log(s-1) = \log\left(1 + \frac{1}{s}\right) = \log(1+t),$$

we have

$$0 = -t + \log(1+t) + a_1\left(\frac{t}{1+t-t^2} - t\right) + a_2\left[\left(\frac{t}{1+t-t^2}\right)^2 - t^2\right] + \cdots.$$

Now $\frac{t}{1+t-t^2} - t$, when expanded in powers of t, becomes $-t^2 + 2t^3 - 3t^4 + 5t^5 - 8t^6 + 13t^7 \cdots$, the coefficients forming a Farey sequence (with initial unity omitted). Similarly $\left(\frac{t}{1+t-t^2}\right)^2 - t^2$ yields $-2t^3 + 5t^4 - 10t^5 + 20t^6 - 38t^7 \cdots$. From $\left(\frac{t}{1+t-t^2}\right)^3 - t^3$, we obtain $-3t^4 + 9t^5 - 22t^6 + 51t^7 + \cdots$. From $\left(\frac{t}{1+t-t^2}\right)^4 - t^4$ we obtain $-4t^5 + 14t^6 - 40t^7 \cdots$. From the next term we have $-5t^6 + 20t^7 - \cdots$. From the next, $-6t^7 + \cdots$. The expansion of $\log(1+t)$ in powers of t is $t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \frac{t^6}{6} + \frac{t^7}{7} - \cdots$. We have then, altogether, the following equation:

$$\begin{aligned} 0 = &\left(-\tfrac{1}{2} - a_1\right)t^2 + \left(\tfrac{1}{3} + 2a_1 - 2a_2\right)t^3 + \left(-\tfrac{1}{4} - 3a_1 + 5a_2 - 3a_3\right)t^4 \\ &+ \left(\tfrac{1}{5} + 5a_1 - 10a_2 + 9a_3 - 4a_4\right)t^5 + \left(-\tfrac{1}{6} - 8a_1 + 20a_2 - 22a_3 + 14a_4 - 5a_5\right)t^6 \\ &+ \left(\tfrac{1}{7} + 13a_1 - 38a_2 + 51a_3 - 40a_4 + 20a_5 - 6a_6\right)t^7 + \cdots. \end{aligned}$$

Equating the coefficients of the several powers of t to zero, we have, at once, by successive solutions and substitutions,

$$a_1 = -\tfrac{1}{2}, \quad a_2 = -\tfrac{1}{3}, \quad a_3 = -\frac{5}{36}, \quad a_4 = -\frac{13}{240},$$
$$a_5 = -\frac{193}{1800}, \quad a_6 = -\frac{947}{7560}, \quad \cdots.$$

From the character of the computation, it is clear that further coefficients could have been obtained in the same manner, although with increasing difficulty. This use of undetermined

coefficients does not of itself throw any light upon the question of convergence. When these values that have been found are placed in the initial series, we have, as desired,

$$n + c = s + \log (s - 1) - \frac{1}{2s} - \frac{1}{3s^2} - \frac{5}{36s^3} - \frac{13}{240s^4} - \frac{193}{1800s^5} - \frac{947}{7560s^6} - \cdots,$$

where s stands for s_n. The constant, c, depends upon the initial values of s_n. To determine c, the values of $n = 15$, $n = 20$, $n = 25$, $n = 30$, $n = 31$ were tried with uniform agreement up to the last digit. Thus c was determined as .64018855.

To test the availability of this asymptotic form for large values of n, s_{100} was computed. This involved the independent computation of log s for values in the neighborhood of $s = 96$. Despite this added work and the fact that repeated interpolation was necessary, the computation was completed by use of a computing machine within a few minutes. The result obtained was

$$s_{100} = 96.09059614.$$

The use of this asymptotic formula implies that s_n is to be regarded as a continuous real function of n capable of asymptotic representation. No inquiry as to convergence was undertaken, but the formula above agreed most convincingly with the results secured by the recursion formula.

3008 [1923, 76]. Proposed by P. R. RIDER, Washington University.

The altitude of a right circular cone is a, the radius of its base is b, and its slant height is c. A string is wrapped n times about the cone, starting at the vertex and ending at the base, in such a manner that for any complete circuit the vertical rise (the cone being supposed to rest on its base) is the same. A bird at the vertex takes the end of the string in its beak and flies around the cone, unwinding the string, keeping it taut and always tangent to the curve of the string as it lies around the cone. Find an expression for the distance that the bird has flown when the string is completely unwound, (a) if it starts at the vertex, (b) if it starts at the base.

Solution by the Proposer.

(a) Consider a moving horizontal line which always passes through the axis of the cone and through a point of the string as it lies wrapped around the cone. Choose the limiting position of this line, as it approaches the vertex of the cone, as the x-axis; choose the axis of the cone as the z-axis, and take the plus direction upward; choose the y-axis perpendicular to the x- and z-axes. The origin will of course be the vertex of the cone.

Let θ be the angle through which the horizontal line has rotated for any given position, and denote by r the segment of the line included between the axis of the cone and the string. Then $r = b\theta/2n\pi$, and if x, y, z are the coördinates of a point on the string,

$$x = r \cos \theta = \frac{b\theta \cos \theta}{2n\pi}, \qquad y = \frac{b\theta \sin \theta}{2n\pi}, \qquad z = -\frac{a\theta}{2n\pi}.$$

It can easily be shown that the direction cosines of the tangent line to the curve are

$$\cos \alpha = \frac{b(\cos \theta - \theta \sin \theta)}{\sqrt{b^2\theta^2 + c^2}}, \qquad \cos \beta = \frac{b(\sin \theta + \theta \cos \theta)}{\sqrt{b^2\theta^2 + c^2}}, \qquad \cos \gamma = \frac{-a}{\sqrt{b^2\theta^2 + c^2}}.$$

If s represents the length of the string from the vertex of the cone to the point (x, y, z), then

$$s = \int_0^\theta \sqrt{x'^2 + y'^2 + z'^2}d\theta = \frac{1}{4bn\pi}\left(b\theta\sqrt{b^2\theta^2 + c^2} + c^2 \log \frac{b\theta + \sqrt{b^2\theta^2 + c^2}}{c}\right). \qquad (1)$$

Denote by X, Y, Z the coördinates of the point on the path of the bird corresponding to the point (x, y, z) on the cone. Then, if we note that the unwound portion of the string extends in the opposite direction from the positive tangent to the curve, we get

$$X = x - s \cos \alpha, \qquad Y = y - s \cos \beta, \qquad Z = z - s \cos \gamma.$$

Differentiation gives

$$\begin{aligned} X' &= x' - s' \cos \alpha - s(\cos \alpha)' = - s(\cos \alpha)', \\ Y' &= y' - s' \cos \beta - s(\cos \beta)' = - s(\cos \beta)', \\ Z' &= z' - s' \cos \gamma - s(\cos \gamma)' = - s(\cos \gamma)'. \end{aligned}$$

The length of the path of the bird is

$$\int_0^{2n\pi} \sqrt{X'^2 + Y'^2 + Z'^2}d\theta = b\int_0^{2n\pi} \frac{s}{b^2\theta^2 + c^2} \sqrt{(\theta^2 + 4)(b^2\theta^2 + c^2) - b^2\theta^2}d\theta,$$

in which s is given by (1).

(*b*) Taking the origin at the center of the base, we have, for suitably chosen axes,

$$r = b(1 - \theta/2n\pi),$$

and

$$x = r\cos\theta = b\left(1 - \frac{\theta}{2n\pi}\right)\cos\theta, \qquad y = r\sin\theta = b\left(1 - \frac{\theta}{2n\pi}\right)\sin\theta, \qquad z = \frac{a\theta}{2n\pi}\cdot$$

The direction cosines of the tangent line to the curve are

$$\cos\alpha = \frac{-2bn\pi\sin\theta - b(\cos\theta - \theta\sin\theta)}{\sqrt{b^2\phi^2 + c^2}}, \qquad \cos\beta = \frac{2bn\pi\cos\theta - b(\sin\theta + \theta\cos\theta)}{\sqrt{b^2\phi^2 + c^2}},$$

$$\cos\gamma = \frac{a}{\sqrt{b^2\phi^2 + c^2}},$$

where $\phi = \theta - 2n\pi$.

The length of the string from the end to the point (x, y, z) is found to be

$$s = \frac{1}{4bn\pi}\left[b\phi\sqrt{b^2\phi^2 + c^2} + 2bn\pi\sqrt{4b^2n^2\pi^2 + c^2} + c^2\log\frac{b\phi + \sqrt{b^2\phi^2 + c^2}}{-2bn\pi + \sqrt{4b^2n^2\pi^2 + c^2}}\right]\cdot \tag{2}$$

If X, Y, Z are the coördinates of the point on the path of the bird corresponding to the point (x, y, z) on the cone, we have

$$X = x - s\cos\alpha, \qquad Y = y - s\cos\beta, \qquad Z = z - s\cos\gamma,$$

and

$$X' = -s(\cos\alpha)', \qquad Y' = -s(\cos\beta)', \qquad Z' = -s(\cos\gamma)'.$$

The distance that the bird flies is found to be

$$b\int_{-2n\pi}^{0} \frac{s}{b^2\phi^2 + c^2}\sqrt{(\phi^2 + 4)(b^2\phi^2 + c^2) - b^2\phi^2}d\phi,$$

where s is defined by (2).

3011 [1923, 146]. Proposed by E. T. BELL, University of Washington.

In a certain paper it is stated that "it is easy to prove that, if $p > 0$ is an integer, the relation $a_1\sin\frac{\pi}{2p} + a_2\sin\frac{2\pi}{2p} + \cdots + a_{p-1}\sin\frac{(p-1)\pi}{2p} + a_p = 0$ necessitates $a_1 = a_2 = \cdots = a_{p-1} = a_p = 0$, the a's being integers." Prove it.

REMARK BY A. PELLETIER, Montreal, Canada.

If none of the a's are negative, the proposition is evident, since all the sines are positive. If the a's may be any integers, then the proposition is false. For let $p = 3$, then

$$a_1\sin(\pi/6) + a_2\sin(2\pi/6) + a_3 = 0$$

is satisfied by $a_1 = 2$, $a_2 = 0$, $a_3 = -1$.

NOTE BY OTTO DUNKEL, Washington University.

If the a's may be any integers, the proposition is false for an infinite number of cases. For let $p \geqq 3$ be an odd positive integer; then

$$2\sum_{k=1}^{k=(p-1)/2}(-1)^k\sin\left[\frac{(2k-1)\pi}{2p}\right] - (-1)^{(p-1)/2} = 0.$$

Here the coefficients of the sines and the independent term are all integers, and they are not all zero. This includes the special case above.

3013 [1923, 146]. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that $\lim\limits_{m\to\infty} \sum\limits_{r=0}^{r=m-1} \dfrac{4m}{4m^2 + (2r+1)^2} = \dfrac{\pi}{4}$.

SOLUTION BY ALEXANDER WIENER, Cornell University.

$$\lim_{m\to\infty} \sum_{r=0}^{r=m-1} \frac{4m}{4m^2 + (2r+1)^2} = \lim_{m\to\infty} \sum_{r=0}^{r=m-1} \frac{1}{m} \frac{1}{1 + \left(\dfrac{2r+1}{2m}\right)^2}.$$

If the interval from 0 to 1 is divided into m equal parts, the second factor in the sum to the right is the value of $1/(1+x^2)$ at the middle point of the interval from r/m to $(r+1)/m$. Hence by Riemann's definition of the definite integral the limit is

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}.$$

Also solved by the PROPOSER.

3014 [1923, 146]. Proposed by the late T. M. BLAKSLEE, Ames, Iowa.

If 1, p, d are the radius and sides of the regular inscribed pentagon and decagon, and if a triangle be formed from these three lengths, determine the angle opposite p without the use of trigonometric tables.

SOLUTION BY A. M. HARDING, University of Arkansas.

We have $2d \cos A = 1 + d^2 - p^2$, where $d = 2 \sin 18°$, $p = 2 \sin 36°$, and A denotes the angle opposite p.

Write x in place of 18°, then

$$\begin{aligned} 4 \sin x \cos A &= 1 + 4 \sin^2 x - 4 \sin^2 2x \\ &= 1 - 4 \sin x \sin 3x. \end{aligned}$$

Multiply both sides by $\cos x$, then

$$2 \sin 2x \cos A = \cos x - 2 \sin 2x \sin 3x = \cos 5x = \cos 90° = 0.$$

Hence $A = 90°$.

Also solved by THEODORE BENNETT, H. C. BRADLEY, E. P. BUGDANOFF, W. H. HILL, E. J. OGLESBY, A. PELLETIER, J. B. REYNOLDS, and the PROPOSER.

3017 [1923, 205]. Proposed by F. W. PERKINS, JR., Cambridge, Mass.

M. Edouard Lucas, in *Récréations Mathématiques* (vol. 1, pp. 41–51), gives a proof of the following theorem: Any labyrinth can be traversed in such a way as to cover each path twice, and only twice, returning finally to the starting point (provided merely that one has some means of marking paths as they are traversed) by following the three rules given below:

(1) On arriving at a new vertex (*i.e.*, one not visited before), leave by a new path if possible; otherwise return by the same path.

(2) On arriving at an old vertex by a new path, return by the same path.

(3) On arriving at an old vertex by an old path, leave by a new path if possible; otherwise by a path marked just once. It can be shown that it is always possible to do this until every path of the labyrinth has been marked twice.

Assuming Lucas' results, show that, if his rules are observed, the two trips on each path are in opposite directions.

NOTE: We may represent any labyrinth geometrically by a collection of points (" vertices "), not necessarily in a plane, which are joined by plane or twisted curves (" paths ") in any manner whatever, provided merely that it is possible to pass from any point to any other along the curves and that the curves do not meet except at vertices.

Solution by the Proposer.

Let us first consider the particular case that the labyrinth is such that it is possible to go from any given vertex to any other by only one route (*i.e.*, the case that the labyrinth is a "tree"). Suppose some particular path p is traversed the first time from the end P to the end P'. Let us now assume that the second time it is traversed in the same direction. In the intervening part of the itinerary, then, we go from P' to P without going along p. There are, therefore, two routes in the labyrinth, connecting P and P'. But this is impossible, since the labyrinth is a tree. This contradiction establishes the desired result for the case that the labyrinth is a tree.

Let us now consider the general case. We note from rules No. 1 and No. 3 that on arriving at an old vertex by a new path we return by the same path, just as if the vertex were a new vertex from which no other path led. Consider an arbitrary itinerary of the labyrinth in accordance with the rules. Let us imagine that all new paths by which we come to old vertices (on this particular itinerary) have their ends at that vertex blocked, so that the ends of these paths may be regarded as new vertices to which, in each case, only one path leads. This will not interfere with the particular itinerary under consideration, which is in accordance with the rules as applied to either the given labyrinth, or to the modified labyrinth. If the modified labyrinth is a tree, then in the given itinerary each path is traversed once in each direction. Hence, our theorem will be proved if we can show that the modified labyrinth is always a tree.

Suppose that this is not always the case. Then the modified labyrinth corresponding to some itinerary of the given labyrinth must have the property that it has two distinct vertices, P, Q, such that it is possible to go from P to Q along either of two routes, r, r', which are, in part at least, distinct. Consider the different stages of the itinerary at which we make our first trips on the various paths of r, r'. Let p be the last one of these paths on which we make the initial trip. Then both ends of p are old vertices when we make this trip; hence, after traversing p, we arrive at an old vertex by a new path.

From the manner of construction of the modified labyrinth, it is clear, however, that in the itinerary in question, we never come to an old vertex by a new path. This contradiction establishes the fact that every possible modified labyrinth is a tree, and so completes the proof of the theorem.

Note by the Editors:—The proof may also be put in two other forms:

(*a*) If any vertex has only one path ending in it, that path must necessarily be described twice in opposite directions. This part of any itinerary may be obliterated without modifying the application of the rules for the rest. Also, if there is a return to an old vertex by a new path, this path must also be described twice in opposite directions and it may be obliterated. If in any given itinerary this process of obliteration be continued, it may be shown that the whole itinerary is obliterated.

(*b*) Lucas has pointed out that before the completed process any vertex, not the initial vertex, has at a moment before or after entering it two or no paths used only once. It may be easily shown that the initial vertex cannot have any path described twice in the same direction. Now by aid of the fact pointed out by Lucas, it may be shown that there can be no first instance of any vertex having a path described twice in the same direction.

NOTES AND NEWS.

It is hoped that readers of the Monthly will coöperate in contributing to the interest of this department by sending items to R. W. Burgess, Brown University, Providence, R. I.

The following additional announcements of Summer Session courses have been received.

University of Wisconsin, June 30–August 8. By Professor E. B. Skinner: Differential geometry; Theory of algebraic numbers. By Professor Arnold Dresden: Analytic projective geometry; higher algebra. By Professor H. W. March: Mathematical formulation of scientific problems; Fourier series. By Professor W. W. Hart: The teaching of secondary mathematics. By Professor

WARREN WEAVER: Differential equations; Definite integrals; Generalized coordinates. By Mr. E. B. MILLER: Theory of equations. Mathematical Research will be directed by Professors SKINNER, DRESDEN, MARCH, and WEAVER.

Johns Hopkins University, July 7 to August 15. In addition to courses in Trigonometry, Analytic geometry, and Calculus, the following course is offered: By Professor F. D. MURNAGHAN: Differential geometry of curves and surfaces.

Between February 28 and March 11, Professor H. E. SLAUGHT made a tour through the states of Louisiana, Mississippi, Alabama and Georgia speaking on behalf of the interests of mathematics as a whole and, in particular, on behalf of the American Mathematical Society and the Mathematical Association of America. He delivered eight addresses in all, two of them before groups of teachers of collegiate mathematics and their guests from numerous departments related to this field, and six before large groups of students interested in mathematics.

At the University of Georgia, he was the guest of the Southeastern Section of the Association on the occasion of its third annual meeting. At the University of Louisiana he was the official representative of the Association at a conference of Louisiana and Mississippi teachers of collegiate mathematics called for the purpose of organizing a Section in those states. This conference was highly successful and resulted in the drafting of a petition for a charter signed by twenty-nine members and applicants for membership. A program of mathematical papers was carried through after the business session and much interest and enthusiasm were in evidence as surety for the success of Section number fourteen in case a charter is granted. The petition will be acted upon by the Trustees at an early date.

At Tulane University, Professor Slaught was the guest of Professor H. E. Buchanan. He found there an active mathematical club consisting of the faculty of Tulane and Sophie Newcomb, their advanced students majoring in mathematics, and high school teachers of mathematics in New Orleans. A good indication of the vitality of this club was shown by the fact that on a Mardi Gras holiday more than one hundred members turned out to hear an address on the "Romance of the Number System."

At the University of Alabama Professor Slaught was the guest of Professor Tomlinson Fort. He found two active mathematical clubs, one of freshmen and one of Seniors, the latter a chapter of Pi Mu Epsilon. The standing and influence of these clubs may be inferred from the fact that on their invitation nearly three hundred students came to hear this address on the Number System. The impression was strongly corroborated that a great field of usefulness is open to the undergraduate mathematical clubs and that every encouragement should be given to the organization of such clubs where they do not now exist.

Mr. C. C. CRAIG, instructor in mathematics at the University of Michigan, is one of 16 Americans to be granted fellowships for 1924–1925 by the American-Scandinavian Society. He plans to study mathematical statistics under Pro-

fessors C. V. L. CHARLIER and S. D. WICKSELL at the University of Lund, Sweden.

The following appointments and promotions in the department of mathematics at Hunter College of the City of New York have been announced: to succeed Professor EMMA M. REQUA, retired, as head of the department, Professor TOMLINSON FORT of the University of Alabama; to be an associate professor, Miss LAO G. SIMONS who has been acting head of the department for the present year; to be an instructor, Miss HELEN KUNTE.

W. E. BYRNE, who has studied at Rensselaer Polytechnic Institute and at the University of Paris, has been awarded an American Field Service Fellowship for French Universities for the year 1924–1925. Applications for the year 1925–1926 should reach the secretary, at 525 West 120th Street, New York City, not later than December 15. 1924.

At Harvard University, Professor G. D. BIRKHOFF will be exchange professor at Pomona, Colorado, and Grinnell Colleges during the first half of the academic year 1924–25. For the year 1924–25, Mr. H. W. BRINKMANN and Mr. J. L. HOLLEY have been appointed Benjamin Peirce instructors, and Mr. P. D. EDWARDS and Mr. MALCOLM MACLAREN, JR. instructors. Dr. J. L. WALSH has been promoted to an assistant professorship of mathematics.

Professor E. R. HEDRICK, of the University of Missouri, has been appointed professor of mathematics at the University of California and will be in charge of the department of mathematics at the southern branch, Los Angeles.

Professor A. N. WHITEHEAD, of the Imperial College of Science and Technology, London, has been appointed professor of philosophy at Harvard University.

Associate Professor CLARA E. SMITH, of Wellesley College, has been promoted to a full professorship of mathematics.

Dr. H. T. DAVIS, of the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Indiana.

Dr. G. A. PFEIFFER, of Columbia University, has been promoted to an assistant professorship of mathematics.

Dr. GEORGE RUTLEDGE, of Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to D. C. GILLESPIE, Cayuga Heights, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

The following are dates of Section meetings of the Association in 1923 (unless otherwise specified):

ILLINOIS, Elgin, May 2–3, 1924

IOWA, Des Moines, November; Ames, April, 1924

KANSAS, Topeka, January 20

KENTUCKY, Center College, April, 1924

MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Annapolis, December 8, 1924

MINNESOTA, Northfield, May 19

MISSOURI, University of Missouri, Columbia, November 30–December 1, 1924

OHIO, Ohio State University, Columbus, March 30–31

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 1924

SOUTHEASTERN, University of Georgia, Athens, Ga., March 7–8, 1924

TEXAS, Fort Worth, November 30–December 1, 1924

MICHIGAN, Ann Arbor, April 3, 1924

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924
NUMBER 6, JUNE

PUBLISHED BY THE ASSOCIATION
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others

THE MATHEMATICAL ASSOCIATION.

The Trustees of the Association have voted to approve the organization of a Louisiana-Mississippi Section. The report of the organization meeting will appear at an early date.

The Trustees have voted in favor of accepting the invitation of Cornell University to hold its summer meeting there in 1925, in conjunction with the summer meeting and the colloquium of the American Mathematical Society.

For the Committee on Arrangements in connection with the annual meeting of the Association in Washington, D. C., next December, President Rietz has appointed the following: H. L. Hodgkins, Chairman; W. D. Cairns, Archibald Henderson, W. D. Lambert, A. E. Landry, F. D. Murnaghan, and E. B. Phelps.

The following thirty-nine persons and six institutions have been elected to membership, on applications duly certified:

To Individual Membership.

Zelda Allen. Natchitoches, La.

H. E. Arnold, A.M. (Wesleyan). Instr., Wesleyan Univ., Middletown, Conn.

L. G. Butler, A.B. (Oregon). Prin., High School, LaConner, Wash.

J. P. Cole, B.S. (La. State Univ.). Instr., La. State Univ., Baton Rouge, La.

A. R. Congdon, A.M. (Nebraska). Asso. Prof., Pedagogy of Math., Univ. of Nebraska, Lincoln, Nebr.

T. F. Cope, B.S. (Tulane). Instr., Tulane Univ., New Orleans, La.

Winnifred D. Daly, A.B. (Tulane). Teacher, Wright High School, New Orleans, La.

Coralie Droz, A.M. (La. State Univ.). Teacher, High School, Baton Rouge, La.

R. W. Frary, B.S. (Va. Military Inst.). Asst. Prof., La. State Normal Coll., Natchitoches, La.

Neda B. Freeman, A.M. (Boston Univ.). Teacher, Irving Coll., Mechanicsburg, Pa.

Amelia Gaulden. Teacher, High School, Bogalusa, La.

Lucye Guilbeau, A.B. (La. State Univ.). Teacher, St. Gabriel High School, Baton Rouge, La.

J. D. Gwaltney, A.B. (Howard Coll., Ga.). Teacher, Sc. and Math., Jefferson Milit. Coll., Washington, Miss.

J. A. Hardin, A.B. (Tennessee). Prof., Centenary Coll., Shreveport, La.

E. J. Hickox, A.M. (Columbia). Prof., Math. and Physics, Internatl. Y. M. C. A. Coll., Springfield, Mass.

H. N. Hubbs, B.E. (Union). Instr., Hobart Coll., Geneva, N. Y.

A. Frances Johnson, Ph.D. (Minnesota). Instr., Univ. of Minnesota, Minneapolis, Minn.

Nellie M. Johnston, A.B. (Kearney State Teachers Coll.). Prin. and Teacher of Math., High School, Gibbon, Nebr.

Lois Karr, A.M. (Wisconsin). Instr., Lindenwood Coll., St. Charles, Mo.

J. A. Manning. Natchitoches, La.

Janice W. Mauldin, M.S. (Chicago). Prof., Grenada Coll., Grenada, Miss.

B. E. Mitchell, Ph.D. (Columbia). Prof., Millsaps Coll., Jackson, Miss.

J. E. Opp, B.S. (Nebraska). Asst. Instr., Univ. of Nebraska, Lincoln, Nebr.

W. A. Peters, A.B. (La. State Univ.). Prin., High School, Zachary, La.

V. S. Pugh, B.S. (La. Polytechnic Inst.). Instr., La. Polytechnic Inst., Ruston, La.

Louis Quarles, A.B. (Michigan). Patent Atty., 68 Wisconsin St., Milwaukee, Wis.

Marion B. Rood, M.S. (Michigan). Teacher, Math. and Physics, Westover School, Middlebury, Conn.

Rev. C. G. Scharf, A.B. (St. Cyril); B.S. (Chicago). Treas. and Instr. in Sc., St. Cyril High School, Chicago, Ill.

J. M. Sharp, A.B. (Mississippi). Prof., Woman's Coll., Hattiesburg, Miss.

C. R. SHERER, A.M. (Nebraska). Instr., Univ. of Nebraska, Lincoln, Nebr.
J. R. SHOPTAUGH, A.B. (Culver-Stockton). Prin., Univ. Demonstration High School, Baton Rouge, La.
J. H. SIMESTER, M.A. (Toronto). Instr., Carnegie Inst. of Technology, Pittsburgh, Pa.
W. B. STOKES, M.S., M.E. (Alabama Polytechnic Inst.). Head of Dept. of Math., Southwestern La. Inst., Lafayette, La.
NORMA E. TOUCHSTONE, A.B. (La. State Univ.). Teacher, High School, Le Compte, La.
B. A. TUCKER, A.B. (Mississippi). Prin., High School, Elton, La.
C. E. WEBB, B.S. (La. State Univ.). Teacher, Univ. Demonstration High School, Baton Rouge, La.
LOUIS WEISNER, Ph.D. (Columbia). Instr., Univ. of Rochester, Rochester, N. Y.
S. J. WIFVAT, B.S. (Drake). Instr., Drake Univ., Des Moines, Iowa.
F. E. WOOD, Ph.D. (Chicago). Asst. Prof., Northwestern Univ., Evanston, Ill.

To Institutional Membership.

LITTLE ROCK COLLEGE, Little Rock, Ark. Professor T. F. Smith, Official Representative.
UNIVERSITY OF SOUTHERN CALIFORNIA, Los Angeles, Calif. Professor H. C. Willett, Official Representative.
INDIANA UNIVERSITY, Bloomington, Ind. Professor S. C. Davisson, Official Representative.
LOYOLA UNIVERSITY, New Orleans, La. Dean F. D. Sullivan, Official Representative.
U. S. MILITARY ACADEMY, West Point, N. Y. Colonel C. P. Echols, Official Representative.
SETON HILL COLLEGE, Greensburg, Pa. Dean M. Francesca, Official Representative.

W. D. CAIRNS, *Secretary-Treasurer.*

A SECOND BUDGET OF EXERCISES ON DETERMINANTS.

By Sir THOMAS MUIR, Rondebosch, South Africa.

This second [1] collection of results, like the first, ranges freely over the whole subject—freely, but not quite impartially, some of the special forms of determinant receiving a little more than their fair share of attention. The skill required for the establishing of the results will be found also to vary over a wide range. On the one hand some of them are merely curious and more or less interesting instances of "evaluation," while on the other, there are a few that are statements of fresh properties and that belong to the borderland of recent research. The difficulties which these latter may present are worth the expenditure of some time and effort, both because of their own value and because of their possible fruitfulness in leading to other properties equally desirable to be known. It is thus hoped that all grades of readers of the MONTHLY who have begun the study of the subject may find something in the budget to suit their respective stages of advancement. Exceedingly few of the results can have been printed before; the two budgets, therefore, with their hundred items are a clear addition to the material equipment of every teacher of the theory of determinants.

1. $$\begin{vmatrix} \alpha+2\beta+2\gamma & \alpha+\beta & \alpha+\gamma \\ \beta+\alpha & 2\alpha+\beta+2\gamma & \beta+\gamma \\ \gamma+\alpha & \beta+\gamma & 2\alpha+2\beta+\gamma \end{vmatrix} = 9\Sigma\alpha\cdot\Sigma\alpha\beta.$$

[1] The first appeared in the MONTHLY for January, 1922, pp. 10–14.

2. If s stand for $\alpha + \beta + \gamma$, then

$$(m+1)s(m^2s^2 - 2ms^2 + 3\Sigma\alpha\beta)$$

$$= \begin{vmatrix} (m-1)s + 2\alpha & \beta + \gamma & \beta + \gamma \\ \gamma + \alpha & (m-1)s + 2\beta & \gamma + \alpha \\ \alpha + \beta & \alpha + \beta & (m-1)s + \gamma \end{vmatrix}$$

$$= \begin{vmatrix} ms - \alpha & \alpha + \beta & \alpha + \gamma \\ \beta + \alpha & ms - \beta & \beta + \gamma \\ \gamma + \alpha & \gamma + \beta & ms - \gamma \end{vmatrix}$$

$$= \frac{m+1}{m} \begin{vmatrix} ms - 2\alpha & \alpha & \alpha \\ \beta & ms - 2\beta & \beta \\ \gamma & \gamma & ms - 2\gamma \end{vmatrix}$$

$$= \frac{m+1}{m} \begin{vmatrix} ms - \gamma - \beta & \gamma & \beta \\ \gamma & ms - \alpha - \gamma & \alpha \\ \beta & \alpha & ms - \beta - \alpha \end{vmatrix}$$

3. If $bfg = cdh$, then

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

can be expressed in terms of three non-collinear elements and their complementary minors, the expressions being

$$\frac{BF}{g} + \frac{FG}{b} + \frac{GB}{f} - \frac{BFG}{bfg}, \qquad \frac{CD}{h} + \frac{DH}{c} + \frac{HC}{d} - \frac{CDH}{cdh}.$$

4. If each element of a chosen row of $|a_1b_2c_3|$ be replaced by the product of four other elements, namely, the corresponding elements in the other rows and the elements not in the same row or column with the said corresponding elements, the resulting determinant equals

$$- \begin{vmatrix} a_1a_2 & a_2a_3 & a_3a_1 \\ b_1b_2 & b_2b_3 & b_3b_1 \\ c_1c_2 & c_2c_3 & c_3c_1 \end{vmatrix}.$$

The same is true if columns be substituted throughout for rows, the immediate equivalent then being

$$- \begin{vmatrix} b_1c_1 & b_2c_2 & b_3c_3 \\ c_1a_1 & c_2a_2 & c_3a_3 \\ a_1b_1 & a_2b_2 & a_3b_3 \end{vmatrix},$$

which in substance does not differ from the previous equivalent.

5. The alternant

$$\begin{vmatrix} x_1 - a_1 & (x_1 - a_1)^2 & x_1 - a_2 & (x_1 - a_2)^2 \\ x_2 - a_1 & (x_2 - a_1)^2 & x_2 - a_2 & (x_2 - a_2)^2 \\ x_3 - a_1 & (x_3 - a_1)^2 & x_3 - a_2 & (x_3 - a_2)^2 \\ x_4 - a_1 & (x_4 - a_1)^2 & x_4 - a_2 & (x_4 - a_2)^2 \end{vmatrix} = 0.$$

6. The expansion of

$$\begin{vmatrix} \cdot & \cdot & a_3 & a_4 & a_5 & a_6 \\ \cdot & \cdot & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & \cdot & \cdot & c_5 & c_6 \\ d_1 & d_2 & \cdot & \cdot & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & \cdot & \cdot \\ f_1 & f_2 & f_3 & f_4 & \cdot & \cdot \end{vmatrix}$$

as an aggregate of products of 2-line minors having been found, a simple *memoria technica* for it is obtainable.

7. The number of positive terms in the determinant got from $|a_{1n}|$ by changing the signs of all the elements on one side of the diagonal is

$$\tfrac{1}{2}(n! + 2^{n-1}).$$

8.

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_1 & b_4 & b_5 & b_4 \\ a_3 & c_2 & a_1 & a_6 & c_5 & a_4 \\ f_3 & d_2 & f_1 & f_6 & d_5 & f_4 \\ e_1 & e_2 & e_1 & e_4 & e_5 & e_4 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 - a_3 & a_6 - a_4 \\ f_1 - f_3 & f_6 - f_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 + a_3 & a_2 + c_2 & a_5 + c_5 & a_6 + a_4 \\ b_1 & b_2 & b_5 & b_4 \\ e_1 & e_2 & e_5 & e_4 \\ f_1 + f_3 & f_2 + d_2 & f_5 + d_5 & f_6 + f_4 \end{vmatrix}.$$

9. If in an n-line determinant two diagonals parallel to the principal diagonal be taken, the one containing h elements and the other $n - h$ elements, the term composed of the said n elements bears the sign $(-1)^{(n+1)h}$. When the minor diagonals are parallel to the secondary diagonal, the exponent of -1 must be increased by $\frac{1}{2}n(n-1)$.

10.

$$\begin{vmatrix} 1 & a + b_1 + c_1 & ab_1 + b_1c_1 + c_1a & ab_1c_1 \\ 1 & b + c_1 + a_1 & bc_1 + c_1a_1 + a_1b & bc_1a_1 \\ 1 & c + a_1 + b_1 & ca_1 + a_1b_1 + b_1c & ca_1b_1 \\ 1 & a + b + c & ab + bc + ca & abc \end{vmatrix}$$

$$= -(a - a_1)(b - b_1)(c - c_1) \begin{vmatrix} 1 & a + a_1 & aa_1 \\ 1 & b + b_1 & bb_1 \\ 1 & c + c_1 & cc_1 \end{vmatrix}.$$

11. $|A_1B_2C_3|$ being the adjugate of $|a_1b_2c_3|$, the permanent

$$\overset{+}{\underset{}{|}}\begin{matrix} a_1^2A_1 & a_2^2A_2 & a_3^2A_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\overset{+}{\underset{}{|}}$$

can be expressed as a determinant.

12. If $|A_1B_2C_3|$ be the adjugate of $|a_1b_2c_3|$, then

$$a_1A_2A_3B_1C_1 + a_2A_3A_1B_2C_2 + a_3A_1A_2B_3C_3 = |a_1b_2c_3| \cdot |a_1a_2 \quad b_2b_3 \quad c_3c_1|.$$

13. $$\begin{vmatrix} a_2a_3a_4 & a_1a_3a_4 & a_1a_2a_4 & a_1a_2a_3 \\ b_2b_3b_4 & b_1b_3b_4 & b_1b_2b_4 & b_1b_2b_3 \\ c_2c_3c_4 & c_1c_3c_4 & c_1c_2c_4 & c_1c_2c_3 \\ d_2d_3d_4 & d_1d_3d_4 & d_1d_2d_4 & d_1d_2d_3 \end{vmatrix} = \begin{vmatrix} b_3b_4|a_2b_1| & b_2b_4|a_3b_1| & b_2b_3|a_4b_1| \\ c_3c_4|a_2c_1| & c_2c_4|a_3c_1| & c_2c_3|a_4c_1| \\ d_3d_4|a_2d_1| & d_2d_4|a_3d_1| & d_2d_3|a_4d_1| \end{vmatrix}.$$

14. $|a_1b_2c_3d_4|^3$ is a factor of

$$\begin{vmatrix} A_2A_3A_4 & A_1A_3A_4 & A_1A_2A_4 & A_1A_2A_3 \\ B_2B_3B_4 & B_1B_3B_4 & B_1B_2B_4 & B_1B_2B_3 \\ C_2C_3C_4 & C_1C_3C_4 & C_1C_2C_4 & C_1C_2C_3 \\ D_2D_3D_4 & D_1D_3D_4 & D_1D_2D_4 & D_1D_2D_3 \end{vmatrix},$$

and $|a_1b_2c_3d_4|^4$ is not.

15. $|a_1b_2c_3d_4|$ is a factor of

$$\begin{vmatrix} a_2A_3A_4 & a_1A_3A_4 & a_4A_1A_2 & a_3A_1A_2 \\ b_2B_3B_4 & b_1B_3B_4 & b_4B_1B_2 & b_3B_1B_2 \\ c_2C_3C_4 & c_1C_3C_4 & c_4C_1C_2 & c_3C_1C_2 \\ d_2D_3D_4 & d_1D_3D_4 & d_4D_1D_2 & d_3D_1D_2 \end{vmatrix},$$

and $|a_1b_2c_3d_4|^2$ is not.

16. $$\begin{vmatrix} 1 & ab + cd & a^rb^r + c^rd^r \\ 1 & ac + bd & a^rc^r + b^rd^r \\ 1 & ad + bc & a^rd^r + b^rc^r \end{vmatrix} = - \, |a^0b^1c^rd^{r+1}|.$$

17. If A, B, C, D stand for

$$\zeta^{1/2}(b, c, d), \quad -\zeta^{1/2}(a, c, d), \quad \zeta^{1/2}(a, b, d), \quad -\zeta^{1/2}(a, b, c),$$

respectively, then the product of the binomial sums of A, B, C, D is equal to

$$-\{\zeta^{1/2}(a, b, c, d)\}^2 \cdot (a + b - c - d)^2(a + c - b - d)^2(a + d - b - c)^2.$$

18. If $(x;\ h_1, h_2, h_3)$ stand for $(x - h_1)(x - h_2)(x - h_3)$,

$$\begin{vmatrix} \cdot & (b_1\colon a_1, a_2, a_3) & (c_1\colon a_1, a_2, a_3) & (d_1\colon a_1, a_2, a_3) \\ (a_1\colon b_1, b_2, b_3) & \cdot & (c_1\colon b_1, b_2, b_3) & (d_1\colon b_1, b_2, b_3) \\ (a_1\colon c_1, c_2, c_3) & (b_1\colon c_1, c_2, c_3) & \cdot & (d_1\colon c_1, c_2, c_3) \\ (a_1\colon d_1, d_2, d_3) & (b_1\colon d_1, d_2, d_3) & (c_1\colon d_1, d_2, d_3) & \cdot \end{vmatrix}$$

is divisible by $\zeta^{1/2}(a_1, b_1, c_1, d_1)$.

19. $$|a^0b^mc^nd^{m+n}| = \begin{vmatrix} (a^m-b^m)(c^n+d^n) & (a^m-c^m)(b^n+d^n) & (a^m-d^m)(b^n+c^n) \\ & (b^m-c^m)(a^n+d^n) & (b^m-d^m)(a^n+c^n) \\ & & (c^m-d^m)(a^n+b^n) \end{vmatrix}.$$

20. $|a^0b^1c^2d^3|\,|a^0b^4c^8d^{12}| - |a^0b^1c^4d^5|\,|a^0b^2c^8d^{10}| + |a^0b^1c^8d^9|\,|a^0b^2c^4d^6| = 0.$

21. Any n-line axisymmetric determinant can be expressed as an axisymmetric in $n!$ different ways.

22. The zero-axial axisymmetric determinant

$$\begin{vmatrix} \cdot & abc & abd & acd & bcd \\ abc & \cdot & abe & ace & bce \\ abd & abe & \cdot & ade & bde \\ acd & ace & ade & \cdot & cde \\ bcd & bce & bde & cde & \cdot \end{vmatrix} = (abcde)^3 \cdot 4(-1)^4.$$

23. If $|a_{16}|$ be axisymmetric, then

$$\frac{\partial^3 |a_{16}|}{\partial a_{12} \partial a_{34} \partial a_{56}} = -2S,$$

where S is the sum of the four minors that have $a_{12}a_{34}a_{56}$ for diagonal.

24.
$$\begin{vmatrix} \frac{1}{1\cdot3\cdot5} & \frac{1}{2\cdot4\cdot6} & \frac{1}{3\cdot5\cdot7} \\ \frac{1}{2\cdot4\cdot6} & \frac{1}{3\cdot5\cdot7} & \frac{1}{4\cdot6\cdot8} \\ \frac{1}{3\cdot5\cdot7} & \frac{1}{4\cdot6\cdot8} & \frac{1}{5\cdot7\cdot9} \end{vmatrix} = \frac{17}{2^{12}\cdot3\cdot5^3\cdot7^3}.$$

25.
$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} \begin{vmatrix} ax^2+2byx+cy^2 & bx^2+2cxy+dy^2 & cx^2+2dxy+ey^2 \\ bx^2+2cxy+dy^2 & cx^2+2dxy+ey^2 & dx^2+2exy+fy^2 \\ cx^2+2dxy+ey^2 & dx^2+2exy+fy^2 & ex^2+2fxy+gy^2 \end{vmatrix}$$
$$= \begin{vmatrix} ax+by & bx+cy & cx+dy \\ bx+cy & cx+dy & dx+ey \\ cx+dy & dx+ey & ex+fy \end{vmatrix}^2$$
$$+ \begin{vmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{vmatrix} \begin{vmatrix} ax^2+2bxy+cy^2 & bx^2+2cxy+dy^2 \\ bx^2+2cxy+dy^2 & cx^2+2dxy+ey^2 \end{vmatrix} y^2.$$

26. The four persymmetric determinants

$P(8_8, 9_8, \cdots, 14_8)$, $P(8_2, 8_3, \cdots, 8_8)$, $-P(9_7, 10_7, \cdots, 13_7)$, $-P(9_3, 9_4, \cdots, 9_7)$ are all equal, r_s standing for $r(r-1)\cdots(r-s+1)/s!$, and

$$P(a, b, c, d, e, f, g) \text{ for } \begin{vmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{vmatrix}.$$

27. Factor
$$\begin{vmatrix} a^2+b^2+c^2+d^2 & 2a^2 & 2a^2 & 2a^2 \\ 2b^2 & b^2+a^2 & b^2 & b^2 \\ 2c^2 & c^2 & c^2+a^2 & c^2 \\ 2d^2 & d^2 & d^2 & d^2+a^2 \end{vmatrix}.$$

28. If Σa stand for $a + b + c + d + e$, then

$$\begin{vmatrix} \Sigma a - 4a & a & a & a & a \\ b & \Sigma a - 4b & b & b & b \\ c & c & \Sigma a - 4c & c & c \\ d & d & d & \Sigma a - 4d & d \\ e & e & e & e & \Sigma a - 4e \end{vmatrix}$$

$$= - \Sigma a\{3(\Sigma a)^2 - 15(\Sigma a)^2\Sigma ab + 50\Sigma a\Sigma abc - 125\Sigma abcd\}.$$

29. Any determinant whose array consists of the arrays of four 3-line centro-symmetric determinants is resolvable into a 2-line and a 4-line determinant.

30. If the principal diagonal of a determinant be a, 0, a, 0, a, 0, a and all the other elements 1, its value is $2(a - 1)^3(a - 3)$.

31. If D be a determinant of the $(2n + 1)$th order whose odd rows have an a where they intersect the diagonal and a b elsewhere and whose even rows have a b where they intersect the diagonal and an a elsewhere, then

$$D = (-1)^n(a - b)^{2n}(nb - na + a).$$

32. If the minor got by deleting the rth row and sth column of

$$\begin{vmatrix} a & 2b & c & \cdot \\ \cdot & a & 2b & c \\ \cdot & b & 2c & d \\ b & 2c & d & \cdot \end{vmatrix}$$

be denoted by $[rs]$, then $[11] - [22] + 2[31] = 0$ and there are other similar linear relations.

33. In the dialytic eliminant of

$$\begin{aligned} a_0x^m + a_1x^{m-1} + \cdots + a_m &= 0, \\ b_0x^n + b_1x^{n-1} + \cdots + b_n &= 0, \end{aligned}$$

the array got by deleting the last row of a's, the last row of b's, and the last column is identical with the array got by deleting the first row of a's, the first row of b's, and the first column: and consequently the eliminant has pairs of identical secondary minors.

34. If the minors got from either of the arrays specified in the preceding by deleting one column at a time be M_0, $-M_1$, M_2, $-M_3$, $\cdots$, then the persymmetric determinant of the M's vanishes when $n = m + 2$.

35. In the dialytic eliminant of

$$\left.\begin{aligned} a_0x^n + a_1x^{n-1} + \cdots + a_n &= 0 \\ b_0x^n + b_1x^{n-1} + \cdots + b_n &= 0 \end{aligned}\right\}$$

(written with an a row and a b row alternately), the deletion of the last two rows and last column gives an array whose primary minors are the elements of the adjugate of Bézout's eliminant.

36. The bordered axisymmetric determinant

$$\begin{vmatrix} \cdot & 1 & 1 & 1 & 1 \\ 1 & 1 & 1+a & 1+b & 1+c \\ 1 & a+1 & a & a+b & a+c \\ 1 & b+1 & b+a & b & b+c \\ 1 & c+1 & c+a & c+b & c \end{vmatrix}$$

is 1/8 of the determinant got from it by making the diagonal elements all zeros.

37. If $|P_1Q_2R_3| \equiv \left\| \begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \\ c_1 & c_2 & \cdots & c_r \end{matrix} \right\| \cdot \left\| \begin{matrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r \\ \beta_1 & \beta_2 & \cdots & \beta_r \\ \gamma_1 & \gamma_2 & \cdots & \gamma_r \end{matrix} \right\|$,

then

$$\begin{vmatrix} \cdot & \cdot & \alpha_x & \beta_x & \gamma_x \\ \cdot & \cdot & \alpha_y & \beta_y & \gamma_y \\ a_x & a_y & P_1 & P_2 & P_3 \\ b_x & b_y & Q_1 & Q_2 & Q_3 \\ c_x & c_y & R_1 & R_2 & R_3 \end{vmatrix} = \sum_{\theta=1}^{\theta=r} \{ |a_x b_y c_\theta| \, |\alpha_x \beta_y \gamma_\theta| \}.$$

38. C being the functional symbol of a circulant, the axisymmetric determinant

$$\begin{vmatrix} 2a^m b^n c^n & -c^{m+2n} & -b^{m+2n} \\ -c^{m+2n} & 2a^n b^m c^n & -a^{m+2n} \\ -b^{m+2n} & -a^{m+2n} & 2a^n b^n c^m \end{vmatrix} = -2a^n b^n c^n C(a^{m+n}, b^{m+n}, c^{m+n}).$$

39. In the circulant $C(x, y, z, w)$,

$$\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial z^2} = -2\frac{\partial^2 C}{\partial y \partial w}.$$

40. If $\begin{vmatrix} abc \\ \alpha\beta\gamma \end{vmatrix}$ stands for the minor whose elements belong to the ath, bth, cth rows and αth, βth, γth columns, then in the case of any 6-line circulant we have

$$\begin{vmatrix} 123 \\ 123 \end{vmatrix} + \begin{vmatrix} 123 \\ 156 \end{vmatrix} + \begin{vmatrix} 123 \\ 426 \end{vmatrix} + \begin{vmatrix} 123 \\ 453 \end{vmatrix} = \begin{vmatrix} 135 \\ 135 \end{vmatrix} + 3\begin{vmatrix} 162 \\ 135 \end{vmatrix},$$

and there are six other 3-line minors that are connected in the same way.

41. The circulant $C(a, b, c)$ remains unaltered for all permutations of the variables: but $C(a, b, c, d)$ only for certain permutations.

42. $$\begin{vmatrix} C(x, b, c) & C(y, b, c) & C(z, b, c) \\ C(x, c, a) & C(y, c, a) & C(z, c, a) \\ C(x, a, b) & C(y, a, b) & C(z, a, b) \end{vmatrix} = 3\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} \begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix}$$

and when $x, y, z = a, b, c$, it has $C(a, b, c)$ for a factor.

43. If any n-line determinant having the sum of each row equal to S (for example, the circulant) be bordered with units, the resulting determinant is equal to $-\frac{n}{S}$ times the original.

44.
$$\begin{vmatrix} |a_1b_2c_3| & |a_1b_2c_4| & |a_1b_2c_5| & |a_1b_2c_6| \\ |a_1b_2d_3| & |a_1b_2d_4| & |a_1b_2d_5| & |a_1b_2d_6| \\ |a_1c_2d_3| & |a_1c_2d_4| & |a_1c_2d_5| & |a_1c_2d_6| \\ |b_1c_2d_3| & |b_1c_2d_4| & |b_1c_2d_5| & |b_1c_2d_6| \end{vmatrix} = 0.$$

45.
$$\begin{vmatrix} |b_2c_3d_4| & |b_1c_3d_4| & |b_1c_2d_4| & |b_1c_2d_3| \\ |a_2c_3d_4| & |a_1c_3d_4| & |a_1c_2d_4| & |a_1c_2d_3| \\ |y_2z_3w_4| & |y_1z_3w_4| & |y_1z_2w_4| & |y_1z_2w_3| \\ |x_2z_3w_4| & |x_1z_3w_4| & |x_1z_2w_4| & |x_1z_2w_3| \end{vmatrix} = |a_1b_2c_3d_4| \cdot |c_1d_2z_3w_4| \cdot |x_1y_2z_3w_4|.$$

46.
$$\begin{vmatrix} |a_1b_2| & |b_1c_3| & |c_1a_4| & |f_2g_3| & |g_2h_4| & |h_3f_4| \end{vmatrix}$$
$$= \begin{vmatrix} |a_1b_2f_3g_4| & |a_1b_2g_3h_4| & |a_1b_2h_3f_4| \\ |b_1c_2f_3g_4| & |b_1c_2g_3h_4| & |b_1c_2h_3f_4| \\ |c_1a_2f_3g_4| & |c_1a_2g_3h_4| & |c_1a_2h_3f_4| \end{vmatrix}.$$

47. If the last two-line minor of every element of the compound determinant
$$\begin{vmatrix} |a_1b_2c_3d_4e_5f_6| & |a_1b_2c_3d_4e_5f_7| & |a_1b_2c_3d_4e_5f_8| & |a_1b_2c_3d_4e_6f_7| & |a_1b_2c_3d_4e_6f_8| & |a_1b_2c_3d_4e_7f_8| \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ |a_1b_2c_3d_4g_5h_6| & |a_1b_2c_3d_4g_5h_7| & |a_1b_2c_3d_4g_5h_8| & |a_1b_2c_3d_4g_6h_7| & |a_1b_2c_3d_4g_6h_8| & |a_1b_2c_3d_4g_7h_8| \end{vmatrix}$$
have zero-elements, the determinant is equal to
$$|a_5b_6c_7d_8|^3 \cdot |e_1f_2g_3h_4|^3 \cdot |a_1b_2c_3d_4|^3.$$

48. If the n-line continuant $K(x, {}^{-1}, x, {}^{-1}, x, {}^{-1} \cdots \cdots)$
$$\begin{vmatrix} x & 1 & \cdot & \cdot & \cdots \\ 1 & x & 1 & \cdot & \cdots \\ \cdot & 1 & x & 1 & \cdots \\ \cdot & \cdot & 1 & x & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$
be denoted by D_n, then
$$D_{31} = (D_8^2 - 2D_7^2 + D_6^2)(D_4^2 - 2D_3^2 + D_2^2)(D_2^2 - 2D_1^2 + 1)(D_1^2 - 2)D_1$$
and the roots of the equation $D_4^2 - 2D_3^2 + D_2^2 = 0$ are $\pm\sqrt{2 \pm \sqrt{2 \pm \sqrt{2}}}$.

49. If the minor diagonals of a continuant be
$$\begin{matrix} a_1, & a_2, & \cdots, & a_{n-1}, \\ a_{n-1}, & a_{n-2}, & \cdots, & a_1, \end{matrix}$$
and the main diagonal be
$$a_1 + a_n, \quad a_2 + a_{n-1}, \quad \cdots, \quad a_n + 1,$$

then the value of the continuant is

$$a_{m+1}\cdots a_{2m}\{a_{m+1}\cdots a_{2m}+2a_m\cdot\text{first }(m-1)\text{-line minor}\},\text{ when } n=2m,$$

and

$$2a_m\cdots a_{2m-1}\times\text{ first }(m-1)\text{-line minor, when } n=2m-1.$$

50. The Hessian of the 4-line circulant is $3\cdot 4^4$ times the square of the circulant.

51. The Hessian of the alternant $|a^0b^1c^3|$ is $2\cdot 3^3|a^0b^1c^2|\cdot C(a, b, c)$.

52. If $X=x^my^nz^pw^q$, $Y=y^mz^nw^px^q$, $Z=z^mw^nx^py^q$, $W=w^mx^ny^pz^q$, show that

$$\frac{\partial(X, Y, Z, W)}{\partial(x, y, z, w)}=\frac{XYZW}{xyzw}C(m, n, p, q).$$

53. If Δ be the discriminant of $ax^2+by^2+cz^2+2fyz+2gzn+2hxy$ and $A, B, \cdots$ be the cofactors of $a, b, \cdots$ in Δ, then

$$\frac{\partial(B, C, F, G, H)}{\partial(b, c, f, g, h)}=a^2\Delta \quad\text{and}\quad \frac{\partial(A, B, C, G, H)}{\partial(a, b, c, g, h)}=2bc\Delta,$$

and consequently

$$\sum\frac{\partial(B, C, F, G, H)}{\partial(b, c, f, g, h)}=(a+b+c)^2\Delta.$$

54. If a 3-line orthogonant be axisymmetric, the sum of the diagonal elements is either -1 or 3.

55. If any column of an orthogonant be replaced by a new column of arbitrarily chosen elements, the determinant so formed contains as a factor the product of the said two columns.

56. If $|\alpha_1\beta_2\gamma_3|$ be a positive unit orthogonant, then

$$|\alpha_1{}^2\beta_2{}^2\gamma_3{}^2| = |\overset{+}{\alpha_1}\beta_2\overset{+}{\gamma_3}|.$$

57. If $|a_1b_2c_3|$ and $|\alpha_1\beta_2\gamma_3|$ be positive unit orthogonants and

$$|a_1+\alpha_1 \quad b_2+\beta_2 \quad c_3+\gamma_3|=0,$$

then the product of the orthogonants is axisymmetric: in other words

$$\Sigma a\beta=\Sigma b\alpha,\qquad \Sigma a\gamma=\Sigma c\alpha,\qquad \Sigma b\gamma=\Sigma c\beta.$$

58. If $|a_1b_2c_3|$ and $|\alpha_1\beta_2\gamma_3|$ be orthogonants whose basic numbers are s and σ respectively, then the square of any row or column of their product is $s\sigma$.

59. $$\begin{vmatrix} m & \alpha & \beta & \gamma \\ a & \cdot & d & e \\ b & d & \cdot & f \\ c & e & f & \cdot \end{vmatrix} = \begin{vmatrix} \alpha & \beta & \gamma \\ & d & e \\ & & f \end{vmatrix}\cdot\begin{vmatrix} a & b & c \\ & d & e \\ & & f \end{vmatrix} - 2df\begin{vmatrix} \alpha & m & \gamma \\ & a & e \\ & & c \end{vmatrix}.$$

60. If a zero-axial skew determinant have each of its elements augmented by x, it is unaltered in value when even-ordered; and, when odd-ordered, it reduces to a term in x whose coefficient is a zero-axial determinant of the next higher order.

61. The $2n$-line Pfaffian whose element in the (r, s)th place is $a_r a_s$ is equal to $a_1 a_2 \cdots a_n$: for example,

$$\left| \begin{array}{ccc} a_1a_2 & a_1a_3 & a_1a_4 \\ & a_2a_3 & a_2a_4 \\ & & a_3a_4 \end{array} \right| = a_1a_2a_3a_4.$$

62. If n of the frame-lines of a $2n$-line Pfaffian intersect one another in zero elements, the Pfaffian is expressible as an n-line determinant. For example, when $n = 4$ and the lines which have zero crossings are the 1st, 3d, 5th, 7th, we have

$$\left| \begin{array}{ccccccc} a_2 & \cdot & a_4 & \cdot & a_6 & \cdot & a_8 \\ & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\ & & c_4 & \cdot & c_6 & \cdot & c_8 \\ & & & d_5 & d_6 & d_7 & d_8 \\ & & & & e_6 & \cdot & e_8 \\ & & & & & f_7 & f_8 \\ & & & & & & g_8 \end{array} \right| = \left| \begin{array}{cccc} a_2 & a_4 & a_6 & a_8 \\ -b_3 & c_4 & c_6 & c_8 \\ -b_5 & -d_5 & e_6 & e_8 \\ -b_7 & -d_7 & -f_7 & g_8 \end{array} \right|.$$

63. Any zero-axial skew permanent of odd order vanishes.

64. If the three-line determinants of the array

$$\begin{array}{cccccc} 1 & 1 & \cdots & 1 & 1 \\ a+b & b+c & \cdots & c+f & f+a \\ ab & bc & \cdots & ef & fa \end{array}$$

be denoted by the numbers of their columns in the array, then

$$125 \cdot 346 = 452 \cdot 613.$$

65. The 4-line determinants of the array

$$\left\| \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e \\ c+d & d+e & e+a & a+b & b+c \\ eab & abc & bcd & cde & dea \end{array} \right\|$$

have the common factor $\Sigma ab(c^2 - e^2)$.

66.

$$\left| \begin{array}{cccc} a_1-\alpha_1 & -\beta_1 & a_2 & \cdot \\ -\alpha_2 & a_1-\beta_2 & \cdot & a_2 \\ b_1 & \cdot & b_2-\alpha_1 & -\beta_1 \\ \cdot & b_1 & -\alpha_2 & b_2-\beta_2 \end{array} \right| = \left| \begin{array}{cccc} 1 & -(a_1+b_2) & |a_1b_2| & \cdot \\ \cdot & 1 & -(a_1+b_2) & |a_1b_2| \\ 1 & -(\alpha_1+\beta_2) & |\alpha_1\beta_2| & \cdot \\ \cdot & 1 & -(\alpha_1+\beta_2) & |\alpha_1\beta_2| \end{array} \right|.$$

67. The inverso-symmetric determinant

$$\begin{vmatrix} 1 & a & b & c & j \\ a^{-1} & 1 & d & e & i \\ b^{-1} & d^{-1} & 1 & f & h \\ c^{-1} & e^{-1} & f^{-1} & 1 & g \\ j^{-1} & i^{-1} & h^{-1} & g^{-1} & 1 \end{vmatrix}$$

is expressible as a quasi-quadric in $\alpha\zeta$, $\beta\epsilon$, $\gamma\delta$, where

$$\begin{aligned} \alpha &= da - b, & \delta &= fb - c, \\ \beta &= ea - c, & \epsilon &= hb - j, \\ \gamma &= ia - f, & \zeta &= ge - j. \end{aligned}$$

68. If to $\beta_1, \beta_2, \beta_3, \cdots$ be given the values

$$2b, \quad 2b^2 + c, \quad 2bc + d, \quad 2bd + e - 2b^4 + \tfrac{1}{2}c^2, \quad \cdots,$$

then the values of the recurrents

$$-\beta_1, \quad \begin{vmatrix} \beta_1 & 1 \\ \beta_2 & \beta_1 \end{vmatrix}, \quad -\begin{vmatrix} \beta_1 & 1 & \cdot \\ \beta_2 & \beta_1 & 1 \\ \beta_3 & \beta_2 & \beta_1 \end{vmatrix}, \quad \begin{vmatrix} \beta_1 & 1 & \cdot & \cdot \\ \beta_2 & \beta_1 & 1 & \cdot \\ \beta_3 & \beta_2 & \beta_1 & 1 \\ \beta_4 & \beta_3 & \beta_2 & \beta_1 \end{vmatrix}, \quad \cdots$$

will be the same as those of the β's, save that $b, c, d, e, \cdots$ will be negative: in other words, the value of the rth recurrent can be got from that of β_r by changing the signs of the letters involved.

69. The sum of the positive terms of the determinant $|11 \cdot 22 \cdot 33 \cdot 44|$ is

$$11 \cdot 22 \cdot 33 \cdot 44 + \Sigma\{11 \cdot (23 \cdot 34 \cdot 42 + 24 \cdot 43 \cdot 32)\} + \Sigma\{(12 \cdot 21)(34 \cdot 43)\}$$

and there is a corresponding expression for the sum of the negative terms.

70. The sum of the elements of the adjugate of the product of $|a_1b_2c_3|$, $|f_1g_2h_3|$, $|l_1m_2r_3|$ is in bilinear form

$$\begin{array}{ccc|l} A_1 + B_1 + C_1 & A_2 + B_2 + C_2 & A_3 + B_3 + C_3 & \\ \hline F_1 & G_1 & H_1 & M_1 + M_2 + M_3 \\ F_2 & G_2 & H_2 & N_1 + N_2 + N_3 \\ F_3 & G_3 & H_3 & R_1 + R_1 + R_3, \end{array}$$

where the capital letters are used in the usual way in connection with an adjugate.

THE TRIGONOMETRY OF CORRELATION.[1]

By DUNHAM JACKSON, University of Minnesota.

Let $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ be two sets of n real numbers each, neither set consisting entirely of zeros. Let

$$r = \frac{\Sigma x_k y_k}{\sqrt{(\Sigma x_k^2)(\Sigma y_k^2)}}, \tag{1}$$

the summation extending in each case from $k = 1$ to $k = n$. If

$$\Sigma x_k = \Sigma y_k = 0, \tag{2}$$

the quantity r is the *coefficient of correlation* of the given sets of numbers.[2] Most of the work that is to be done here is entirely independent of the restriction (2), and it will be understood that that restriction is not imposed unless expressly mentioned.

By way of geometric representation, it is customary to think of the x's and y's as the coördinates of n points $(x_1, y_1), \cdots, (x_n, y_n)$ in a plane; and this representation is very important for an understanding of the algebraic relations involved. Even more suggestive in some respects is a "geometric representation" not by means of n points in space of two dimensions, but by means of two points in space of n dimensions, with coördinates $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$. It will perhaps best serve the purpose of the moment if the properties of n-dimensional space that are needed are accepted as given merely by analogy. There would be no prohibitive difficulty in filling out the analogies into real demonstrations. But when the facts have once been suggested, it is probably easiest to demonstrate them by straightforward algebraic processes. Algebraic proofs of most of the facts presented here will be found in the writer's paper on the "Algebra of Correlation" in a recent number of the MONTHLY[3] (*1924*, 110–121).

When $n = 3$, the sets of numbers (x_1, x_2, x_3) and (y_1, y_2, y_3) can actually be represented by two points P and Q in space. If the origin of coördinates is designated by O, one of the first things learned in solid analytic geometry is that the formula (1) gives the *cosine of the angle* between the lines OP and OQ. Of course the case of an n actually equal to 3 would be trivial for statistical

[1] Presented to the Minnesota Section of the Association, May 24, 1924. The same principles were set forth in a paper presented to the American Mathematical Society, under a different title, April 19, 1924. For the underlying ideas, *cf.* the words of Karl Pearson quoted by Professor Huntington in this MONTHLY (*1919*, 422). See also James McMahon, Hyperspherical goniometry; and its application to correlation theory for n variables, *Biometrika*, vol. 15 (1923), pp. 173–208. The point of view adopted here, to be sure, is somewhat different from that of Professor McMahon's paper.

[2] More generally, it is the coefficient of correlation of $(x_1 + \bar{x}, x_2 + \bar{x}, \cdots, x_n + \bar{x})$ and $(y_1 + \bar{y}, y_2 + \bar{y}, \cdots, y_n + \bar{y})$, if $\bar{x}$ and $\bar{y}$ are any two numbers; the original x's and y's are then the deviations of the new numbers from their respective means $\bar{x}$ and $\bar{y}$.

[3] This paper will be cited by the letter A.

purposes; but it suggests more general relations which are by no means trivial. We shall say, by analogy or by definition, that the sets of numbers $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ are the *coördinates* of two *points* P and Q, that the origin O is the point with coördinates $(0, 0, \cdots, 0)$, and that r, as defined by (1), is the *cosine of the angle* between the lines OP and OQ in n-dimensional space. The angle itself will be regarded as defined by the equation

$$\theta = \cos^{-1} r,$$

and will be taken positive (or zero) and not greater than 180°.

If this interpretation is to be acceptable, the first requirement is that the numerical value of r belong to the interval from -1 to $+1$. It is a fact of fundamental importance, readily proved by algebra,[1] that the number defined by the formula (1) always does satisfy this condition.

A fact which is even more immediately recognized algebraically is that the value of r is not changed if the numbers $(x_1, x_2, \cdots, x_n)$ are replaced by $(cx_1, cx_2, \cdots, cx_n)$, the multiplier c (> 0) being the same for all the numbers. The geometric interpretation also is immediate. The point P' with the coördinates $(cx_1, \cdots, cx_n)$ is a point *on the line* OP, its distance from the origin being such that $OP'/OP = c$. The angle POQ is then the same as the angle $P'OQ$, and the cosine of the angle—the value of r—is the same in both cases. A negative c would replace the angle by its supplement, and would reverse the sign of r. The y's could of course be multiplied by a common factor in the same way.

Let σ and τ be the *standard deviations*[2] of the x's and the y's:

$$\sigma = \sqrt{(\Sigma x_k^2)/n}, \qquad \tau = \sqrt{(\Sigma y_k^2)/n}.$$

The distances OP and OQ are $\sigma\sqrt{n}$ and $\tau\sqrt{n}$ respectively. Let

$$s_k = x_k/\sigma, \qquad t_k = y_k/\tau,$$

and let S and T be the points $(s_1, \cdots, s_n)$ and $(t_1, \cdots, t_n)$, respectively situated on OP and OQ at the distance $\sqrt{n}$ from the origin. If x_k and y_k in (1) are replaced by σs_k and τt_k, the formula becomes simply $r = (\Sigma s_k t_k)/n$. Another expression which is sometimes useful comes at once[3] from an application of the law of cosines in the isosceles triangle SOT:

$$\overline{ST}^2 = \overline{OS}^2 + \overline{OT}^2 - 2\overline{OS}\cdot\overline{OT}\cos\theta;$$

that is,

$$\Sigma(s_k - t_k)^2 = n + n - 2nr, \qquad r = 1 - \frac{1}{2n}\Sigma(s_k - t_k)^2.$$

If T' is the point $(-t_1, \cdots, -t_n)$, the cosine of the angle SOT' is $-r$, and the

[1] *Cf.* the paper A, p. 113.

[2] The name is applicable only when the conditions (2) are satisfied; otherwise the formulas are to be used on their own merits.

[3] The "proof" can of course be given in a variety of ways; this form, which is probably the simplest, was suggested to me by Professor R. W. Brink. For an algebraic demonstration, *cf.* A, p. 117.

law of cosines in the triangle SOT' gives

$$r = -1 + \frac{1}{2n}\Sigma(s_k + t_k)^2.$$

One of the most important problems in the statistical treatment of the numbers (x_k), (y_k) is the determination of a multiplier λ in such a way that the expression

$$\Sigma(y_k - \lambda x_k)^2 \tag{3}$$

shall be a minimum. This λ (when the equations (2) are fulfilled) is the slope of what is called a *line of regression.* The name applies to the two-dimensional representation, with which we are not concerned in this paper; but the interpretation in the n-dimensional figure is no less simple (and is independent of (2)). The point P' with coördinates $(\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$, for arbitrary λ, is an arbitrary point on the line OP. The distances OP, OP', and OQ are $\sigma\sqrt{n}$, $\lambda\cdot\overline{OP} = \lambda\sigma\sqrt{n}$, and $\tau\sqrt{n}$ respectively. The expression (3) is the square of the distance $P'Q$. The problem is then to determine λ so that this distance shall be a minimum. The minimum is of course attained when P' is the foot of the perpendicular from Q on OP. This means that $\overline{OP'} = \overline{OQ}\cos\theta = r\tau\sqrt{n}$, or, by substitution of the value given above for $\overline{OP'}$,

$$\lambda\sigma\sqrt{n} = r\tau\sqrt{n}, \qquad \lambda = (\tau/\sigma)r.$$

The minimum value of (3) is

$$\overline{P'Q}^2 = (\overline{OQ}\sin\theta)^2 = n\tau^2\sin^2\theta = n\tau^2(1 - r^2).$$

A similar calculation can be performed with the perpendicular from P on OQ, corresponding to the determination of the other line of regression in the two-dimensional diagram.[1] It is to be noticed that the figure to which the above discussion relates is itself merely a two-dimensional figure, though it has to be regarded as situated in n-dimensional space.

The power of the method is still more forcibly illustrated in connection with the idea of partial correlation. Let $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ and $(z_1, \cdots, z_n)$ be three sets of numbers, corresponding to three points P, Q, R. In dealing with the lines OP, OQ, OR, we shall be concerned with a three-dimensional figure, situated in n-dimensional space, to be sure, but intelligible without reference to its n-dimensional background. The coefficient of partial correlation between x and y, when z is the only other variable to be taken into account, is defined as follows. Let λ and μ be determined so that the sums $\Sigma(x_k - \lambda z_k)^2$, $\Sigma(y_k - \mu z_k)^2$ are reduced to the smallest possible values. When λ and μ are so defined, let

$$u_k = x_k - \lambda z_k, \qquad v_k = y_k - \mu z_k.$$

Then the coefficient of partial correlation[2] between the x's and the y's is the

[1] For the algebraic reasoning, *cf. A*, p. 119.

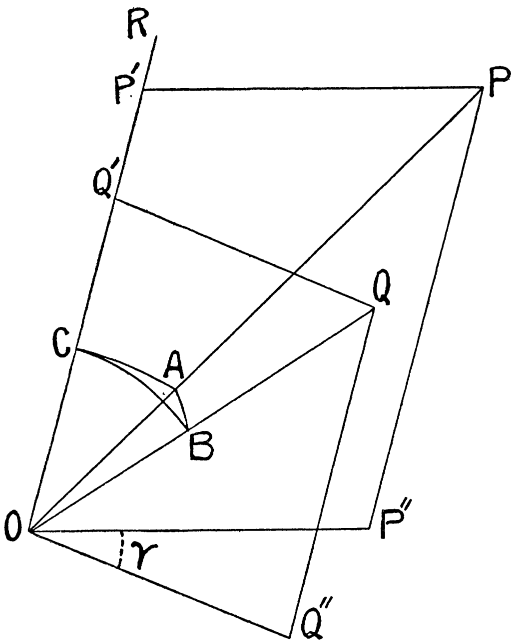
[2] The established use of the term involves the hypothesis that $\Sigma x_k = \Sigma y_k = \Sigma z_k = 0$, but the substance of the discussion, apart from the name, is independent of this restriction. In

ordinary coefficient of correlation between the u's and the v's,

$$r' = \frac{\Sigma u_k v_k}{\sqrt{(\Sigma u_k^2)(\Sigma v_k^2)}}\,.$$

For practical calculation, it is important to know that r' can be expressed[1] in terms of the ordinary coefficients of correlation between the x's, the y's, and the z's, without explicit evaluation of the u's and v's. It is for the form of this expression that a geometric suggestion is to be sought.

The points P' and Q', with the coördinates $(\lambda z_1, \lambda z_2, \cdots, \lambda z_n)$, $(\mu z_1, \mu z_2, \cdots, \mu z_n)$, are the feet of the perpendiculars from P and Q on OR (see figure). The numbers u_k and v_k are the components of the vectors $P'P$ and $Q'Q$, or, if OP'' and OQ'' are drawn from O equal and parallel to $P'P$ and $Q'Q$ respectively, the u's and the v's are the coördinates of the points P'' and Q''. *Then r' is the cosine of the angle $\gamma = P''OQ''$, which measures the dihedral angle between the planes POR and QOR.*



Let the angles QOR, POR, and POQ be denoted now by a, b, and c, and their cosines by r_{23}, r_{13}, and r_{12}, so that r_{12} is the number given by the formula (1), while r_{13} and r_{23} are the corresponding coefficients formed for the x's and z's and for the y's and z's respectively. Let a sphere be constructed with its center at O. Let the lines OP, OQ, and OR pierce the sphere at the points A, B, and C, the traces of the planes QOR, POR and POQ forming the sides of a spherical triangle ABC. The angular measures of these sides are a, b, and c, while the dihedral angle γ is the angle ACB of the spherical triangle. By the law of cosines in spherical trigonometry,

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma,$$

$$\cos \gamma = \frac{\cos c - \cos a \cos b}{\sin a \sin b}\,;$$

order that the formula may have a meaning, the special case that the u's or the v's are all zero has to be ruled out; that is, it must be assumed that neither the x's nor the y's are proportional to the z's.

[1] For an algebraic proof, *cf.* *A*, p. 120.

that is,

$$r' = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}}, \tag{4}$$

which is the desired formula.

A relation of inequality is suggested by the fact that a side of a spherical triangle can not be greater than the sum of the other two sides. In the triangle under consideration, $c \leqq a + b$, whence, if $a + b \leqq \pi$,

$$\cos c \geqq \cos (a + b) = \cos a \cos b - \sin a \sin b, \tag{5}$$

that is,

$$r_{12} \geqq r_{13}r_{23} - \sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}. \tag{6}$$

The equality holds, of course, only if the triangle degenerates so that A, B, and C are all on the same great circle; in other words, if the four points O, A, B, C lie in one (two-dimensional) plane. If $a + b > \pi$, (5) is obtained from the fact that $a + b + c \leqq 2\pi$, so that

$$c \leqq 2\pi - a - b, \qquad \cos c \geqq \cos (2\pi - a - b) = \cos (a + b).$$

Two other relations similar to (6) are obtained by permuting the r's. Algebraically, (6) is an immediate consequence of (4), since $r' \geqq -1$. The relation

$$r_{12} \leqq r_{13}r_{23} + \sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}, \tag{7}$$

deduced from the fact that $r' \leqq +1$, says merely that $c \geqq |a - b|$, and adds nothing geometrically to what has already been observed. All six relations, the three like (6) and the three like (7), can be inferred from the single symmetrical form

$$r_{12}^2 + r_{13}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23} \leqq 1, \tag{8}$$

which is equivalent to (6) and (7) taken together.[1] The reader whose experience in mathematical analysis has gone beyond the elementary stages will notice that (8), after a slight amount of formal rearrangement, reduces to a statement that a certain three-rowed Gramian determinant is positive or zero.

By way of more detailed notation, let the number defined by the formula (4) be denoted by r_{12}', and the other two expressions which result from permutation of the r's on the right, the cosines of the other angles α and β of the spherical triangle, by r_{23}' and r_{13}'. From the law of cosines in the polar triangle of ABC,

$$\cos \gamma = - \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c,$$

$$r_{12} = \cos c = \frac{r_{12}' + r_{13}'r_{23}'}{\sqrt{(1 - r_{13}'^2)(1 - r_{23}'^2)}}, \tag{9}$$

with symmetric expressions for r_{13} and r_{23}. Analogous to (8) is the relation [2]

$$r_{12}'^2 + r_{13}'^2 + r_{23}'^2 + 2r_{12}'r_{13}'r_{23}' \leqq 1.$$

[1] *Cf.* Yule, *Introduction to the theory of statistics*, p. 250.

[2] *Cf.* Yule, *op. cit.*, pp. 249–250. The formula (9) and the analogous formulas for r_{13} and r_{23} were not mentioned in the paper A. They may be verified directly by substituting the ex-

In conclusion, a remark may be made with regard to the equations (2), which have been left out of account throughout most of the paper. Geometrically they mean of course that the points P and Q are restricted to a certain $(n-1)$-dimensional hyperplane. It has been insisted already that this restriction bears only on the use of the term *coefficient of correlation,* and is irrelevant to the substance of the work. Let $(X_1, X_2, \cdots, X_n)$ be any set of n real numbers, not all equal to each other, and let $(Y_1, Y_2, \cdots, Y_n)$ be any other set, similarly unspecialized. The arithmetical mean $\bar{x}$ of the X's may be defined as the number which makes the expression $\Sigma(X_k - \bar{x})^2$ a minimum, and the mean $\bar{y}$ of the Y's may be similarly defined.[1] If x_k and y_k stand for the numbers $X_k - \bar{x}$ and $Y_k - \bar{y}$, the equations (2) are satisfied. The notation in an earlier paragraph leading to the definition of the coefficient of *partial* correlation reduces to the definition of the *ordinary* coefficient of correlation, if $(x_1, x_2, \cdots, x_n)$ are replaced by $(X_1, X_2, \cdots, X_n)$, $(y_1, y_2, \cdots, y_n)$ by $(Y_1, Y_2, \cdots, Y_n)$, $(z_1, z_2, \cdots, z_n)$ by $(1, 1, \cdots, 1)$, λ and μ by $\bar{x}$ and $\bar{y}$, u_k and v_k by x_k and y_k, and r' by r.

A THEOREM IN THERMODYNAMICS.

By J. E. TREVOR, Cornell University.

1. Introduction. With the aid of the fundamental principles of thermodynamics it is found that the quantity of heat absorbed by any body in any reversible change of its thermodynamic state is equal to the integral of an expression θdS, where S is a single-valued function of the independent variables that determine the states of thermodynamic equilibrium of the body, θ is a single-valued function of the temperature alone, and the line integral is taken over the path of the change of state. The difficulty that many students more or less unconsciously find in seeking to comprehend the establishment and content of this formulation is largely obviated when the mathematical reasoning employed in its deduction is clearly stated. For this reason I hope that the following study of a troublesome stage of the argument may have some interest.

Assuming that the ideas of a reversible path and a Carnot cycle are familiar, I propose that any succession of Carnot cycles advancing through progressive temperature intervals, whereby the quantity of heat absorbed in the operation of each cycle after the first is equal to that developed at the same temperature in the immediately preceding cycle, shall be termed a "sequence." The term "any sequence" shall be understood to include any sequence having but one term, *i.e.*, any Carnot cycle; and the work and heat absorbed or developed by the working body or bodies in the operation of a sequence shall be termed the work and heat absorbed or developed "by the sequence." The uniform temperature

pression (4) and the corresponding expressions for r_{13}' and r_{23}' in the right-hand members. It will be found that the relation (8), with the fact that the r's do not exceed 1 numerically, eliminates any ambiguity with regard to the square roots.

[1] *Cf. A*, p. 118.

t of the working body shall be understood with reference to an arbitrarily selected temperature scale. It may then be asserted that the principles of thermodynamics lead without difficulty to a set of statements that may be postulated in the following form.

2. The Data. The symbols $W(q, t_1, t_2)$ and $W(q, t_2, t_1)$ shall denote the work absorbed by a sequence absorbing the heat q at the temperature t_1, respectively at the temperature t_2, and operating between the temperatures t_1 and t_2, irrespective of whether t_2 is greater than, equal to, or less than t_1.

If Q_1 is the (positive, zero, or negative) heat *absorbed* at the temperature t_1 by any sequence operating between the temperatures t_1 and t_2, where t_1, t_2 are any two values of the temperature t subject to the condition $t_2 > t_1$, the work absorbed by the sequence is a single-valued continuous function of the independent real variables Q_1, t_1, t_2,

$$W(Q_1, t_1, t_2) \equiv W(1, t_1, t_2) \cdot Q_1 \qquad (t_2 > t_1), \tag{I}$$

where the single-valued continuous function $W(1, t_1, t_2)$ is positive. Further, the (positive, zero, or negative) heat Q_2 *developed* by the sequence at the temperature t_2 is equal to the sum of the quantities $W(Q_1, t_1, t_2)$ and Q_1,

$$Q_2 = [W(1, t_1, t_2) + 1]Q_1 \qquad (t_2 > t_1). \tag{II}$$

If t_r is an arbitrary constant value of the independent real variable t, we have

$$\lim_{t \to t_r} W(1, t, t_r) = 0 \qquad (t < t_r); \tag{III}$$

and we have that W satisfies the functional relation

$$W(1, t_r, t) + 1 = \frac{1}{W(1, t, t_r) + 1}, \tag{IV}$$

for all values of t.

3. The Problem. In pursuing the development of the theory, under the guidance of a simple and obvious analogy, we are led to proceed as follows. Let t_s be an arbitrary constant value of t such that $t_s > t_r$, and let k be an arbitrary positive integer. Then, if Q_r is the heat absorbed at t_r by any sequence of k equal-work cycles operating between t_r and t_s, the work kw absorbed by the sequence is, by (I),

$$kw = W(1, t_r, t_s) \cdot Q_r. \tag{V}$$

Further, in any sequence operating between t_r and the general temperature t and absorbing the heat Q_r at t_r, let Q be the heat developed at t. Hereupon, by use of the foregoing data, we seek to establish that the quantity w defined by (V) is a single-valued continuous function $w(Q, t)$ of the variables Q, t considered as independent; and we seek to establish that the quantity θ defined by [1]

$$\theta \cdot w = Q \tag{VI}$$

[1] It appears eventually that (VI) does not define θ for $Q = 0$, but that θ may be defined for this case by the limit of the ratio Q/w as Q approaches zero.

is a positive single-valued continuous function of the single variable t. The solution of this purely mathematical problem is as follows.

4. The Solution. If any sequence operating between the arbitrary constant temperature t_r and the general temperature t *absorbs* the heat Q_r at t_r and *develops* the heat Q at t, the equation (I) asserts that the work absorbed by the sequence is

$$W(Q_r, t_r, t) \equiv W(1, t_r, t) \cdot Q_r \qquad (t > t_r),$$
$$W(-Q, t, t_r) \equiv W(1, t, t_r)(-Q) \qquad (t < t_r),$$

where the factor W in each second member is positive, single-valued, and continuous. Let us investigate the function $W(1, t_r, t)$ for all values of t. For $t < t_r$ we know that $W(1, t, t_r)$ is positive, single-valued, and continuous; and hence by (IV) that $W(1, t_r, t)$ is negative, single-valued, and continuous; and that $W(1, t_r, t) + 1 > 0$. It follows that, for $t < t_r$, the value of $W(1, t_r, t)$ lies between 0 and -1. For $t > t_r$, as t approaches t_r, the limit of $W(1, t_r, t)$ is found by (IV) from

$$[W(1, t_r, t) + 1]^2 = 1 \qquad (t = t_r).$$

Hence the limit is 0 or -2, and hence is zero since the continuous $W(1, t_r, t)$ is positive for $t > t_r$. For $t < t_r$, as t approaches t_r, the limit of $W(1, t_r, t)$ is found by (IV) and (III) to be zero. These results establish that the function $W(1, t_r, t)$ is single-valued and continuous for all values of t; that it is positive for $t > t_r$, zero for $t = t_r$, and negative for $t < t_r$; and that its negative values lie between 0 and -1.

Now let us consider the relation between the quantities Q_r and Q. By (II) we have

$$Q = [W(1, t_r, t) + 1]Q_r \qquad (t > t_r), \tag{1}$$

$$-Q_r = [W(1, t, t_r) + 1](-Q) \qquad (t < t_r). \tag{2}$$

By (IV), the equation (2) may be written

$$Q_r[W(1, t_r, t) + 1] = Q \qquad (t < t_r). \tag{2a}$$

For $t > t_r$, and for $t < t_r$, we thus have that Q_r is single-valued and continuous in Q, t. And from both (1) and (2a) we find

$$\lim_{t \to t_r} Q_r = \lim_{t \to t_r} \{Q \cdot [W(1, t_r, t) + 1]^{-1}\} = Q.$$

If we now define Q_r at $t = t_r$ by its limit, we have that

$$Q_r = \frac{Q}{W(1, t_r, t) + 1}, \tag{3}$$

for all values of t. Since the denominator of the fraction is positive, single-valued, and continuous, this function $Q_r(Q, t)$ is single-valued and continuous in the variables Q, t considered as independent.

On substituting the function $Q_r(Q, t)$ for Q_r in the equation (V) defining w,

and writing W_{rs} for the positive constant $W(1, t_r, t_s)$, we obtain

$$kw = \frac{W_{rs} \cdot Q}{W(1, t_r, t) + 1}; \tag{4}$$

whence it appears that w is a single-valued continuous function of the independent variables Q, t. Solving (4) for Q/w,

$$\frac{Q}{w} = \frac{k}{W_{rs}}[W(1, t_r, t) + 1] \qquad (Q \neq 0). \tag{5}$$

By (4), when $Q = 0$ we have $w = 0$, and the ratio Q/w becomes indeterminate. Let us, then, define the ratio Q/w for $Q = 0$ by the second member of (5), which is the limit of this ratio as Q approaches zero. The ratio is now a positive continuous function of t alone. On substituting it in the equation (VI) defining θ, we find

$$\theta = \frac{k}{W_{rs}}[W(1, t_r, t) + 1].$$

Here k and W_{rs} are positive constants, and the bracket is positive, single-valued, and continuous. So θ is a positive single-valued continuous function of the single variable t. This result, and the conclusion expressed by (4), are the theorems it was sought to establish.

5. Continuation. The states of any continuous region of states of thermodynamic equilibrium of a given body are determined by certain independent variables. The heat absorbed by the body on any path of change of state within the region is expressed by the corresponding line integral of a linear differential form in these variables. In the further development of the theory the results just obtained are employed to show that the function $1/\theta(t)$ is an integrating factor of this differential form, and indeed that it is the only integrating factor that is a function of the variable t alone. This establishes the formulation cited at the beginning of this paper.

ON THE SOLUTION OF ALGEBRAIC EQUATIONS WITH RATIONAL COEFFICIENTS.

By GLENN JAMES, Southern Branch, University of California.

Since algebraic polynomials with rational coefficients may contain real factors of degree higher than the first when they do not contain real linear factors, it is sometimes desirable to remove roots from equations in sets rather than singly. This can be done by means of a generalization of the ordinary factor theorem.

Attacking the solution of equations from this viewpoint, this article makes the solution of an equation depend upon finding roots of auxiliary equations each of which has at least one real root. And these auxiliary equations may have rational roots when the original equations do not.

1. Theorem: *If $R(x)$ be the remainder obtained by dividing $F(x)$ by $f(x)$ and if all the roots of $f(x) = 0$ are among the roots of $F(x) = 0$, then $R(x)$ is identically zero.*

Proof: Denote the integral part of $F(x)/f(x)$ by $Q(x)$. Then

$$F(x) = Q(x) \times f(x) + R(x),$$

whence $R(x) = 0$ for all values of x for which $f(x)$ and $F(x)$ both vanish. But $R(x)$ is of lower degree than $f(x)$, therefore $R(x) \equiv 0$.

2. On the Solution of Cubic Equations. If the cubic polynomial $x^3 + a_1x^2 + a_2x + a_3$ be divided by $x^2 + b_1x + b_2$, the quotient and remainder are, respectively,

$$x + (a_1 - b_1) \tag{1}$$

and

$$[a_2 - b_2 - (a_1 - b_1)b_1]x + a_3 - (a_1 - b_1)b_2. \tag{2}$$

Now if the roots of

$$x^2 + b_1x + b_2 = 0 \tag{3}$$

are roots of

$$x^3 + a_1x^2 + a_2x + a_3 = 0, \tag{4}$$

the above theorem states that

$$a_2 - b_2 - (a_1 - b_1)b_1 = 0 \tag{5}$$

and

$$a_3 - (a_1 - b_1)b_2 = 0. \tag{6}$$

The elimination of b_2 between these two equations gives

$$b_1^3 - 2a_1b_1^2 + (a_1^2 + a_2)b_1 - a_1a_2 + a_3 = 0. \tag{7}$$

This auxiliary equation in b_1 has a rational root whenever the original cubic does. Moreover, when its real root, or an approximation to it, has been found, one can write out the roots of the original cubic by means of (1), (2), (3), and (5). They are

$$b_1 - a_1 \quad \text{and} \quad \frac{-b_1 \pm \sqrt{(4a_1 - 3b_1)b_1 - 4a_2}}{2}.$$

If the coefficient of the second degree term in (4) were zero, the reduced cubic and its auxiliary equation would be identical and the roots of the reduced cubic would be

$$b_1 \quad \text{and} \quad \frac{-b_1 \pm \sqrt{-3b_1^2 - 4a_2}}{2}.$$

3. On the Solution of Quartic Equations. If the quartic polynomial

$$x^4 + a_2x^2 + a_3x + a_4 \tag{8}$$

be divided by $x^2 + b_1x + b_2$, the quotient and remainder are, respectively, $x^2 - b_1x + a_2 - b_2 + b_1^2$ and $[a_3 + 2b_1b_2 - a_2b_1 - b_1^3]x + a_4 - a_2b_2 + b_2^2 - b_1^2b_2$.

The elimination of b_2 between the two equations

$$a_3 + 2b_1b_2 - a_2b_1 - b_1^3 = 0 \tag{9}$$

and

$$a_4 - a_2b_2 + b_2^2 - b_1^2b_2 = 0 \tag{10}$$

gives [1]

$$b_1^6 + 2a_2b_1^4 + (a_2^2 - 4a_4)b_1^2 - a_3^2 = 0. \tag{11}$$

Solving this equation for b_1^2, that is for b_1 and $-b_1$, and determining the corresponding values of b_2 from (9) gives two quadratic factors of (8), whence the roots of the quartic equation

$$x^4 + a_2x^2 + a_3x + a_4 = 0 \tag{12}$$

are

$$\frac{-b_1 \pm \sqrt{b_1^2 - 2a_2 - \dfrac{2a_3}{b_1}}}{2} \quad \text{and} \quad \frac{b_1 \pm \sqrt{b_1^2 - 2a_2 + \dfrac{2a_3}{b_1}}}{2}$$

provided $b_1 \neq 0$. However, if this proviso does not hold, $a_3 = 0$ by (9), and (12) is quadratic in x^2.

The discriminants of equation (12) can be evaluated, without making the detailed substitutions, by a method that will be taken up in the next article.

4. The Evaluation of the Discriminant of a Quadratic Factor. If $x^2 + a_1x + a_2$ be divided by $x + b_1$, the quotient and remainder are, respectively,

$$x + (a_1 - b_1) \tag{13}$$

and

$$a_2 - (a_1 - b_1)b_1. \tag{14}$$

Since the condition that the sum of the roots of

$$x^2 + a_1x + a_2 = 0 \tag{15}$$

be zero is $a_1 = 0$, if we reduce the roots of (15) by $-a_1/2$, the roots of the reduced equation will be $\pm\sqrt{-R_1}$, where R_1 is the first remainder obtained in the reduction process. The roots of (15) are then $-a_1/2 \pm \sqrt{-R_1}$.

In the case of the cubic, the condition for a pair of roots whose sum is zero is found by placing b_1 equal to zero in equations (5) and (6) of Art. 2 and eliminating b_2. The result is

$$a_3 - a_1a_2 = 0. \tag{16}$$

Also from (5) $b_2 = a_2$, when $b_1 = 0$. Hence if the cubic (4) has a pair of roots whose sum is zero, they are $\pm\sqrt{-a_2}$.

If condition (16) be not satisfied, we can solve for b_1 as in Art. 2, then proceed to reduce the roots of (4) by $-b_1/2$ until we have reached the second remainder.

[1] Descartes' method of obtaining this auxiliary equation involves the determination of three parameters. See Dickson's *Elementary Theory of Equations*, p. 42.

Call this remainder R_2. The roots of the original equation are then

$$\frac{-b_1}{2} \pm \sqrt{-R_2} \text{ and } b_1 - a_1,$$

where b_1 and a_1 refer to the original cubic.

If the quartic polynomial

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \tag{17}$$

be divided by

$$x^2 + b_1x + b_2 \tag{18}$$

and the coefficients of the remainder equated to zero, there results

$$a_3 - (a_1 - b_1)b_2 - [a_2 - b_2 - (a_1 - b_1)b_1]b_1 = 0 \tag{19}$$

and

$$a_4 - [a_2 - b_2 - (a_1 - b_1)b_1]b_2 = 0. \tag{20}$$

Substituting $b_1 = 0$ in (19) and (20) and eliminating b_2 between the two resulting equations gives

$$a_1^2a_4 - a_1a_2a_3 + a_3^2 = 0. \tag{21}$$

If this condition be not satisfied, we reduce the roots of

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{22}$$

by $-b_1/2$ and find two of the roots of (22) to be $-(b_1/2) \pm \sqrt{-R_2/R_4}$, where R_2 and R_4 are the second and fourth remainders obtained in the process of reducing the roots of the quartic by $-b_1/2$.

5. On the Solution of an Equation of the *n*th Degree. The division of the polynomial

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \tag{23}$$

by

$$x^2 + b_1x + b_2 \tag{24}$$

gives a remainder

$$A_nx + B_n \tag{25}$$

[see (2) Art. 2 for $A_3x + B_3$], where A_n and B_n are defined as follows:

$$A_2 = a_1 - b_1, \qquad A_n = B_{n-1} - A_{n-1}b_1, \qquad n > 2. \tag{26}$$

$$B_2 = a_2 - b_2, \qquad B_n = a_n - A_{n-1}b_2, \qquad n > 2. \tag{27}$$

Equating to zero the coefficients in (25) gives

$$A_n = 0 \tag{28}$$

and

$$B_n = 0. \tag{29}$$

Eliminating b_2 between (28) and (29) gives an auxiliary equation in b_1 which *always has at least one real root.* The degree of this equation will be ${}_nC_2$, although the effective degree may be lower as in the case of the quartic equation.

Having approximated a value of b_1, we evaluate the discriminant of the corresponding quadratic factor by a method similar to that of Art. 4. Substituting

zero for b_1 in (28), we get

$$a_1b_2^{(n-2)/2} - a_3b_2^{(n-4)/2} + a_5b_2^{(n-6)/2} \cdots + (-1)^{(n-2)/2}a_{n-1} = 0, \quad n \text{ even}, \tag{30}$$

or

$$b_2^{(n-1)/2} - a_2b_2^{(n-3)/2} + a_4b_2^{(n-5)/2} \cdots + (-1)^{(n-1)/2}a_{n-1} = 0, \quad n \text{ odd}. \tag{31}$$

If the original equation has a pair of roots whose sum is zero, those roots are $\pm\sqrt{-b_2}$, where b_2 is among[1] the roots of (30) or (31), whichever applies. Otherwise we reduce the roots of the original equation by $-b_1/2$ and write the corresponding equation of the form of (30) or (31). If b_2 be a properly chosen root of the latter, two of the roots of the original equation are

$$-(b_1/2) \pm \sqrt{-b_2}.$$

In conclusion it is of interest to note that the nature of the roots of (23) can be studied by means of a *discriminant equation* obtained by eliminating b_1 and b_2 between (28), (29) and the equation

$$b_1^2 - 4b_2 = K, \tag{32}$$

where $b_1^2 - 4b_2$ is the discriminant of (24). The original equation has a pair of equal roots, a pair of complex roots or n real roots, respectively, according as this discriminant equation in K has a zero root, a negative or complex root, or ${}_nC_2$ positive roots.

A NEW METHOD FOR THE DETERMINATION OF THE GROUP OF ISOMORPHISMS OF THE SYMMETRIC GROUP OF DEGREE N.

By H. A. BENDER, University of Illinois.

Our leading text books on group theory which determine the group of isomorphisms of the symmetric group of degree n make use of the following two theorems. First, the symmetric group of degree n ($n \neq 6$) contains n and only n subgroups of order $(n-1)!$ forming a single conjugate set. The symmetric group of degree 6 contains 12 subgroups of order $5!$, which are simply isomorphic with one another and form two sets of conjugates of 6 each. Second, if G is a transitive substitution group of degree n and index n, then the group of isomorphisms of G can be represented as a transitive substitution group of degree n which contains G as an invariant subgroup.

In this article we shall study the group of isomorphisms of the symmetric group of degree n from the standpoint of independent generators of the symmetric group.[2]

It is known that the symmetric group G of degree n can be generated by a cyclic substitution s of degree $n-1$ and a transposition t which connects any

[1] Not all the roots need be valid, since when taken with $b_1 = 0$ they may not satisfy (29). They can, of course, be tested by substitution in this equation with $b_1 = 0$ or in the original equation (23).

[2] O. Hölder, *Mathematische Annalen*, vol. 46 (1895), p. 345.

one of the $n - 1$ letters in s with the remaining letter.[1] The cyclic substitution s can be selected in $\frac{n!}{n-1}$ different ways, and for each cyclic substitution s the transposition t can be selected in $n - 1$ different ways, hence the symmetric group of degree n can be generated in $n!$ different ways. Thus it follows that the symmetric group G can be made isomorphic with itself in $n!$ different ways such that all these isomorphisms are inner isomorphisms.[2] Moreover, these isomorphisms constitute the group of inner isomorphisms.

Since the central of a symmetric group is the identity, it follows that the group of inner isomorphisms is simply isomorphic with this symmetric group. If this is not the group of isomorphisms, it is an invariant subgroup of the group of isomorphisms and the remaining isomorphisms are outer, or contragredient, isomorphisms.

We shall now consider the possibility of automorphisms of G in which t corresponds to substitutions of order 2 which are composed of transpositions. It is evident that the number of conjugates under G of such substitutions must be equal to the number of transpositions in G.

If we equate and simplify the number of transpositions in G to the number of substitutions of order two and degree four, six and $2r$ respectively, we have

$$(n-2)(n-3) = 2\cdot 2!, \qquad (n-2)(n-3)(n-4)(n-5) = 2^2\cdot 3!,$$
$$(n-2)(n-3)\cdots(n-2r+1) = 2^{r-1}\cdot r!.$$

The first equation is not satisfied for real values of n. The second equation is satisfied for $n = 6$ and for no other integral value of n. In general, suppose $n = 2r$, then the third equation reduces to

$$(2r-2)(2r-3)\cdots(r+1) = 2^{r-1},$$

and for $r > 3$,

$$(2r-2)(2r-3)\cdots(r+1) > (r+1)^{r-2} > 2^{r-1}.$$

Thus we have shown that in all possible automorphisms of the symmetric group of degree n ($n \neq 6$) transpositions must correspond to transpositions.

We shall next consider the possibility of an automorphism of G ($n \neq 6$) in which the cyclic substitution s corresponds to a non-cyclic substitution s_1 of order $n - 1$. It is evident that s_1 can not be of degree less than n. If the substitution s_1 is of degree n, then the product of this substitution and a transposition will unite two of the cycles of s_1 into one cycle or decompose one of its cycles into two cycles according as the letters of the transposition appear in two cycles or in the same cycle respectively. Thus we have shown that outer isomorphisms of the symmetric group are possible only in the case $n = 6$.

It is evident that in the case $n = 6$ the substitution s of order 5 must correspond to cyclic substitutions of order 5 in every automorphism of G.

We shall now consider some of the conditions necessary for a cyclic substitu-

[1] *Cf.* R. D. Carmichael, *Quarterly Journal*, vol. 49 (1922), p. 226.

[2] This may also be shown by using other sets of independent generators.

tion s of degree 5 and a substitution t_1 of order 2 and degree 6 to generate the symmetric group of degree 6.

The product of s and t_1 will omit two letters if the two substitutions contain two pairs of adjacent letters, and hence will not be of order 6. That is, the product

$$abcde \cdot ab \cdot cd \cdot ef = bdfe$$

will omit a and c. Likewise, t_1 and the powers of s can not have a pair of adjacent letters in common. Hence, for a given s the substitution t_1 can be selected in but five ways. For $abcde$ the five substitutions are

$$ab \cdot ce \cdot df, \qquad ac \cdot bf \cdot de, \qquad ad \cdot bc \cdot ef, \qquad ae \cdot bd \cdot cf, \qquad af \cdot be \cdot cd.$$

That s and t_1 so defined will generate the symmetric group of degree 6 is shown in the *Quarterly Journal*, vol. 49 (1922), p. 235.

Thus we have shown that *the group of isomorphisms I of the symmetric group of degree n ($n \neq 6$) is the group of inner isomorphisms and is the symmetric group of this degree. For $n = 6$ the order of the group of isomorphisms is* $\frac{6!}{5}(5 + 5)$, *or twice the order of the symmetric group of degree six, and exactly one half of the isomorphisms are outer isomorphisms.*

It can be shown that the group of isomorphisms of the symmetric group G of degree n ($n \neq 6$) can be generated by the two isomorphisms S and T satisfying the following conditions:

$$S^{-1}sS = s, \qquad S^{-1}tS = s^{-1}ts, \qquad T^{-1}sT = t^{-1}st, \qquad T^{-1}tT = t.$$

For $n = 6$ let us consider the automorphisms of G in which t corresponds to t_1. Since all the substitutions of the same type as t or t_1 are conjugate under G, it follows that t_1 in turn must correspond to a substitution which is one of the conjugates of t under G. This being true for the remaining sets of conjugate substitutions, it follows that the square of any outer isomorphism, as well as the product of any two outer isomorphisms, is an inner isomorphism.

Furthermore, the square of any outer isomorphism T_1 which leaves s invariant is an inner isomorphism which may be obtained by transforming G by s or by a power of s. Since s is in the alternating group of G and all the isomorphisms brought about by transforming G by the operators of the alternating group form a subgroup in the I of G simply isomorphic with the alternating group, it follows that the square of T_1 is in the subgroup simply isomorphic with the alternating group. Since the commutators of an inner and an outer isomorphism are in the subgroup simply isomorphic with the alternating group, it follows that the square of every outer isomorphism is in the subgroup simply isomorphic with the alternating group, and hence this subgroup and one half of the outer isomorphisms constitute an invariant subgroup in the I of G. *Thus the group of isomorphisms of the symmetric group of degree* 6 *contains* 3 *invariant subgroups of order* 6!, *each having the common subgroup of order* 360.

QUESTIONS AND DISCUSSIONS.

Edited by C. F. Gummer, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems (especially new problems) which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. An Elementary Solution of a Problem of Diophantus.

By H. E. J. Curzon, Goldsmith's College, London, England.

[Extract from a letter to Professor R. D. Carmichael.]

It is required to find three rational numbers such that the product of any pair of those numbers plus or minus the sum of the three numbers shall, in each of the six cases, be the square of a rational number.[1]

Let the required numbers be a, b, c. Then the problem will be solved if there can be found a rational solution of the four equations

$$a + b + c = -2xyz; \tag{1}$$

$$bc = y^2z^2 + x^2; \qquad ca = z^2x^2 + y^2; \qquad ab = x^2y^2 + z^2. \tag{2}$$

If equations (2) are to hold good, then

$$a(b - c) = (y^2 - z^2)(x^2 - 1); \qquad b(c - a) = (z^2 - x^2)(y^2 - 1); \quad \text{etc.} \tag{3}$$

If it be now assumed that a, b, c have the forms

$$a = -(x + 1)(y + z); \quad b = -(y + 1)(z + x); \quad c = -(z + 1)(x + y); \tag{4}$$

then it follows that

$$(b - c) = -(x - 1)(y - z); \qquad c - a = -(y - 1)(z - x);$$
$$a - b = -(z - 1)(x - y);$$

and therefore equations (3) are satisfied by the values of a, b, c assumed in (4).

In order to find the conditions that equations (4) shall be consistent with equations (1) and (2), the values of a, b, c in terms of x, y, z as given in (4) must be substituted in (1) and (2).

Let

$$x + y + z = p, \qquad yz + zx + xy = q, \qquad xyz = r.$$

Then, substituting in (1),

$$-2q - 2p = -2r, \qquad i.e., \qquad p + q = r. \tag{5}$$

Substituting in the first equation of set (2),

$$\{yz + (y + z) + 1\}\{yz + x(y + z) + x^2\} = y^2z^2 + x^2;$$

[1] Heath, Sir T. L., *Diophantus of Alexandria*, 2d edition, 1910, 16, p. 164.
Tannery, *Diophantus Alexandrinus*, 1893.
Carmichael, *Diophantine Analysis*, 1915, 5, p. 112.

that is,

$$y^2z^2 + xyz(x + y + z) + (y + z)(y + x)(z + x) + (yz + zx + xy) + x^2 = y^2z^2 + x^2.$$

That is,

$$rp + pq - r + q = 0. \tag{6}$$

Eliminating r from (5) and (6),

$$p(p + 2q - 1) = 0.$$

Hence equations (4) are consistent with (1) and (2) if $p = 0$ and $q = r$.[1]

The problem is thus reduced to finding a solution of the cubic equation

$$t^3 + rt - r = 0, \tag{7}$$

in which, for a certain selected value of r, all three values of t shall be rational.

Assume that λ is one rational root of (7). Then $r = \lambda^3/(1 - \lambda)$ and (7) must have the form

$$(1 - \lambda)t^3 + \lambda^3 t - \lambda^3 = (t - \lambda)\{(1 - \lambda)t^2 + \lambda(1 - \lambda)t + \lambda^2\} = 0.$$

Now the roots of

$$(1 - \lambda)t^2 + \lambda(1 - \lambda)t + \lambda^2 = 0 \tag{8}$$

are rational if $(\lambda - 1)(\lambda + 3)$ is the square of a rational quantity, and this is the case if λ has the form $[(m^2 + mn + n^2)/(mn + n^2)]$. The roots of equation (8) are then

$$\frac{\lambda(\lambda - 1) \pm \sqrt{\lambda^2(\lambda - 1)(\lambda + 3)}}{2(1 - \lambda)} = -\frac{\lambda}{2}\left(1 \pm \sqrt{\frac{\lambda + 3}{\lambda - 1}}\right) = -\frac{\lambda}{2}\left(1 \pm \frac{m + 2n}{m}\right).$$

Hence x, y, z may be given the forms

$$\frac{m^2 + mn + n^2}{n(m + n)}, \quad \frac{m^2 + mn + n^2}{m(m + n)}, \quad -\frac{m^2 + mn + n^2}{mn}.$$

Whence a, b, c take the forms below, on using (4):

$$\frac{(m + n)^2 + n^2}{n^2(m + n)^2}(m^2 + mn + n^2), \quad \frac{m^2 + (m + n)^2}{m^2(m + n)^2}(m^2 + mn + n^2),$$
$$\frac{m^2 + n^2}{m^2n^2}(m^2 + mn + n^2).$$

The solution may then be stated symmetrically:

Let l, m, n be three integers whose sum is zero. Then all sets of three numbers having the form

$$\frac{m^2 + n^2}{2m^2n^2}(l^2 + m^2 + n^2), \quad \frac{n^2 + l^2}{2n^2l^2}(l^2 + m^2 + n^2), \quad \frac{l^2 + m^2}{2l^2m^2}(l^2 + m^2 + n^2),$$

satisfy the required conditions.

$l = 1$, $m = 2$, $n = -3$ gives the set 91/36, 70/9, 35/4.

[1] If the alternative $p + 2q = 1$ is followed, then the cubic that arises is such that all its roots cannot be rational.

The solution that is arrived at by following the geometrical methods of Diophantus may be expressed as follows: Find integers b_1, c_1; b_2, c_2; b_3, c_3 such that $b_1c_1 = b_2c_2 = b_3c_3$, while $(b^2 + c^2)$ is in each case a complete square.

Let

$$(b_2^2 + c_2^2)(b_3^2 + c_3^2) + (b_3^2 + c_3^2)(b_1^2 + c_1^2) + (b_1^2 + c_1^2)(b_2^2 + c_2^2) = K.$$

Then the set

$$\frac{K}{2b_1c_1(b_1^2 + c_1^2)}, \qquad \frac{K}{2b_2c_2(b_2^2 + c_2^2)}, \qquad \frac{K}{2b_3c_3(b_3^2 + c_3^2)}$$

is a solution, as can easily be verified by elementary algebra.[1] One set of integers for the b's and c's are 40, 42; 24, 70; 15, 112. In this case $K = 131299224$ and the set

$$\frac{781543}{67280}, \qquad \frac{781543}{109520}, \qquad \frac{781543}{255380},$$

will be found to satisfy the initial requirements of the problem.

The solutions obtained by the analytic and the geometric method respectively have no apparent connexion.

II. A Note on Checking a Solution of a Triangle.

By E. J. Moulton, Northwestern University.

Suppose we are solving an oblique plane triangle, given two sides and the included angle, a, b, C. We may use the formulas

$$(A + B)/2 = (180° - C)/2; \tag{1}$$

$$\tan (A - B)/2 = \frac{a - b}{a + b} \tan (A + B)/2; \tag{2}$$

$$A = (A + B)/2 + (A - B)/2; \qquad B = (A + B)/2 - (A - B)/2; \tag{3}$$

$$c = a \sin C/\sin A; \tag{4}$$

and check with the formula

$$c = b \sin C/\sin B. \tag{5}$$

Suppose that an error is made in using (1), and no other error is made. Will the check formula (5) enable us to detect the error? It may be a little surprising at first that it does not. A proof follows:

The value obtained from (1) for $(A + B)/2$ through the error would have been the correct value for an angle C_1 instead of C. If we were to solve the triangle, given a, b, C_1, we would obtain the values of A and B actually found through the error. The third side c_1 would satisfy both equations

$$c_1 = a \sin C_1/\sin A; \qquad c_1 = b \sin C_1/\sin B. \tag{6}$$

Hence for the determined values of A and B, which were in error,

$$a/\sin A = b/\sin B. \tag{7}$$

[1] Tannery, *Diophantus Alexandrinus*, 1893.

It is now seen that the (incorrect) values of c found from (4) and (5) will be equal, and consequently formula (5) does not serve to detect errors in the use of (1).[1]

Moral. Insert between formulas (3) and (4) the check formula

$$A + B + C = 180°. \tag{3'}$$

III. On the Treatment of Maxima and Minima in Calculus Texts.

By A. A. Bennett, University of Texas.

It is frequently, perhaps usually, difficult to distinguish in mathematics between a treatment coördinated with respect to the subject matter and one whose thread of unity is supplied by the method used, since method and material so often coincide in mathematics. There are to be sure certain striking instances to the contrary that come to mind. In the discussion of ruler and compass constructions, where the algebraic character of the solution is to be emphasized, one comes across the concept of the transcendental number, π. In the integration of rational expressions one meets with the elementary algebraic discussion of partial fractions, usually omitted from the freshman course. In the study of the monodromic group of a linear differential equation, the classification of square matrices bobs up.

While one encounters the subject of maxima and minima in a text on calculus, one might inquire whether the topic has been introduced as affording scope for the application of the general theory under study, or whether on the other hand this is a field of interest on its own account, and related to the rest of the topics treated by a tenuous similarity of method. An author is surely justified in insisting upon discussing examples only in so far as these may adequately illustrate some feature of the principal topic under consideration. Thus a book on analytical geometry may rightly sidestep all discussion of oblique asymptotes, tangents, radius of curvature and other topics, if the author sees proper, merely because the simplest approach to these topics involves more advanced notions, although these questions naturally arise in connection with the simpler properties of the loci discussed. But does an author have the privilege of apparently dismissing a subject as though the final word had been said, when some of the most obvious examples cannot be treated by the methods explained? Has not the student a right to demand that the limitations of method be at least suggested?

Nearly every text on differential calculus gives the impression of treating elementary problems on maxima and minima completely. Of course the obvious query as to the effect that a change in the choice of the independent variable might have upon the vanishing of the first derivative and upon the problem as a whole is left alone, while most of the problems are such that several convenient arguments are available. How algebraic functions with more than one degree

[1] A similar situation arises in spherical trigonometry. If a, b, and C are given, and A, B, and c are found from the formulas $\tan (A + B)/2 = \cos (a - b)/2 \cdot \sec (a + b)/2 \cdot \cot C/2$, $\tan (A - B)/2 = \sin (a - b)/2 \cdot \operatorname{cosec} (a + b)/2 \cdot \cot C/2$, $\sin c = \sin C \sin a/\sin A$, the check $\sin c = \sin C \sin b/\sin B$ will fail to reveal an error in reading log cot $C/2$.

Editor.

of freedom are to be handled is a question never answered and wisely never raised. The distinction between problems of the sort considered in the calculus of variations and those to be handled by elementary calculus directly is never pointed out. Should the ambitious student later take up this more advanced topic, he would be informed that certain methods available in the finite case are there also applied, but, of course, he would never have heard of these methods in the calculus class. But these are all minor points. The fact that many problems are accompanied by tacit inequalities seems to be usually ignored. An actual problem may be one for which the interval of significance is restricted while the algebraic equation employed in formulating the problem cannot take account of these bounds directly. Additional inequalities are needed and may entirely alter the character of the solution. For example, the maximum or minimum may occur at the end of some interval which might be regarded as an interval of definition so far as the particular problem is concerned. The derivative may be infinite or cease for other reasons to exist. Frequently the problem is one which does not use derivatives at all. Even when the regular analytical machinery is available the problems in American texts are frequently capable of elementary synthetic treatment by use of the more familiar loci of plane geometry. This remark applies to the familiar exercise on the position of a point in a line from which a given segment perpendicular to the line subtends the maximum angle.

A glance at the following examples some of them reduced to ridiculous simplicity will serve to emphasize a few of these objections. These same difficulties become serious when the problems are of similar sort but not so simple.

1. Find the maximum of x.
2. Find the minimum positive real number.
3. What is the minimum value of y not less than $x^2 + 1$ and x^3 simultaneously?
4. Find the point the sum of whose distances from two given distinct points is a minimum.
5. Find the maximum and minimum real values of $e^{1/x}$.
6. Find the minimum of $1 + |x - 1|$.
7. Find the positive integer whose square exceeds three times the integer by as little as possible.
8. Find the smallest number of coins (U. S., 1924) with which one can pay a bill of thirty cents.
9. What is the smallest number of straight lines which can be used to dissect the interior of a circle into exactly ten portions?
10. Find the broken line of the smallest number of segments which will pass through every square on a chess board at least once.

RECENT PUBLICATIONS.

REVIEWS.

The Teaching of Geometry in Schools, a report prepared by a sub-committee of the (British) Mathematical Association and accepted by the general teaching committee, November 3, 1923. London, Bell and Sons, 1923.

The reviewer approaches his task with diffidence because it is singularly difficult for an American to give a fair and judicious account of a British educational report; too many prejudices and preconceptions stand in the way. The difficulty is increased when the report deals with mathematics. The radicals among us dismiss the subject with the remark "What can you expect of a people who cling to the use of Euclid?" The conservatives reply "What can you expect of a people who invented the Perry movement?" There is a diversity of gibes but the same spirit. All of which is unfortunate, for the report may have valuable material and may be conceived in a much less narrow spirit than that exhibited by the scorner.

The present report is a careful, sane, and helpful document. Although there are no names of outstanding mathematical celebrities among the signers, there is plenty of evidence of sound scholarship. One gets the impression that they were actuated by a desire to benefit eductation by making the subject of geometry more helpful and fruitful, rather than by the ambition to weary the reader with generalities concerning the cultural and social significance of exact science in a democratic system of education.

The real heart of the report lies in the contention for a spiral form of teaching geometry. The spiral has three turns, which they call the three stages: (*a*) experimental, (*b*) deductive, (*c*) systematic. About the first little need be said, the methods advocated are observational, the problems discussed are those which are picturesquely described as "Boy Scout Geometry." It is worthy of mention in this connection that the committee consistently uses the word "boys" to mean "pupils."

The second stage, the deductive, is of more interest. Here the pupil will be required to prove theorems and originals (*Anglice* riders).

"At this stage the systematising instinct is not strongly developed. On the other hand, great interest may be aroused by the search for geometrical truths. It is better at this stage to steer into the unknown than to attempt the proof of propositions that are judged obviously true, the more striking the result, the greater the interest aroused. The reasoning powers have become strong enough to use the deductive method . . . . A good number of the simpler theorems will be assumed from observation . . . . By the end of stage (*b*) the boy should know the interesting theorems of plane geometry, he should be able to devise constructions and to solve easy riders, he should be able to apply his knowledge to simple solid figures, he should have some grasp on logical method.[1]

"A course ending with stage (*b*) might be stimulating, but would certainly be ragged and unfinished. With dull boys probably nothing more can be accomplished, the Committee believes that they will derive more benefit from a frankly preliminary course than from an old-fashioned course on Euclidean lines. On the other hand, able boys will feel the need of rounding off and consolidating their study. This is the purpose of stage (*c*)."

[1] The quotation is from pp. 15 ff. It is a little depressing to learn that the committee expects that these valuable accomplishments will normally be acquired between the ages of 12 and 15.

The committee is familiar with the work of our own committee of 1911 and quotes with especial approval Professor Cajori's account of the history of geometry. Indeed it is evident that their recommendations concerning elementary geometry are entirely in agreement with the findings of our own recent National Committee on Mathematical Requirements, although it would seem that at each stage the British Committee expects the greater results.

Two other stages are suggested rather half-heartedly: (*d*) modern geometry, geometrical conics, solid geometry; (*e*) the philosophy of geometry. These topics, except solid geometry, are outside the realm of practical discussion, at least so far as American schools are concerned. American teachers will marvel at the inclusion of geometrical conics, or wonder what the subject may be about anyway. In fact we have a conspiracy of silence on the topic in the United States; it is looked upon as subversive of good morals, and likely to undermine the credit of elementary geometry. The reviewer well remembers his feeling of absolute guilt when, at a neighboring university, he gave in one hour the geometrical proofs of practically all the theorems about the ellipse taken up in the usual first course in analytics.

After the outline of stages, attention is turned to some miscellaneous topics. The raciness of the style tempts one to give a few specific quotations. On pages 18 and 19 occurs a discussion of the number of experimental demonstrations which should be allowed in stage (*b*):

"The main point is that something of the sort should be done before the boy comes to the proof, as a *hors d'œuvre* to whet the appetite for the solid part of the feast. . . . The other class of theorems that may, it is suggested, be left over for a demonstration in stage (*c*) contains theorems which are neither fundamental nor, to the mind of the boy of 14, in need of proof—Equal chords are equally distant from the center—the fact is too obvious to be interesting, and to dwell on it would be to delay progress, and allow interest to cool.

"Every opportunity should be taken to generalize the results obtained, this is a feature characteristic of modern as contrasted with Greek mathematics. Euclid [1] gives no explanation as to the reason why proofs take the form in which they are presented, that is, the synthetic proof is given but not the analysis by which it was obtained. The synthetic form is suitable for final statement, but in teaching the analysis should not be omitted."

This question of generalization leads to the observation that a part of the report is not arranged in very logical order. On page 11 under the heading "Organization of derived propositions" we find the power of a point with regard to a circle very prettily deduced, with pertinent suggestions as to the value of the concept of directed measurement. This is admirable; we have been waiting long and patiently for a writer of an elementary geometry who shall have the courage to put Euclid III 35 *If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other* and Euclid III 36 *If a point be taken outside a circle, and from that point there fall on the circle two straight lines, one cutting the circle and the other touching it, then the rectangle, contained by the whole of the straight line which cuts the circle and the segment on it between the point and the convex circumference,*

[1] American teachers will surely agree with this criticism of Euclid, at least if they have ever looked into that classic.

will be equal to the square on the tangent into a single theorem to the effect that *The product of the distances from a given point to two points on a circle collinear with it is independent of the direction of the joining line.* But why do the authors introduce the concept of this generalization on page 11, when the general topic does not come till page 22 and the term *power of a point with respect to a circle*, though casually mentioned here, appears among the neglected details on page 66? There is, by the way, one detail which they mention which we hope will always be neglected:

> "The theorem that if two figures are directly congruent a point may be found by rotation about which one may be made to coincide with the other is a very easy rider or series of riders on congruent triangles."

The difficulty about this easy rider, which is another way of saying that every motion of the plane is a rotation, is that of course it is not necessarily true.

A third part of the report deals with "Disputed points." The first of these is congruence and superposition. The reviewer is in most hearty sympathy with the attitude taken on this important matter. The committee begins by quoting from Veronese[1] a passage which the reviewer found years ago to be a lamp unto his feet: "If we say that two bodies are equal when they can be superposed by means of movement without deformation we are committing a *petitio principii*, for the definition of rigidity assumes a criterion for equality of spaces, and if the only criterion available is derived from superposition, there is no escape from the vicious circle." It is justly observed that for the sake of brevity it is convenient to speak of putting a figure in a certain position, drawing a circle, etc.

Let us look more closely into the matter. Where do we actually make use of superposition in our usual proofs? There seem to be only two cases. The second is the proof that in the same circle or equal circles equal central angles determine equal arcs. But the committee justly points out (pp. 31, 32) that until the length of an arc has been actually defined we must depend on intuition entirely for our concept of what we mean by equal arcs. At a much later stage when we define the length rigorously by a limiting process (if we ever really do so), an accurate proof will be available. The first use of superposition is in the *Pons Asinorum*, the first congruence axiom for triangles.[2] Now Veronese and Hilbert have pointed out the great advantage of taking this theorem as one of the axioms. The committee would also favor such a course, that is, for stage (*c*). For stage (*b*) they would go further and give a congruence axiom along the following lines: *Two figures are equal if the measurements necessary to specify them uniquely are equal.* Or otherwise *Any figure (plane or solid) can be exactly reproduced anywhere.* For stage (*b*) the committee would go still further and assume: *Any figure can be reproduced anywhere on any enlarged or diminished scale.* It is pointed out that the independence of these two can be shown, and the second one enables us to eliminate the troublesome parallel axiom.

Teachers will certainly differ in their readiness to follow such suggestions.

[1] *Fondamenti di Geometria*, Padua, 1891, page 259, note 1.

[2] For a discussion see Heath, *Euclid*, vol. I, pp. 247 ff. Cambridge, 1908.

The reviewer is in doubt about the second assumption, but is in hearty sympathy with the idea of using the first. He has a wistful feeling that the committee had a splendid chance, which they failed to take, of recommending geometers to forget the cacophonous words *congruent* and *congruence*. Oh why did we ever allow these terms to escape from Pandora's box to trouble our lives! Behold a definition from one of our geometries, equally renowned for its popularity, and for the scholarly standing of its author:[1]

"If two figures have exactly the same size and shape, they are called *congruent figures*. . . . *In two congruent figures the parts of one figure are equal respectively to the parts of the other figure.*" The distinction seems to be that the term "congruent" applies to figures while the humbler and older term "equal" applies to parts of figures. Suppose that we have two congruent parallelograms, and divide them into corresponding pairs of triangles by corresponding diagonals, are the triangles equal because they are parts of congruent figures, or congruent because corresponding measurements are equal?

The next topic is limits, this term indicating limiting processes. The reviewer is obliged to confess that he can not quite understand what the report is driving at in this matter. We read, for instance (p. 47):

> "The conclusion to which the Committee is driven is that there is no need to discourage the use of limits in the informal stages of geometry, but that they are not suitable to the stage of formal demonstration. Generalization should always be as wide as the state of knowledge permits, but tangents should be included whenever possible as particular cases, not as limiting cases of chords."

The reviewer is at a loss to see how a tangent which is an unbounded straight line can be a particular case of a chord, which is a line-segment bounded by two points. He is even less happy in the discussion of the next topic, the incommensurable case. The committee says (p. 51):

> "Until Pythagoras' theorem is reached, the student need have no idea that irrationals exist. He knows that in arithmetic some numbers have squares and some have not, but this is no more puzzling than the fact that the world contains black cats, and cats which are not black, and the observation does not plunge him at once into the gloom where all is grey. . . . In geometry the incommensurables suddenly appear before the eyes, for it is inconceivable that the side of a square should have a definite length and the diagonal of the same square should not, whatever difficulties may appear when measurement is attempted.
>
> "This is one reason why the teacher of geometry must accept the burden of deciding to what extent the question of incommensurables is to be faced. Another reason is that when irrational numbers are introduced, the fundamental operations of addition and multiplication are no longer possible in the senses in which they are defined for rational numbers, and fresh definitions have to be found; the ordinary geometrical operations, on the other hand, are undisturbed by questions of relative commensurability."

These words form an auspicious beginning, but the reviewer finds the subsequent developments less satisfactory, he never feels quite sure whether the pea is under the geometrical shell or the arithmetical one. The committee would assume that every line segment has a numerical length, and every rectangle a numerical area, but whereas addition is defined as a combination of lengths, multiplication consists in passing from a length to an area. To an American

[1] *Essentials of Plane and Solid Geometry* by D. E. Smith, 1893, p. 21.

this seems artificial, and prompted by a desire to conform to the Euclidean tradition of always saying "rectangle" instead of "product." It is certain that in future in our country the incommensurable case will either be omitted entirely or postponed to the last stage of school geometry, and it is likely to be taken up as the geometrical equivalent of irrational number. But the simplest way to handle an irrational number would seem to be to define it as an endless non-periodic decimal, and to assume that the laws of operation for such expressions are the same as those for terminating decimals.

The remaining chapters of the report are occupied with matters of minor importance. The argument against an agreed order has no bearing in America where no one ever thinks of suggesting such an order. It is pleasant to note that the committee does not seem to stand in that dread of examiners and their requirements which has been characteristic of British writers in the past. It may be said with some truth that the thesis often upheld in American educational theory, that the best person to examine a pupil is the teacher who has prepared that pupil, is partly born of the fear that many have of the showing that their pupils will make when committed to the tender mercies of others. On the other hand, it is hard to escape the feeling that in Great Britain the examination seems to be looked upon as a great and glorious end in itself, rather than as an ill-devised means of accomplishing something we do not yet know how to do otherwise.

In chapter V it is suggested that elementary trigonometry, including particularly the rules for right triangles, and the laws of sines and cosines, be introduced into the study of geometry, but in this matter American opinion has led the way, and such a recommendation has nothing novel for us.

The reviewer must close by expressing his great satisfaction that in Great Britain the immediate future of geometry should be decided by its friends, sane and wise and practical friends but preëminently friends.

J. L. COOLIDGE.

Relativity and Gravitation. By T. P. NUNN. London, University of London Press, 1923. 162 pages. Price 6 shillings.

This book by Professor Nunn has for its secondary title "An elementary treatise upon Einstein's theory." It calls attention to the fact that our literature upon the theory of relativity may be divided into works which are written for readers who have little or no mathematics and works of a more serious nature intended for those who have had considerable training in the mathematical sciences. The author states that his present contribution to the theory "seeks to fill a modest place between the two groups." He says that "the level of difficulty may be indicated by saying that it should be well within the scope of anyone who has read mathematics up to, or nearly up to, the pass standard required for a B.Sc. degree." It is, he says, "to be regarded as an exposition of the elements written by a layman for other laymen."

Now there are two interesting things which will at once strike any American

reviewer as he considers the American system of education and the achievements of our students. The first is that here is a book based upon a critical study of Einstein's own papers, and yet made by a university professor of education. It seems a little odd; professors of education in our universities do not cultivate such fields. The second thing is that probably an uncomfortably large per cent of students in this country who have recently received the degree mentioned, and who have "read mathematics up to, or nearly up to, the pass standard required," would find considerable difficulty in reading the book. It is as Père Bosmans recently said in a review of an American work—the book is addressed "à un public instruit, très instruit même, mais cependant à un public qui n'a pas été coulé dans le moule classique de nos humanités grécolatines," and he might have said "of our modern humanities as well."

It is therefore instructive for us to see a book written by such an author and addressed to laymen of college training, and to ask if, after all, we are on the best educational track.

It can hardly be expected in a review of such a work that more should be given than a general statement of the contents and of the style in which it is written. The successive chapters discuss the following leading topics: (1) Absolute and relative motion, (2) The restricted theory of relativity, (3) The general theory of relativity, (4) The Lorentz transformation and some applications, (5) The space-time invariant, (6) Some mathematical notes, (7) The geodesic law of motion, (8) The gravitation potentials, (9) The crucial phenomena, (10) The tensor method, (11) Restriction (or contraction) of tensors, (12) Tensor-differentiation, (13) The law of gravitation.

Professor Nunn has introduced his exposition of the theory with much the same simplicity as that which characterized so acceptably Professor Birkhoff's recent Lowell lecture upon the subject. The reading of the chapters upon the nature of absolute and relative motion, the restricted theory of relativity, and even the general theory, requires no knowledge of mathematics beyond what a high school furnishes. The chapter on the Lorentz transformation requires an initial college course in physics and in the integral calculus. In the second half of the book, however, the ordinary college graduate would meet with some difficulty, although those who have taken mathematics with a view to proceeding later to the master's degree would find in the vector analysis, the free use of determinants, the polar coördinates, the differential equations, the physical formulas (which they may or may not have studied), and the use of the tensor method, material with which they are, in general, fairly familiar.

Altogether the book is for students of mathematics in our junior or senior year rather than for the average layman who may possess a college degree. It should have a place in all college libraries and on the shelves of all students who expect to proceed to the study of graduate mathematics. Its style is lucid and its exposition logical.

David Eugene Smith.

The Theory of Determinants in the Historical Order of Development. By SIR THOMAS MUIR. London, Macmillan and Company.

1906, vol. I.—The Period 1693 to 1841, 475 pages. Price 21 shillings.
1911, vol. II.—The Period 1841 to 1860, 503 pages. Price 21 shillings.
1920, vol. III.—The Period 1861 to 1880, 491 pages. Price 35 shillings.
1923, vol. IV.—The Period 1881 to 1900, 508 pages. Price 40 shillings.

For many years there has probably existed among mathematicians a growing conviction that it is of greater importance at the present stage of development of mathematics and its history that we should possess exhaustive historical accounts of special fields than that we have a general history of all mathematics. Such an exhaustive treatment implies that the author planned to take into account all papers written on his subject, the apparently unimportant ones as well as the obviously important, and that he made all reasonable efforts to secure such completeness. It seems nearly obvious that every branch of mathematics must have its detailed treatment in this sense before we shall be properly prepared for an authentic history of the whole of mathematics. The lack of such broad preparatory work—which can only be slowly built up by the efforts of many mathematicians and historians—is without doubt one main reason why such works as Cantor's monumentally designed *Geschichte der Mathematik* must be considered to a certain extent disappointing, in spite of the immense amount of work and learning involved in their preparation.

The number of such exhaustive treatments is slowly increasing. Within the last few years we have had Dickson's fundamental *History of the Theory of Numbers,* the *Report of the Committee on Algebraic Numbers,* and, quite recently, the fourth and final volume of Muir's *Theory of Determinants in the Historical Order of Development.*

The difficulty of compiling a complete record of even a comparatively small domain will be appreciated when one realizes that this summary of the relatively modern and special field of determinants covers, in four volumes of about equal thickness, nearly two thousand pages, and represents work by the author extended over more than forty years.

The earliest paper mentioned is one by Leibniz, 1693, the next two, Fontaine, 1748, and Cramer, 1750. Prior to 1800, eleven papers, all told, are listed; the authors are, besides the three just named, Bézout, Vandermonde, Laplace, Lagrange, Hindenburg.

The tremendous increase in the rate of production becomes apparent when one compares the number of titles reviewed in each volume. Thus, in the century and a half 1693–1841, there were 97 papers, while, in the successive twenty-year periods after 1841, there were 222, 588, and 875 papers respectively. In the successive volumes the treatment has undergone a natural tendency toward greater condensation. However, even in the concluding volume, at least the content of every paper is indicated, sometimes in a single sentence, sometimes in a report covering several pages; the position of the paper with respect to previous work is stated; and in very many cases a valuation of the paper in discreet form is attempted.

We quote the greater part of the preface:

"With the issue of this volume, dealing with the period 1880–1900, my effort to present a historical account of the Theory of Determinants up to the close of the nineteenth century comes to an end. May it give to present and future students all the help which some such book would certainly have given to me half a century ago.

"The work connected with the preparation of the volume has been more than ordinarily onerous. Considerably over 800 writings of one kind and another have had to be dealt with, that is to say, about a half more than in the case of the third volume. In view of the fact that each volume deals with a twenty-year period, the increase thus indicated in the attention given to the study of determinants is most striking, even when one makes allowance for the considerably increased facilities for publication during the latter period.

"On first thoughts the bulk of the fourth volume ought, therefore, to be about a half more than that of the third; but then we have to reflect that the increased number of students and the increased facilities referred to are not accompanied by the like increase of original matter. It is only that workers rush into print a little more hastily than formerly, and that scientific societies and editors do not always sufficiently exert themselves to withstand the rush,—a state of matters that after all may be well-ordered in the interest of instruction and the spread of knowledge. Be this as it may, it is an undoubted fact that the number of writings which the chronicler of scientific progress can without unkindness dismiss in a sentence grows apace. All that can be fairly expected of him is to try his hardest to be just in his judgments, and, in the interest of his readers, to guard against hurtful condensation.

"The bibliographical basis of the work, as is probably now well known among students of the subject, is the six lists of writings published in the *Quarterly Journal of Mathematics* at intervals from 1881 to 1916. A seventh list was added in 1920. Since in the compiling of every one of these lists after 1900 writings belonging to the preceding century continued to be carefully sought for, it is hoped that little matter of serious importance has been passed over that was needed for the History. . . ."

Chapter headings of volume IV are as follows:

I. Determinants in general, 1880–1900, 76 pp.—II. Text-books, 20 pp.—III. Determinants and linear equations, 11 pp.—IV. Axisymmetric determinants, 30 pp.—V. Symmetric determinants that are not axisymmetric, 4 pp.—VI. Alternants, 60 pp.—VII. Compound determinants, 22 pp.—VIII. Recurrents, 18 pp.—IX. Wronskians, 9 pp.—X. Jacobians, 8 pp.—XI. Skew determinants and Pfaffians, 25 pp.—XII. Orthogonants and latent roots, 28 pp.—XIII. Persymmetric determinants, 20 pp.—XIV. Bigradients, 20 pp.—XV. Hessians, 4 pp.—XVI. Circulants, 40 pp.—XVII. Continuants, 18 pp.—XVIII. Multilineants, 12 pp.—XIX. N-dimensional determinants, 2 pp.—XX. Bordered determinants, 7 pp.—XXI. Determinants having invariant factors (from 1851 to 1900), 19 pp.—XXII. The less common special forms, 44 pp.—Alphabetical list of authors, 11 pp.

A very appreciative review of the third volume was published in the MONTHLY (*1920*, 419) by Professor R. C. Archibald.

Sir Thomas is entitled to the sincere congratulations of mathematicians for having successfully carried through his ambitious plan of giving, up to about 1900, a complete record of the writings on determinants.

A. J. KEMPNER.

Mechanics of Particles and Rigid Bodies. By J. PRESCOTT. London, Longmans, Green and Co. 1923. 8vo, vii + 538 pages. Price $4.75 net.

This is a second edition of this book and because it has been written and printed with the usual care of the British writer and publisher one has to search closely to find even a typographical error and then it is usually only a defective type. There are almost no errors even of this kind in the book and such as do occur are not worth calling especial attention to. The book contains 538 pages

make them familiar; numerous examples of attraction are clearly worked out in the text. Newton's laws are presented with careful explanation and kinematics in a plane with applications to central orbits gives a most satisfactory preparation for students taking astronomy. The treatment of a rigid body in three dimensions is carried out far enough to furnish the methods for working a number of excellent exercises and to introduce all the fundamental notions. The book ends with a good chapter on units and dimensions. Curiously enough, there is an appendix giving elementary properties of conics which one would think familiar to any student capable of entering such a course.

One of the splendid things about this book, common to nearly all British mathematical texts, is the collection of problems and exercises. They are not fancy, made-up problems, for most of them are signed as having been posted as examination questions at various institutions, as is their custom. The Briton in this respect simply has us beaten to a finish. The reviewer thinks it unfortunate that we do not give the time, contemplation and the practice in our undergraduate work necessary to acquire such skill and power to answer questions on examination as many of these problems require.

It is with pleasure that we recommend this book to American students with the assurance that they will be benefited by its careful reading and prepared thereby to pursue work in any direction they may elect in scientific work.

W. H. ECHOLS.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

ANNALS OF MATHEMATICS, second series, volume 24, no. 4, June, 1923: "An introduction to the theory of elliptic functions" by G. Mittag-Leffler, 271–351; "Cyclotomic quintisection for all primes of the form $10n + 1$ between 1900 and 2100" by P. O. Upadhyaya, 352–354; "Geodesic lines in Riemann space" by R. Henderson, 355–358; "A functional equation from the theory of the Riemann $\zeta(s)$-functions" by A. Arwin, 359–366; "The geometry of paths and general relativity" by L. P. Eisenhart, 367–392.

AMERICAN JOURNAL OF MATHEMATICS, volume 47, no. 1, January, 1924: "On the theory of numbers and generalized quaternions" by L. E. Dickson, 1–16; "A study of the rational involutorial transformations in space which leave a web of sextic surfaces invariant" by J. O. Osborn, 17–36; "On the reduction of differential parameters in terms of finite sets, with remarks concerning differential invariants of analytic transformations" by O. E. Glenn, 37–54; "On the isodyadic septimic equations" by J. C. Glashan, 55–69.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 10, no. 2, February, 1924: "Sets of completely independent postulates for cyclic order" by E. V. Huntington, 74–78; "The development of a frequency function and some comments on curve fitting" by E. B. Wilson, 79–84; March, 1924: "New topological invariants expressible as tensors" by J. W. Alexander, 99–101; "On certain new topological invariants of a manifold" by J. W. Alexander, 101–103; "Condition that an electron describe a geodesic" by A. Bramley, 103–107.

SCHOOL SCIENCE AND MATHEMATICS, volume 24, no. 3, March, 1924: "Mathematics" by E. G. Burgess, Jr., 264–272; "Mathematical shortcomings of the Greeks" by G. A. Miller, 284–287; "Disguised facts" by W. V. Lovitt, 287–290; "The cyclic quadrilateral" by R. Morris, 296–300.

JOURNAL DE MATHÉMATIQUES, volume 2, no. 4, 1923: "Theory of non-analytic functions of a complex variable" by E. R. Hedrick, L. Ingold, W. D. A. Westfall, 327–342.

MONIST, volume 34, no. 1, January, 1924: "The structure of exact thought" by R. D. Carmichael, 63–95.

SCIENTIFIC MONTHLY, volume 18, no. 4, April, 1924: "The origin, nature and influence of relativity" by G. D. Birkhoff, 408–421.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3082. Proposed by J. E. TREVOR, Cornell University.

Find three determinants whose arrays are odd-rowed concentric magic squares of order n, and whose absolute values are successively $n\cdot n!$, and $(1\cdot3\cdot5\cdot7\cdots n)^2$, and $(n!)^2$.

A magic square is here understood to be an array of n^2 numbers such that the sum of the elements of each row, of each column, and of each principal diagonal is the same number; while a concentric magic square is a magic from which successive magic squares are obtained by successive removal of the bounding rows and columns.

3083. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that if three distinct normals to the parabola, $y^2 = 4px$, are concurrent, the sum of their slopes is zero, and that if the sum of three numbers is zero, they are the slopes of concurrent normals to the parabola, $y^2 = 4px$. If two of the three concurrent normals are perpendicular and the third bisects the angle between them, show that they meet at $(3p, 0)$, $(8p, \sqrt{5}p)$ or $(8p, -\sqrt{5}p)$.

3084. Proposed by H. S. UHLER, Yale University.

Given that a and b are constants, evaluate the integral

$$\int\{[x^3 \arcsin (b/x)]/\sqrt{a^2 - x^2}\}dx.$$

3085. Proposed by E. T. BELL, University of Washington.

Is there any simple expression, in terms only of n and its divisors, for the sum $\Sigma(-1)^{(x+y)/2}xy$ extended to all odd integers, x, y, satisfying $x^2 + y^2 = n$?

3086. Proposed by A. S. WIENER, Cornell University.

Prove that the determinant of the nth order

$$\begin{vmatrix} x_1^2+\lambda & x_1x_2 & x_1x_3 & x_1x_4 & \cdots & x_1x_n \\ x_1x_2 & x_2^2+\lambda & x_2x_3 & x_2x_4 & \cdots & x_2x_n \\ x_1x_3 & x_2x_3 & x_3^2+\lambda & x_3x_4 & \cdots & x_3x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1x_n & x_2x_n & x_3x_n & x_4x_n & \cdots & x_n^2+\lambda \end{vmatrix}$$

is divisible by λ^{n-1} and find the other factor.

3087. Proposed by H. W. BAILEY, Champaign, Illinois.

Given a polygon of n sides with vertices $(x_1, y_1), \cdots (x_n, y_n)$. Set up a determinant of the nth order which shall represent its area.

3088. Proposed by C. J. COE, University of Michigan.

If a sphere of radius $2r$ is cut by a circular cylinder of radius r so that the center of the sphere lies on the surface of the cylinder, show that the length of the curve of intersection of the two surfaces is twice that of the curve of intersection of this cylinder with a plane cutting its elements at an angle of 45°.

SOLUTIONS.

2966 [1922, 179]. Proposed by OTTO DUNKEL, Washington University.

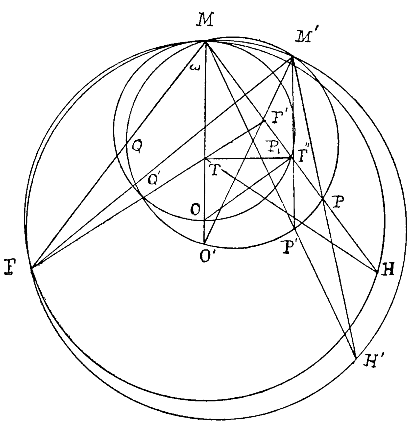
When a curve produces a caustic by the reflection of the rays proceeding from a fixed point, δ and δ' the lengths of the incident and reflected rays, R the radius of curvature of the curve, and ω the angle of incidence, satisfy the equation (see *1920*, 225)

$$\frac{1}{\delta}+\frac{1}{\delta'}=\frac{2}{R\cos\omega}.$$

Give a geometrical proof of this relation, using the harmonic properties of the figure, and obtain in this manner a simpler and different geometrical derivation from that given in Humbert's *Cours d'Analyse*, volume 1, page 77.

Solution by the Proposer.

Let F be the fixed point on the concave side of the curve; FM and FM' the rays to two neighboring points M and M' on the curve; MO' and $M'O'$ the normals to the curve meeting at O'; MF'' and $M'F''$ the reflected rays meeting at F''. Construct a circle through M, M' and O' cutting MF and $M'F$ in Q and Q', respectively, and cutting MF'' and $M'F''$ in P and P', respectively. Draw MP' and $M'P$ meeting in H': then H' and F'' are conjugate points with respect to the circle. Since O' bisects the arcs QP and $Q'P'$, the arc QQ' is equal to the arc $P'P$, and hence the angle MFM' is equal to the angle $MH'M'$. Thus a circle can be passed through F, M, M' and H'. Now as M' approaches M, O' approaches the center of curvature O for the point M; the points F'', P, H' approach limiting positions F', P_1, H on the reflected ray MF''. Hence H and F' are conjugate points with respect to the limit circle on the diameter $MO = R$, and this circle passes through P_1. The limiting position of the second circle is a circle through M and F with its diameter



along MO and passing through H. Since the angle FMO = angle $OMH = \omega$, we have $MF = MH = \delta$. From the harmonic range $MF'P_1H$ we have

$$\frac{1}{MH}+\frac{1}{MF'}=\frac{2}{MP_1} \qquad \text{or} \qquad \frac{1}{\delta}+\frac{1}{\delta'}=\frac{2}{R\cos\omega}.$$

This proof applies whether F' lies on the same side of the curve as F or on the opposite side: in the first case δ' is positive and in the second it is negative.

Draw FF' cutting MO in T, and then the straight lines TP_1 and TH. Since by the symmetry of the figure TM bisects the supplement of angle $F'TH$ (or the angle itself) and the pencil $T(MF'P_1H)$ is harmonic, it follows that TP_1 is perpendicular to MT.

2976 [1922, 225]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

The base of a variable triangle is fixed, the opposite vertex describing a straight line. Find the locus of the symmedian point, the locus of the center of the nine-point circle, and the envelope of the Euler line.

2934 [1921, 467]. This problem is included in the above.

SOLUTION BY OTTO DUNKEL, Washington University.

Let the x-axis be along the fixed base BC and the origin at its middle point so that the coördinates of B and C are, respectively, $(-a, 0)$ and $(a, 0)$. Let the variable vertex A have the coördinates (λ, μ) and suppose that it moves on the line $\mu = m\lambda + b$, $m = \tan \tau$.

The symmedian point, say L, lies on the straight line through the middle point of a side and the middle point of the corresponding altitude of the triangle.

Its distances from the sides of the triangle are proportional to the lengths of the corresponding sides.

From the first property we have at once $2\lambda y = \mu x$, where (x, y) are the coördinates of L. The second property gives

$$y[(\lambda - a)^2 + \mu^2] - 2a(\lambda - a)y + 2a\mu(x - a) = 0,$$

which reduces by means of the first result to

$$y[\lambda^2 + 3a^2 + \mu^2] - 2a^2\mu = 0. \tag{1}$$

Inserting the first result in $\mu = m\lambda + b$, we find

$$\lambda = \frac{bx}{2y - mx}, \qquad \mu = \frac{2by}{2y - mx};$$

and after removing $y/(2y - mx)^2$ the equation for L becomes

$$b^2x^2 + 3a^2(2y - mx)^2 + 4b^2y^2 - 4a^2b(2y - mx) = 0. \tag{2}$$

The form of this equation shows that the locus is an ellipse with $2y - mx = 0$ as its tangent at the origin.

Either one of the two properties might have been used alone to derive the equation; or this third property might have been used. Let BT and CT be the tangents to the circumscribing circle of ABC at B and C and let BL and CL be two lines such that $B(TCLA)$ and $C(TBLA)$ are harmonic pencils. Then L the intersection of BL and CL is the symmedian point.

Turning now to the center of the nine-point circle, let H be the intersection of the altitudes AM and BN. Then from the similarity of the triangles AMC and BMH, it follows that

$$MH = (a^2 - \lambda^2)/\mu.$$

Let K be the center of the circumscribing circle, then $2OK = HA$; and if the coördinates of the middle point of HK are (x, y), then $2x = \lambda$ and

$$y = \frac{OK + MH}{2} = \frac{HA + 2MH}{4} = \frac{\mu^2 - \lambda^2 + a^2}{4\mu}. \tag{3}$$

If $m = 0$, μ is a constant and the locus is the parabola

$$y = \frac{\mu^2 + a^2 - 4x^2}{4\mu}. \tag{4}$$

The case $m = \infty$ is trivial. Consider the remaining cases, and eliminate λ from (3). There results

$$\mu[(m^2 - 1)\mu - 4m^2y + 2b] + m^2a^2 - b^2 = 0, \qquad \mu = 2mx + b. \tag{5}$$

The form of this equation shows that the locus is a hyperbola with the asymptotes

$$x = -\frac{b \cot \tau}{2}, \qquad y = -\cot 2\tau x + \frac{b}{4 \sin^2 \tau}, \qquad m = \tan \tau, \tag{6}$$

and the center $(-\tfrac{1}{2}b \cot \tau, \tfrac{1}{2}b \cot^2 \tau)$.

The coördinates of the center of gravity are $(\lambda/3, \mu/3)$. Hence the equation of the line KH is

$$(1 + m^2)\lambda^3 + [2m(b - y) - (m^2 + 3)x]\lambda^2 + [b^2 - a^2 - 2b(y + mx)]\lambda + (3a^2 - b^2)x = 0. \tag{7}$$

The envelope of this line is obtained by imposing the condition for a pair of equal roots in λ. This condition is

$$27A^2D^2 - B^2C^2 - 18ABCD + 4AC^3 + 4DB^3 = 0,$$

where A, B, C, D are the coefficients of λ^3, λ^2, λ, λ^0 in equation (7). The locus is therefore a curve of the fourth order.

3019 [1923, 206]. Proposed by J. B. REYNOLDS, Lehigh University.

Find the equation of the curve on the cylinder $x^2 - y^2 = a^2$ such that the tangent to it cuts the xy-plane in a lemniscate as the point of tangency moves along the curve.

SOLUTION BY THE PROPOSER.

Solution by vectors. Let the equation of the curve be

$$r = a \cosh t\boldsymbol{i} + a \sinh t\boldsymbol{j} + u\boldsymbol{k},$$

then the vector

$$r' = a \sinh t\boldsymbol{i} + a \cosh t\boldsymbol{j} + p\boldsymbol{k},$$

where $p = du/dt$, will be parallel to the tangent to the curve. So for the vector equation of the tangent we may write

$$r_t = a(\cosh t + s \sinh t)\boldsymbol{i} + a(\sinh t + s \cosh t)\boldsymbol{j} + (u + ps)\boldsymbol{k},$$

s being a scalar quantity.

Where r_t cuts the xy plane we must have $u + ps = 0$ or $s = -u/p$ giving for r_t for the plane curve of intersection

$$r = a(\cosh t - \frac{u}{p} \sinh t)\boldsymbol{i} + a\ (\sinh t - \frac{u}{p} \cosh t)\boldsymbol{j}.$$

This curve will be a lemniscate if

$$\frac{u}{p} = \tanh 2t \qquad \text{or} \qquad \frac{du}{u} = \coth 2tdt.$$

Integrating, we get $u = c\sqrt{\sinh 2t}$ which gives for the required curve

$$r = a \cosh t\boldsymbol{i} + a \sinh t\boldsymbol{j} + c\sqrt{\sinh 2t}\,\boldsymbol{k}$$

and the curve in which the tangent intersects the xy-plane is

$$r = \frac{a \cosh t}{\cosh 2t}\boldsymbol{i} - \frac{a \sinh t}{\cosh 2t}\boldsymbol{j}$$

or the lemniscate $\rho^2 = a^2 \cos 2\theta$.

NOTE BY THE EDITORS: There is little, if any, advantage in using vectors as the work is almost identically the same without them. It has not been shown that there is only one lemniscate which can be obtained as the trace of the tangent in the xy-plane.

3021 [1923, 206]. Proposed by N. ALTSHILLER-COURT, University of Oklahoma.

Prove that the two lines joining the points of intersection of two orthogonal circles to any point of one of them meet the other circle in two diametrically opposite points, and conversely.

SOLUTION BY L. V. ROBINSON, University of Chicago.

Let the circles be PCQ and QCR the tangents of which are AC and BC, respectively, C and Q being the points of intersection. Draw PC and PQ intersecting the circle QCR in O and R. Draw QR and QC.

Then since $\angle ACB = 90°$, it also follows that

$$\angle ACP + \angle BCR = 90°.$$

Since $\frac{1}{2}$ arc PC measures both angles PQC and PCA, and $\frac{1}{2}$ arc CR measures both the angles BCR and CQR,

$$\angle ACP = \angle PQC; \qquad \angle BCR = CQR.$$

$$\therefore \quad \angle PQR = 90°.$$

which means that $\angle OQR$ is measured by a quadrant. It is evident, then, that OR must be the diameter of the circle QCR.

The converse also easily follows. Given OR the diameter of the circle QCR, draw CR and QO intersecting the circle PCQ in some point P. Draw CP and CR and prove PCR is a straight line. As before, it is obvious that

$$\angle ACP = \angle PQC; \qquad \angle BCR = \angle CQR.$$

But since

$$\angle PQC + \angle CQR = \angle OQR = 90°,$$

$$\angle ACP + \angle BCR = 90°;$$

and since $\angle ACB = 90°$,

$$\angle ACP + \angle ACB + \angle BCR = 180°,$$

and PCR is a straight line. Hence the lines drawn through Q and O and through C and R meet the other circle in a common point P.

Also solved by THEODORE BENNETT, A. BOGARD, RUFUS CRANE, A. PELLETIER, A. V. RICHARDSON, J. B. REYNOLDS, W. W. WEBER, and MABEL M. YOUNG.

3022 [1923, 206]. Proposed by M. J. SPINKS, Wilmington, Ohio.

Given that ABC is an equilateral spherical triangle right-angled at C, prove that $\sec A = 1 + \sec a$.

SOLUTION BY L. V. ROBINSON.

Since ABC is equilateral it is also equiangular. But since C is a right angle, all the angles are right and hence each side is 90°. Thus the relation in the problem is satisfied. It will now be shown that the relation is true when the equal angles are not right. By the law of cosines we have $\cos a = \cos^2 a + \sin^2 a \cos A$. Since $\cos a$ is not zero, let us divide this equation by $\cos^2 a$ and then replace $\tan^2 a$ by $\sec^2 a - 1$. There results

$$\sec A = \frac{\sec^2 a - 1}{\sec\ a - 1} = \sec a + 1.$$

Since A and C approach 90° simultaneously, we see that as C approaches a right angle, a and hence b and c approach 90°.

Also solved by J. B. REYNOLDS.

3023 [1923, 206]. Proposed by E. T. BELL, University of Washington.

The equation $x^p + y^p + z^p = 0$ is possible in integers x, y, z prime to the odd prime p, if

$$\frac{1}{2}\left[\frac{1}{2}N_2(p) - \frac{1}{3}N_3(p) + \frac{1}{4}N_4(p) - \cdots + \frac{1}{p-1}N_{p-1}(p)\right] + 1$$

is divisible by p, where $N_r(n)$ is the number of representations (order essential) of n as a sum of r square integers with roots $\gtreqless 0$.

SOLUTION BY THE PROPOSER.

The condition is merely Wieferich's criterion in another form: the equation is possible if $(2^{p-1} - 1)/p^2$ is an integer. For if m is odd, we have

$$m \sum_{r=1}^{m} \frac{(-1)^{r-1}}{r} N_r(m) = 2\zeta_1(m), \tag{1}$$

where $\zeta_1(m)$ = the sum of all the divisors of m. If in (1) we put $m = p$, and note that $N_p(p) = 2^p$, we get

$$\frac{1}{2}N_2(p) - \frac{1}{3}N_3(p) + \frac{1}{4}N_4(p) - \cdots + \frac{1}{p-1}N_{p-1}(p) = 2\left[\frac{2^{p-1}-1}{p} - 1\right],$$

since $N_1(p) = 0$. Hence if $(2^{p-1} - 1)/p^2$ is an integer, we get the stated condition. The identity (1) follows by equating coefficients of q^m in

$$q\frac{d}{dq}\log\left(1 + 2\sum_{n=1}^{\infty} q^{n^2}\right) \equiv q\frac{d}{dq}\left[\sum_{n=1}^{\infty}\log\left(1 - q^{2n}\right) + 2\sum_{n=1}^{\infty}\log\ \left(1\ -\ q^{2n-1}\right)\right], \tag{2}$$

which follows from the well-known identity (see any text on elliptic functions)

$$1 + 2\Sigma q^{n^2} = \Pi(1 - q^{2n}) \times \Pi(1 + q^{2n-1})^2.$$

On the left of (2) the logarithm is expanded and rearranged as a power series in q before differentiation, and the obvious relation $N_r(n) = 2^r N_r'(n)$ is used, where $N_r'(n)$ = the number of representations of n as a sum of r squares of integers with roots > 0.

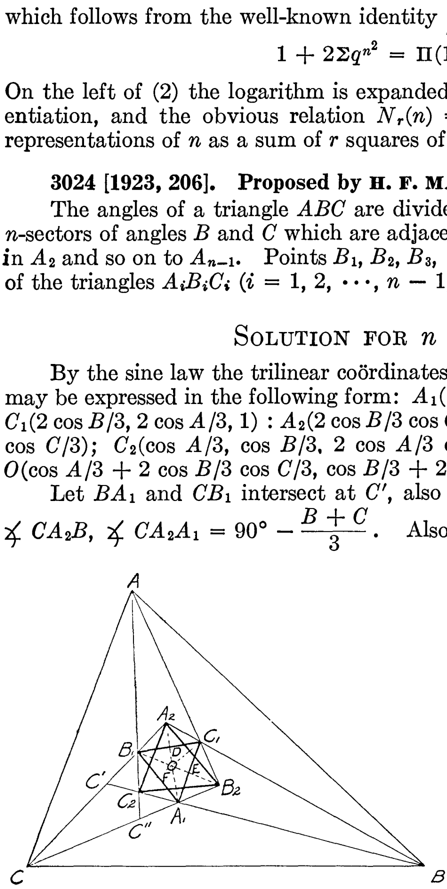
3024 [1923, 206]. Proposed by H. F. MACNEISH, College of the City of New York.

The angles of a triangle ABC are divided into n equal parts ($n = 3, 4, 5, \cdots$) and the two n-sectors of angles B and C which are adjacent to side BC intersect in A_1, the next two n-sectors in A_2 and so on to A_{n-1}. Points $B_1, B_2, B_3, \cdots C_1, C_2, C_3, \cdots$ are similarly determined. Which of the triangles $A_iB_iC_i$ ($i = 1, 2, \cdots, n - 1$) are equilateral?

Solution for $n = 3$ by the Proposer.

By the sine law the trilinear coördinates of points $A_1, A_2, B_1, B_2, C_1, C_2$ are easily found and may be expressed in the following form: $A_1(1, 2\cos C/3, 2\cos B/3)$; $B_1(2\cos C/3, 1, 2\cos A/3)$; $C_1(2\cos B/3, 2\cos A/3, 1)$: $A_2(2\cos B/3\cos C/3, \cos B/3, \cos C/3)$; $B_2(\cos A/3, 2\cos A/3\cos C/3, \cos C/3)$; $C_2(\cos A/3, \cos B/3, 2\cos A/3\cos B/3)$. Then A_1A_2, B_1B_2, C_1C_2 concur at point $O(\cos A/3 + 2\cos B/3\cos C/3, \cos B/3 + 2\cos A/3\cos C/3, \cos C/3 + 2\cos A/3\cos B/3)$.

Let BA_1 and CB_1 intersect at C', also AB_1 and CA_1 intersect at C''. Since A_2A_1 bisects $\measuredangle CA_2B$, $\measuredangle CA_2A_1 = 90° - \dfrac{B + C}{3}$. Also $\measuredangle A_1C'A_2 = \dfrac{2C + B}{3}$; hence the sum $\measuredangle CA_2A_1 + \measuredangle A_1C'A_2 = 90° + C/3$ and the third angle of triangle $C'A_1A_2$, *i.e.*, $\measuredangle C_2A_1A_2$ equals $90° - C/3$. Similarly $\measuredangle C_2B_1B_2$ equals $90° - C/3$. Therefore, $\measuredangle C_2A_1A_2 = \measuredangle C_2B_1B_2$, hence $\triangle OB_1C_2 = \triangle OA_1C_2$ and $B_1C_2 = A_1C_2$. Then $\triangle C_1C_2A_1 = \triangle C_1C_2B_1$; hence, $A_1C_1 = B_1C_1$. By cyclic interchange of letters A, B, C we obtain $A_1C_1 = A_1B_1$ and triangle $A_1B_1C_1$ is equilateral.



The 3 isosceles $\triangle$ $A_1B_1C_2$, $B_1C_1A_2$, $C_1A_1B_2$ have equal bases, but $\measuredangle A_1C_2B_1 = 180° - \frac{2}{3}(A + B)$, $\measuredangle B_1A_2C_1 = 180 - \frac{2}{3}(B + C)$, $\measuredangle C_1B_2A_1 = 180 - \frac{2}{3}(A + C)$; hence unless the triangle ABC is equilateral, their vertex angles are not all equal; hence the altitudes A_2D, B_2E, C_2F are not all equal. A_1D, B_1E, C_1F the altitudes of the equilateral $\triangle A_1B_1C_1$ are equal; therefore $OD = OE = OF$ and hence OA_2, OB_2, and OC_2 are not all equal in general. In $\triangle A_2OB_2$, since $\measuredangle A_2OB_2 = 120°$, $\overline{A_2B_2}^2 = \overline{OA_2}^2 + \overline{OB_2}^2 + OA_2 \cdot OB_2$; and in $\triangle A_2OC_2$ since $\measuredangle A_2OC_2 = 120°$, $\overline{A_2C_2}^2 = \overline{OA_2}^2 + \overline{OC_2}^2 + OA_2 \cdot OC_2$. Hence unless the triangle ABC is equilateral, the triangle $A_2B_2C_2$ is not equilateral.

3025 [1923, 275]. Proposed by E. H. CLARKE, Hiram College.

Sum the infinite series

$$1 + \frac{2^k x^2}{2!} + \frac{3^k x^4}{4!} + \frac{4^k x^6}{6!} + \cdots + \frac{n^k x^{2n-2}}{(2n-2)!} + \cdots,$$

k a positive integer.

Solution by H. L. Slobin, University of New Hampshire.

The summation of this series may be effected by the method explained on page 209 of Crystal's *Algebra, Part II*, for the summation of series of the form $\Sigma \varphi_r(n)\, x^n/n!$, where $\varphi_r(n)$ is a polynomial of the rth degree in n with constant coefficients. In the proposed series suppose that $k = 3$. The method consists in the determination of the coefficients in the identity

$$(2n)^3 = A_0 + A_1(2n - 2) + A_2(2n - 2)(2n - 3) + A_3(2n - 2)(2n - 3)(2n - 4);$$

then by means of this result the given series can be broken up into four series of known form.

Obviously $A_3 = 1$, then A_0, A_1, A_2 may be determined in turn by setting $2n = 2, 3, 4$. In this way it is found that

$$n^3 = 1 + \frac{19}{2^3}(2n - 2) + \frac{9}{2^3}(2n - 2)(2n - 3) + \frac{1}{2^3}(2n - 2)(2n - 3)(2n - 4).$$

Hence

$$\sum_1^\infty \frac{n^3x^{2n-2}}{(2n-2)!} = \sum_1^\infty \frac{x^{2n-2}}{(2n-2)!} + \frac{19x}{2^3}\sum_2^\infty \frac{x^{2n-3}}{(2n-3)!} + \frac{9x^2}{2^3}\sum_2^\infty \frac{x^{2n-4}}{(2n-4)!} + \frac{x^3}{2^3}\sum_3^\infty \frac{x^{2n-5}}{(2n-5)!}$$
$$= \left(1 + \frac{9x^2}{2^3}\right)\cosh x + \left(\frac{19x + x^3}{2^3}\right)\sinh x.$$

For the case of $k = 2$ it is found that

$$\sum_1^\infty \frac{n^2x^{2n-2}}{(2n-2)!} = \left(1 + \frac{x^2}{2^2}\right)\cosh x + \frac{5x}{2^2}\sinh x.$$

Whatever positive integral value k may have, the corresponding A coefficients may be determined in exactly the same way, and consequently the summation can be performed in the manner explained.

3027 [1923, 275]. Proposed by C. N. SCHMALL, New York City.

A parabola whose base (double ordinate) is h and altitude k has a circle inscribed of diameter d, and a circle circumscribed, of diameter D. Show that $D + d = h + k$. Note that k can not be $< h/2$.

Solution by S. E. Field, University of Michigan.

Given the parabola $y^2 = 4px$ with its base the line $x = k$. The coördinates of the extremities of the base are $(k, \pm h/2)$. For the inscribed circle, let the center be $(a_1, 0)$. The radius will be $r = k - a_1$ and its equation will be $x^2 + y^2 - 2a_1x - k^2 + 2ka_1 = 0$. The condition that this circle shall be tangent to the parabola $y^2 = 4px$ is $(2p - a_1)^2 - (2ka_1 - k^2) = 0$, and this reduces to $a_1 = 2p + k - \sqrt{4pk} = 2p + k - h/2$ (discarding the positive sign of the radical since it will obviously put the center of the circle outside the parabola). Then, $r = (h/2) - 2p$ and $d = h - 4p$.

For the circumscribed circle, let the center be $(a_2, 0)$. Since this circle passes through the points $(0, 0)$, $(k, \pm h/2)$, its radius will be

$$R = a_2 = \sqrt{(k - a_2)^2 + h^2/4}$$

from which

$$a_2 = \frac{k + 4p}{2} = R \quad \text{and} \quad D = k + 4p.$$

Hence, $D + d = h + k$.

The abscissa of the points of tangency of the inscribed circle is $k - h/2$.

NOTE.—If $k < h/2$, the circle touches the parabola only at its vertex and the above argument does not apply.

Also solved by A. BOGARD, H. W. BAILEY, MICHAEL GOLDBERG, A. M. HARDING, WILLIAM HOOVER, A. PELLETIER, C. K. ROBBINS, J. B. REYNOLDS, and A. V. RICHARDSON.

3029 [1923, 275]. Proposed by J. ROSENBAUM, Milford, Conn.

To locate two points, D and E, on the sides AB and BC of a triangle ABC such that $AD : DE : EC$ shall be equal to $p : q : r$, where p, q, and r are given line segments.

The above is a generalization of problem 2816 (*1920*, 134).

Solution by A. V. Richardson, Bishop's College.

Take any point X on AB, and from C lay off CY on CB such that $AX : CY = p : r$. Now determine a length XZ such that $AX : XZ = p : q$ and with this length as radius describe a circle with the center X. Let the parallel to AC through Y cut this circle in Z and draw ZS parallel to BC cutting AC in S. Let the line AZ cut CB in E, and draw ED parallel to ZX cutting AB in D. Then D and E are the required points. For $AD/AX = DE/XZ = AE/AZ = EC/YC$.

Note by Otto Dunkel.

If the points D and E may be taken on the extensions of the sides AB and CB, there may be four solutions. This will be the case if $q^2 \neq p^2 + r^2 \pm 2pr \cos B$ and $q^2 > (p \sin A - r \sin C)^2$. If $q^2 < (p \sin A - r \sin C)^2$, there is no solution. It is assumed that none of the lengths p, q, r are zero. For certain triangles and certain values of $p : q : r$ there are two solutions in which the points D and E lie within the respective segments AB and CB.

Also solved by Michael Goldberg, C. K. Robbins and F. L. Wilmer.

3030 [1923, 275]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the envelope of the bisector of the angle that a given segment subtends at a variable point of a given line.

Solution by William Hoover, Columbus, Ohio, and Otto Dunkel, Washington University.

A more general problem will be considered. Let A, B be a pair of fixed points; E, F a second pair of fixed points not in the same straight line with AB; l a fixed straight line. A variable point P of the line l is joined to A, B, E, F by straight lines and the pair of lines which separate harmonically the lines PA, PB as well as PE, PF will be considered. If E and F are the circular points at infinity, this pair of lines become the bisectors of the given problem.

Let l and EF, produced, meet in C; take ABC as the triangle of reference and choose the coördinates so that the equation of l is $x_1 + x_2 = 0$. The equation of EF will be of the form $x_1 + kx_2 = 0$. The equations of AP and BP may then be written

$$x_3 - \lambda x_1 = 0, \qquad x_3 + \lambda x_2 = 0. \tag{1}$$

A point Q on EF is determined by its intersection with a line $ax_1 + bx_2 + x_3 = 0$. The equation of PQ may be found by determining p, q, r, s so that

$$p(x_3 - \lambda x_1) + q(x_3 + \lambda x_2) \equiv r(x_1 + kx_2) + s(ax_1 + bx_2 + x_3) = 0. \tag{2}$$

It is found in this way that the equation of PQ is

$$(ka - b + \lambda)(x_3 - \lambda x_1) + (b - ka - k\lambda)(x_3 + \lambda x_2) = 0. \tag{3}$$

Let the points E, F be given by $b = 0$, $a = 0$, respectively. Then the equations of EP and FP are

$$\begin{aligned} (ka + \lambda)(x_3 - \lambda x_1) - k(a + \lambda)(x_3 + \lambda x_2) &= 0, \\ (\lambda - b)(x_3 - \lambda x_1) + (b - k\lambda)(x_3 + \lambda x_2) &= 0. \end{aligned} \tag{4}$$

The pair of lines which separate harmonically both (1) and (4) are given by

$$(ka + \lambda)(\lambda - b)(x_3 - \lambda x_1)^2 - k(a + \lambda)(k\lambda - b)(x_3 + \lambda x_2)^2 = 0. \tag{5}$$

A factor λ may be taken out and equation (5) becomes

$$\begin{aligned} \lambda^3(x_1^2 - k^2x_2^2) + \lambda^2[(ka - b)(x_1^2 - kx_2^2) - 2x_3(x_1 + k^2x_2)] \\ + \lambda[(1 - k^2)x_3^2 - 2x_3(ka - b)(x_1 + kx_2) + kab(x_2^2 - x_1^2)] \\ + x_3[x_3(1 - k)(ka - b) + 2kab(x_1 + x_2)] = 0. \end{aligned} \tag{6}$$

The envelope is obtained by eliminating λ from (6) by use of its derivative with respect to λ. The result would be complicated and not particularly interesting. From (6) we see that from a point there may be drawn three tangents to the envelope.

3035 [1923, 337]. Proposed by R. M. MATHEWS, Wesleyan University.

Generalize projectively and prove that the envelope of the bisectors of the angles between corresponding lines of two perspective pencils is a curve of the third class.

Solution by the Proposer.

The projective statement is: *The two pairs of lines which join a point P to two fixed pairs of points A, A' and B, B' determine an involution at P the double lines of which envelop a curve of the third class when P describes a line.*

The four points determine a pencil of conics; each conic C cuts the line p on which P moves in its own pair of corresponding points P and P' of an involution on p. To find the double lines in the radial involutions $P(AA', BB')$ and $P'(AA', BB')$, let AB meet $A'B'$ at C while AB' and $A'B$ meet at C'; the line $l \equiv CC'$ cuts conic C at L and L'. Then PL and PL' are the double lines at P; similarly for P'. Now the involutions (PP') and (LL') of points which the pencil of conics gives on p and l are not only projective but are in what Schroeter called "half perspective" position.[1] That is, the conic through the intersection D of p and l gives a point P' on p and L' on l; and D as of p corresponds to P', while as of l it corresponds to L', and in the projectivity of the involution ranges it corresponds once to itself. Now Schroeter has shown that the lines which join the corresponding points of two range involutions in half perspective position envelop a curve of the third class, if the point D, as the envelope of the pencil of lines through it, be discarded.

The general theorem above dualizes as follows:

The two pairs of points in which a line p is cut by two fixed pairs of lines a, a' and b, b' determine an involution the double points of which describe a cubic when p rotates around a point.

NOTE: See the solution of **3030**, p. 312, for an analytical treatment of this problem. If the points A, A', B, B' be on a straight line, then the envelope consists of the pair of points which separate harmonically the two pairs of points A, A' and B, B', and the intersection of p with AA'.

3039 [1923, 337]. Proposed by J. K. WHITTEMORE, Yale University.

Given a conic S, a point A, and a line l. Through A is drawn a variable line cutting S in P and Q. Find the envelope of a conic which is tangent to S at P and Q and which is tangent to l.

I. SOLUTION BY R. A. JOHNSON, Hamline University.

Using trilinear coördinates, we take the fixed point A as $(0, 0, 1)$, the fixed line l as $x_3 = 0$, and the conic S as

$$(xx) \equiv a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0.$$

An arbitrary line, k, through A is given by

$$x_1 + \lambda x_2 = 0,$$

where λ is any constant.

The equation

$$v(xx) + u(x_1 + \lambda x_2)^2 = 0,$$

where u and v are constants, represents a conic having double contact with S where the line, k, intersects S. We must determine $u : v$ so that this conic will also be tangent to l. Setting $x_3 = 0$, we have

$$(a_{11}v + u)x_1^2 + 2(a_{12}v + \lambda u)x_1x_2 + (a_{22}v + \lambda^2 u)x_2^2 = 0.$$

The condition for tangency is

$$(a_{11}v + u)(a_{22}v + \lambda^2 u) - (a_{12}v + \lambda u)^2 = 0.$$

Setting $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, this reduces to

$$(a_{11}\lambda^2 - 2a_{12}\lambda + a_{22})uv + v^2 D = 0, \qquad (a_{21} = a_{12}).$$

The root $v = 0$ yields a trivial solution, line k counted twice.

If $D = 0$, so that S is tangent to l, we have $u = 0$, and the variable conic reduces to S itself.

If $D \neq 0$,

$$v = a_{11}\lambda^2 - 2a_{12}\lambda + a_{22}, \qquad u = -D,$$

and the variable conic specified by the problem is

$$(a_{11}\lambda^2 - 2a_{12}\lambda + a_{22})(xx) - D(x_1 + \lambda x_2)^2 = 0.$$

Arranging this equation according to powers of λ, we have to determine the envelope as λ varies. This is most easily accomplished by equating to zero the discriminant of the quadratic

$$\begin{vmatrix} a_{11}(xx) - Dx_2^2 & a_{12}(xx) + Dx_1x_2 \\ a_{21}(xx) + Dx_2x_1 & a_{22}(xx) - Dx_1^2 \end{vmatrix} = 0,$$

[1] Schroeter, H., *Theorie der ebenen Kurven dritter Ordnung*, Leipzig, 1888. § 6.

which reduces to

$$D(xx)[(xx) - a_{11}x_1^2 - 2a_{12}x_1x_2 - a_{22}x_2^2] = 0$$

or

$$D(xx)x_3(2a_{13}x_1 + 2a_{23}x_2 + a_{33}x_3) = 0.$$

The envelope then consists of S, l, and *a straight line whose equation is*

$$2a_{13}x_1 + 2a_{23}x_2 + a_{33}x_3 = 0.$$

This line passes through the common point of l and the polar of A with regard to S.

Consider briefly the metric interpretation of this result. We may take A as the origin and l as the line at infinity; and the problem is that of the envelope of a set of *parabolas* doubly tangent to a given conic. It is evident that the envelope is parallel to the polar of A with respect to the conic, and half as far from A as this polar.

II. Solution by Otto Dunkel, Washington University.

Let C be a conic tangent to l at L and having double contact with S such that the chord of contact PQ $(= d)$ passes through A. If B is the point on d such that $APBQ$ is a harmonic range and T is the intersection of the common tangents at P and Q, then the fixed line BT $(= a)$ is the polar of A with respect to S, also with respect to C. Let M be the intersection of a with l and consider the second tangent ML' to C with the point of tangency L'. Since M lies on a, its polar $L'L$ with respect to C must pass through A. Thus the pencil MA, l, a, ML' is harmonic; and hence ML' is a fixed straight line which every C touches. Thus the envelope of C consists of the parts S, l and ML'. There can be no other parts to the envelope. For each C determines a point L, and L' is then found as the intersection of AL with the fixed line ML'. Since L and L' count as four points, only one other condition, tangency with S, can be adjoined for the determination of each C.

If S is tangent to l it must also touch ML'; hence C must coincide with S, since they have in common four points in P and Q and two common tangents, l and ML'.

Also solved by W. B. Carver, William Hoover and C. K. Robbins.

3040 [1923, 402]. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Given the radius, R, of a sphere rolling down two intersecting straight lines including the angle 2α and equally inclined to the horizon; show that the locus of the center of the sphere is an ellipse of semi-axes $R \csc \alpha$, R.

Solution by W. S. Barlow, Detroit, Michigan.

Place the two intersecting lines Om and On in the XZ plane so that their bisector coincides with the x-axis, and their point of intersection O is at the origin.

Let x, y be the coördinates of the center of the sphere. Line y is perpendicular to OX and line w is drawn to the point of tangency between the sphere and Om. Line w is therefore perpendicular to y and Om.

Then,

$$w^2 + y^2 = R^2 \quad \text{and} \quad (x^2/R^2 \csc^2 \alpha) + (y^2/R^2) = 1.$$

Also solved by S. F. Bibb, Philip Fitch, H. Halperin, A. Pelletier, J. B. Reynolds, C. K. Robbins, Thaddeus Slonczewski, H. B. Wilcox, and the Proposer.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Dr. ELIHU THOMSON, of the General Electric Company, has received the Kelvin gold medal, which was founded in 1914 by British and American engineers and is awarded triennially by the presidents of representative British societies.

Dr. W. D. LAMBERT, of the United States Coast and Geodetic Survey, has been elected recording secretary of the Washington Academy of Sciences.

Professor J. E. HODGSON of the University of West Virginia died on April 11, 1924. He was fifty-four years of age and had been Professor of Mathematics since 1913, having been appointed Associate Professor in 1912.

Assistant Professor ARTHUR RANUM, of Cornell University, has been promoted to a full professorship of mathematics.

Professor R. W. BURGESS, of Brown University, has been granted leave of absence for the academic year 1924–1925, and has accepted for the year a statistical position with the Western Electric Company in New York City.

Miss FRANCES M. WRIGHT, of Brown University, has been appointed instructor of mathematics at the University of Oklahoma for the academic year 1924–1925.

Miss EVELYN P. WIGGIN, of Brown University, has been appointed instructor of mathematics and physics at Hood College, Frederick, Maryland, for the academic year 1924–1925.

The National Council of Teachers of Mathematics held its annual meeting in conjunction with the Department of Superintendence of the National Education Association in Chicago on February 23, 1924. The following papers were presented: "A Better Use of Tests," by W. D. REEVE of Teachers College, Columbia University; "Memory and Marks in Mathematics," by ETHEL LUCCOCK, Northwestern High School, Detroit, Michigan; "The Laboratory Method in the Class Room," by CHARLES STONE, University of Chicago High School. At the banquet in the evening, with Professor H. E. SLAUGHT presiding, the following papers were read: "Reliability of Teachers' Marks," by RALEIGH SCHORLING, University of Michigan High School, Ann Arbor, Michigan; "Teaching Pupils the Conscious Use of the Technique of Thinking," by ELSIE P. JOHNSON, Oak Park High School, Oak Park, Illinois. The National Council of Teachers of Mathematics is a federation of secondary mathematics associations whose official organ is *The Mathematics Teacher*. The membership of this organization is between 3000 and 4000.

The International Mathematical Congress.

The International Mathematical Congress will be held at Toronto, Canada, August 11th to 16th, 1924, under the auspices of the Royal Canadian Institute and the University of Toronto.

This is the first meeting of the Congress to be held on the American continent.

Special prominence will be given at this meeting to the engineering and other practical applications of mathematics, and contributions have been invited from mathematical physicists and engineers engaged in mathematical investigations of engineering problems.

The proceedings of the Congress will be printed and it is hoped that they may form a complete contemporary account of the pure and applied mathematical sciences.

Seventy scientific institutions on the American continent, and ninety in Europe and elsewhere have arranged to send one or more delegates to the Congress.

The Congress will meet in the following sections:

Section *I:* Algebra, theory of numbers, analysis.
Section *II:* Geometry.
Section *III:* (*a*) Mechanics, mathematical physics.
(*b*) Astronomy, geophysics.
Section *IV:* (*a*) Electrical, mechanical, civil and mining engineering.
(*b*) Aeronautics, naval architecture, ballistics, radiotelegraphy.
Section *V:* Statistics, actuarial science, economics.
Section *VI:* History, philosophy, didactics.

This arrangement of sections is designed to afford, in the sphere of applied mathematics, full opportunity for consideration, not only of those questions in which interest is purely scientific, but also of practical problems of engineering, the solution of which contributes directly to the cause of progress.

Additional information may be obtained from Professor J. K. SYNGE, Royal Canadian Institute, Toronto, Canada.

At the conclusion of the Congress many of the European delegates will join the members of the British Association for the Advancement of Science in an excursion across Canada to Vancouver on the Pacific coast.

The British Association will hold its 92d annual meeting in Toronto from August 6th to August 13th.

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to D. C. GILLESPIE, Cayuga Heights, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

The following are dates of Section meetings of the Association in 1923 (unless otherwise specified):

ILLINOIS, Elgin, May 2–3

IOWA, Iowa State College, Ames, May 2–3

KANSAS, Topeka, February 2

KENTUCKY, Center College, April

MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA Annapolis, December 8, 1923

MICHIGAN, Ann Arbor, April 3

MINNESOTA, Hamline University, St. Paul, May 24

MISSOURI, Kansas City, November or December

OHIO, Ohio State University, Columbus, April 4–5

ROCKY MOUNTAIN, Laramie, April, 1925

SOUTHEASTERN, University of Georgia, Athens March 7–8

TEXAS, San Antonio, November 29–30

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of

America the sum of .... Dollars,

to be known as the..

....Fund, and to be used

for[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witnesses: Signature..

....

....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924

NUMBER 7, SEPTEMBER

PUBLISHED BY THE ASSOCIATION,
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others

ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION.

The eighth annual meeting of the Rocky Mountain section was held at the Steel Works Y. M. C. A., Pueblo, Colorado, on March 28 and 29. There were sixteen present, including the following eight members of the Association: I. M. DeLong, Philip Fitch, G. W. Gorrell, G. H. Light, S. L. Macdonald, J. Q. McNatt, H. E. Russell, C. H. Sisam. The section voted to hold the next meeting at the University of Wyoming. The following officers were elected: J. C. Fitterer, chairman; S. L. Macdonald, vice-chairman; Philip Fitch, secretary; G. H. Light, treasurer.

On Friday evening Mr. F. E. Parks, Manager of the Steel Works, delivered an address of welcome. He pointed out the advantages to all concerned of having the members of the section as guests of the company. Professor S. L. Macdonald responded to this address in a fitting manner, assuring Mr. Parks that the company's problems were also those of the section, and expressed the members' appreciation of the company's generous hospitality. An organ recital, followed by an address by Mr. D. K. Dunton, concluded the evening session. On Saturday morning the members visited the Steel Plant, Mr. Louis Deesz of the company officiating as guide.

The following nine papers were read:

(1) "Report of the Cincinnati meeting" by Professor H. E. Russell.

(2) "The undergraduate mathematics club" by Professor S. L. Macdonald.

(3) "To compute the radius of the circle inscribed in the area bounded by the arcs of three mutually tangent circles" by Mr. J. Q. McNatt.

(4) "Misleading definitions of 'f' in the elementary theory for finding the envelope of $f(x, y, c) = 0$" by Professor I. M. DeLong.

(5) "A problem in probability" by Professor G. W. Gorrell.

(6) "Pedal curves and related envelopes" by Mr. Philip Fitch.

(7) "On curves whose first polars have a rectilinear component" by Professor C. H. Sisam.

(8) "Times of rising and setting of the planets" by Dean H. A. Howe.

(9) "Magic squares of the first nine orders" by Professor F. H. Loud.

In the absence of the authors, the papers by Dean Howe and Professor Loud were read respectively by Professors Russell and Sisam.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. In his report, Professor Russell commented on the attendance and interest of the Cincinnati meeting and dwelt briefly on the salient features of some of the more interesting papers.

2. Professor Macdonald's paper dealt with the advisability of having undergraduate mathematics clubs and showed how interest in mathematics was stimulated by such organizations. The meeting of pupils with a common interest often reveals qualities in them that would otherwise be dormant.

3. The paper by Mr. McNatt demonstrated an interesting method of computing the radius of the circle inscribed in the area bounded by three mutually tangent circles, in terms of the radii of these circles.

4. Professor DeLong pointed out that there were definitions of "f," as applied to the elementary theory for finding the envelope of $f(x, y, c) = 0$, that were misleading and remarked that some authors of works on calculus had made no attempt to clarify the subject.

5. Professor Gorrell compared methods of attacking problems in probability and discussed the advantages of having more than one viewpoint of a problem.

6. In his paper, Mr. Fitch demonstrated a short method for finding the equation of a pedal curve and proved the following properties: (a) The pedal of a given curve with respect to a fixed point is the envelope of a family of circles described on the radii vectores from the fixed point to the given curve as diameters. (b) The fixed point is a conjugate point of this envelope. (c) The caustic of a given curve with respect to a fixed point is a translation of the evolute of its pedal for that point.

7. In this paper, Professor Sisam determined the equations of the noncomposite algebraic plane curves which have the property that every line through a fixed point is a component of a first polar with respect to the curve.

8. Dean Howe's paper dealt with the computation of the approximate times of rising and setting of the planets. The object of the method set forth is to render it possible for a student in elementary descriptive astronomy, by using data easily taken from the American Ephemeris, to obtain the time of rising or setting of any planet on any day of the year, with an error not exceeding three minutes. No logarithmic or trigonometric work is needed. The place for which the computation is to be made is supposed to be in the northern hemisphere, and to have a latitude no greater than 60°. Denver was chosen to illustrate the process.

From the Greenwich time when the planet crosses the Greenwich meridian on the given date, the Denver time when it crosses the Denver meridian is obtained by a simple interpolation. Then the problem is quickly finished by using the tables for sunrise and sunset at the end of the Ephemeris, making allowance for the fact that these tables are for the upper limb of the sun, instead of the center. In those infrequent cases where a planet's distance from the celestial equator exceeds twenty-three and a half degrees, an extrapolation is necessary.

9. In this paper, Professor Loud derived several new and interesting ways for forming magic squares.

PHILIP FITCH, *Secretary.*

NINTH ANNUAL MEETING OF THE OHIO SECTION.

The ninth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on April 4, 1924, in connection with the meetings of the Ohio College Association and allied societies. Chairman W. E. Anderson presided, being relieved by V. B. Caris for an interval.

Sixty persons were registered, the following thirty-three being members of the Association:

R. B. Allen, W. E. Anderson, G. N. Armstrong, C. L. Arnold, Grace M. Bareis, C. T. Bumer, W. D. Cairns, V. B. Caris, E. H. Clarke, H. L. Coar, O. L. Dustheimer, B. C. Glover, G. P. Harmount, H. W. Kuhn, H. B. Lemon, Anna D. Lewis, E. S. Manson, C. N. Moore, C. C. Morris, M. A. Nordgaard, S. E. Rasor, P. L. Rea, C. N. Reynolds, Hortense Rickard, W. G. Simon, S. A. Singer, K. D. Swartzel, M. O. Tripp, R. B. Wildermuth, F. B. Wiley, J. H. Weaver, C. O. Williamson, B. F. Yanney.

At the business session the secretary reported a membership of eighty-five and nine institutional members as against eighty-five and ten, respectively, last year. Officers elected for this year are: Chairman, Professor Harris Hancock, University of Cincinnati; Secretary-Treasurer, Professor G. N. Armstrong, Ohio Wesleyan University; Third member of the executive committee, Professor R. B. Wildermuth, Capital University. Professor R. B. Allen was reëlected to serve on the program committee, his term of service being three years. A collection of $12.00 was taken at the meeting for the immediate uses of the Section. Professor R. B. Wildermuth was named to fill the place of Professor A. D. Pitcher, deceased since the last meeting, upon the committee to issue the usual letters to the high schools of the state on "Why Elect Mathematics?"

The Section dinner, with about fifty present, served in one of the dining rooms of Campbell Hall, was very successful. The evening session was held in the same room and was devoted to the last two numbers on the program.

The following ten papers were presented at the two sessions:

(1) Chairman's Address: "Some methods of creating and maintaining interest in mathematics" by Professor W. E. Anderson, Miami University.

(2) "Business statistics" by Mr. Edmond E. Lincoln, Ph.D., Chief Statistician, The Western Electric Company, New York (by invitation).

(3) "Related variables with coefficient of correlation equal to zero" by Professor C. N. Moore, University of Cincinnati.

Business and Intermission.

(4) "Some problems in the teaching of the mathematics of investment" by Professor C. N. Reynolds, Jr., University of West Virginia.

(5) "Some notes on mathematics in the investment houses" by Professor G. N. Armstrong, Ohio Wesleyan University.

(6) "Some dual theorems of the quadratic function" by Professor C. L. Arnold, Ohio State University.

(7) "An experiment in sectionizing freshman mathematics" by Professor Hazel Schoonmaker, The Western College for Women.

(8) "The use of prognostic tests in sectionizing freshmen in mathematics" by Professor M. A. Nordgaard, Antioch College.

(9) Reports of the Yanney Committee on the Mathematical Situation in Ohio, continued from 1923. Professor C. N. Moore, University of Cincinnati, General Chairman.

(10) Discussion: "What should be done with, for, and to the freshmen having one unit of algebra?"

Professor Schoonmaker being absent due to an accident, her paper was read by Professor Anna D. Lewis.

Abstracts of the papers follow below, the number corresponding to the numbers in the list of titles:

1. Professor Anderson pointed out that the teacher of college mathematics is confronted with the problem of creating interest in case that interest has not been created in high school and of maintaining that interest in case it has already been created. Among factors which enter are the teacher, the subject matter, and the student himself. The preparation of the teacher should include a thorough course in physics, and a knowledge of astronomy, in order that he may the more fully appreciate the practical application of mathematics and thus be able to make it a more real, live subject to the student. He should likewise appreciate the application to the various fields of engineering.

Not only does the content of the course mean much, but in the freshman course especially, the sequence of subjects and the methods of their presentation as well as the amount of time devoted to each play an important rôle. Honor problems should stimulate the better students to greater endeavor with a consequent increase in interest. Honor courses, to which only the students having attained a high standing in mathematics are eligible, present a fine opportunity for the teacher to do work along some lines in which he is particularly interested.

2. (Owing to the widening interest of mathematicians in both the theory and the practice of statistics in business, Dr. Lincoln, who did his college work in Ohio and in Oxford as a Rhodes Scholar from Ohio, was invited to appear on the program.)

The need for study and practical application of business statistics was clearly demonstrated in the years immediately following the World War. Until 1920 prices had been rising for about 25 years and it was comparatively easy for concerns with indifferent management to make money. The recent industrial collapse, and the period of price uncertainty which followed, have caused leaders in business to think more seriously on economic questions. Hence the profession of "business economist" is gradually developing.

The "business economist" or "statistician," in order to do his work effectively, must master the facts of his business and must build upon the work of the accountant. He must take a comprehensive view of the various functions and departments of the organization and study their relations with a view to making

higher profits for the company or improving the service at the same or lower costs. Internal performance must be related to external business conditions. The study of one is ineffective without the study of the other. The principal job of the economist, therefore, is that of forecasting the future of business and the future of his own industry. This work necessitates the use of refined statistical methods combined with a wide economic background and the capacity to "think straight and see far." Generalizations are frequently unsafe; each industry must be studied in its peculiar relations to the general business movement. Prices, wages, interest rates and sales performance must always be thought of in terms of the future based on the experience of the past.

It is extremely difficult to apply definite quantitative measurements to the work of the business economist. Much that he does must necessarily be of a "developmental" sort and much of his work is in the nature of "insurance." Sometimes he may be able to save money directly, through giving sound advice at the proper time. More commonly, however, his chief function is to act as a balance wheel in "steadying" the administration policies in order to avoid losses through unscientific management. His job is to help level the peaks and fill up the valleys in business. Perhaps the test of the value of his service is the non-fulfillment of his prophecies, due to the fact that the concern which he advises by taking thought avoids the disasters which have been foreseen.

This general statement was followed by specific reference to the type of work which is being attempted in the General Statistical Department of the Western Electric Company.

3. In the usual discussions of the coefficient of correlation, this quantity is expressed by means of a formula in terms of n observed pairs of values of two variables x and y. In order to penetrate more deeply into the connection between the value of r and the degree of relationship between the variables, it is desirable to derive a formula which expresses r in terms of the underlying variables on which x and y themselves depend. If we assume that these variables are independent and that x and y are approximately linear functions of them, we obtain the formula $r = \Sigma a_{1n}a_{2n}s_n^2/\sqrt{\Sigma a_{1n}^2 s_n^2 \Sigma a_{2n}^2 s_n^2}$, where the a's are the coefficients in the linear expressions and the s's are the standard deviations of the independent variables. This formula was first given by Professor C. N. Moore in a paper in volume XLII of *Science*. In the present paper he makes use of the formula to set up certain simple examples of closely related variables having a coefficient of correlation equal to zero. In all the cases considered the regression is either exactly linear or very approximately so, and hence the ordinary interpretation of the correlation coefficient would lead to the conclusion that the variables x and y are unrelated.

4. In this discussion, Professor Reynolds first presented (in a more elaborate form) the suggestions he had published in the MONTHLY (*1922*, 122). He then suggested an arrangement of the elementary formulæ for insurance premiums, in two-dimensional array, according to the insurance protection afforded by the policy on the one hand, and according to the payment plan on the other. With

this arrangement of formulæ as a basis he suggested that students be instructed to break all premium problems up into problems in variation, the premium varying directly with a factor determined by the protection afforded by the policy and inversely with a factor determined by the payment plan adopted.

5. The purpose of Professor Armstrong's paper was to give some notion of how much actual use the investment houses make of the mathematics of finance in their ordinary work. The material came from illustrations accumulated during the past few years, and from recent inquiries directed to representative financial institutions. Requests were made for illustrations of actual practice in making up bids and selling prices, in deciding upon conditions for calling and refunding securities, in handling sinking funds, and in treating other problems of a mathematical nature. While these inquiries were answered carefully, very few actual illustrations were supplied.

Two examples of handling bond issues were presented in some detail. Conclusions were that mathematical theories find comparatively small employment in the routine of investment houses, their work being mostly confined to the use of bond tables; that the order in which merit would probably be rewarded is, ability to sell bonds, ability to forecast business conditions and the trend of the money market, ability to use the mathematics of finance.

The United States Treasury department furnished interesting material connected with government bonds, income tax, and "bonus bill" investigations.

6. The special quadratic $y = x^2 + bx + c$ denotes a unit parabola in (x, y) and a line in (b, c). If a point (x, y) traverse the parabola $y = px^2 + hx + k$, the corresponding lines envelop the parabola $c - k = [1/4(1 - p)](b - h)^2$. Reciprocally, if a point (b, c) traverse the parabola $c - k = [1/4(1 - p)](b - h)^2$, the corresponding parabolas envelop $y = px^2 + hx + k$.

In this correspondence *line* and *parabola* are duals while *point* corresponds to *point*. Thus the points of tangency mutually correspond. If $y = 0$, then $c = \frac{1}{4}b^2$ is the important special case of the discriminant.

7. Miss Schoonmaker's paper consisted first of a survey of the department of mathematics at the Western College for Women since 1915–16, the last year in which there was only one full-time instructor in the department. Next was a description of a method of sectionizing which was being tried out, and a statement of the results already obtained.

8. Professor Nordgaard reviewed briefly his experience with the grouping of freshmen on a scholarship basis at the University of Maine in 1915 and at Grinnell College in 1918–21; in the former the classification took place at the end of the first semester, and in the latter at the end of six weeks. He then described the plan he has used at Antioch College the last two years: On registration day every freshman takes a prognostic test, consisting of 25 to 30 short questions in arithmetic, algebra, and geometry, to be answered in an hour; based on the results, five or six groups are formed, each sufficiently homogeneous to have its own mode of presentation and to progress at its own pace. The schedule committee holds open for the mathematics department a block of class periods inside of which

students may shift without having a conflict with other studies. It was found necessary to shift only 5 per cent. during the semester.

To investigate the relative reliability of the prognostic test, all freshmen took the same class examination on the assignment covered by all the sections; between placement tests and uniform class tests the coefficient of correlation was .531 for the first year and .534 for the second year. A lower correlation obtained between the semester's class work and class tests taken at the end of five weeks.

Professor Nordgaard also brought out the correlation obtaining between the scores of the placement test and the semester ranks, the placement scores and the ratings of the Thurston intelligence tests, the Thurston ratings and the semester ranks. As a means of reliability he used the scale worked out by A. R. Crathorne in Chapter X of the *Report by the National Committee on Mathematical Requirements.*

9. The Section has six committees on the mathematical situation in Ohio. Four of these reported at length in 1923. Professor E. H. Clarke, chairman of the committee on college entrance requirement, presented some new material.

10. This subject for discussion was suggested by the new conditions presented due to lowering of entrance requirements in mathematics in many Ohio colleges, making it necessary for those institutions to adopt a policy as regards deficiencies and elementary mathematics. Professor Yanney, who proposed the subject, led the discussion. He opposes the colleges burdening themselves with courses for making up deficiencies. Professor Wiley is optimistic about further "experimenting" on the freshmen. At Denison they have been using student assistants for deficients with success. Professor Cairns intimated that Oberlin College might soon eliminate all applicants deficient in algebra. Professor Rasor suggested that mathematics teachers should not be too backward in "advertising" their studies. Professor Swartzel took an optimistic view of present conditions. Professor Wildermuth advocated more acquaintance with the principles of teaching if we wish to impress the leaders of education. Others contributed interesting suggestions. Miss Gertrude Silver, teacher in the North high school of Columbus, made an effective plea for the college teachers to study more sympathetically the problems of the high school pupils arising out of the high pressure of modern life, especially in the cities.

G. N. ARMSTRONG, *Secretary-Treasurer.*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION.

The third annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of Georgia, at Athens, Georgia, on March 7–8, 1924. On Friday night a special dinner was given in honor of Professor H. E. Slaught with Professor R. P. Stephens presiding as toastmaster. There were about fifty present at this dinner including representatives of several departments of the University of Georgia as well as the members of the Association. All were delighted to honor the "Daddy of the Association" in this way. The following seventeen members were present: Eli Allison, E. A. Bailey, D. F. Barrow, S. M. Barton, T. R. Eagles, Floyd Field, Tomlinson Fort, Miss Leslie Gaylord, J. C. Hinton, J. W. Lasley, Jr., A. V. Martin, J. F. Messick, A. B. Morton, W. T. Peed, W. W. Rankin, Jr., H. E. Slaught, R. P. Stephens.

The following officers were elected for 1924–1925: Chairman, TOMLINSON FORT, University of Alabama; Vice-chairman, S. M. BARTON, University of the South; Secretary-treasurer, W. W. RANKIN, Jr., Agnes Scott College. Professor T. R. EAGLES and Miss LESLIE GAYLORD were appointed to act with the sec.-treas. as a Program Committee. The next meeting will be held at Birmingham, Alabama.

The following papers were presented:

(1) "The Association, its ideals, accomplishments and prospects" by Professor H. E. SLAUGHT.

(2) "Imaginary and infinite elements in undergraduate and high school teaching" by Professor TOMLINSON FORT.

(3) "Early teaching of mathematics at the University of Georgia" by Professor R. P. STEPHENS.

(4) "Mathematics and other sciences" by Professor H. E. SLAUGHT.

(5) "Some contacts of projective geometry with elementary mathematics" by Professor J. W. LASLEY, Jr.

(6) "The cultural value of mathematics" by Professor W. W. RANKIN, Jr.

(7) Discussion: "What can the Association do to aid individual departments" led by Professor TOMLINSON FORT.

Abstracts of the papers follow below, the numbers corresponding to the numbers given in the list of titles:

1. At the banquet given in his honor on Friday evening, Professor Slaught outlined the development of mathematics in America since the opening of Johns Hopkins University and showed how important a part had been played by the American Mathematical Society. He then set forth the rôle of the Mathematical Association of America and showed how much it had contributed in eight short years of its existence. After outlining the type of work done by each of these organizations, he called attention to the fine spirit of coöperation existing between the two, each in its own field carrying forward the development of mathematics and the teaching of mathematics.

He called upon all who are devotees of mathematics loyally to support both organizations in every way possible, calling attention to the strenuous condition

at the present time owing to the enormous increase in the cost of printing. He urged that so far as possible every one teaching mathematics in the colleges and universities should be a member of both the Society and the Association, especially as the combined annual dues in the two organizations are not as great as the dues in many other national scientific societies. Professor Slaught made a strong appeal and aroused the entire group to a finer and deeper interest in mathematics.

2. The discussion of Prof. Fort followed the lines indicated in a note published in this MONTHLY (*1923*, 255–256).

3. According to Professor Stephens, great emphasis was put upon mathematics in the early curriculum—one fourth of the students' entire time was supposed to be given to the courses in mathematics. The first entrance requirement in mathematics was arithmetic through the "rule of three" which was imposed about 1815. This was gradually raised till 1857 when all of arithmetic, elementary algebra through quadratics, and three books of Legendre were required. The first college course adopted in 1801 embraced arithmetic, algebra, trigonometry with many applications to surveying and navigation, and astronomy. The scope of the requirement in mathematics was gradually enlarged till in 1850 both analytic geometry and calculus were included in the required curriculum.

4. Professor Slaught spoke briefly but straight to the point. He pointed out the fundamental relation which exists between mathematics and such sciences as biology, chemistry, physics, sociology and others. He further suggested that a subject was only entitled to call itself a science when the knowledge of this subject could be reduced to mathematical formulæ.

5. Practically all of the advanced students of mathematics, Professor Lasley maintained, are training to be teachers of the subject. The various fields of elementary mathematics were taken up in turn and in detail some of the topics on which projective geometry throws light were pointed out. The analytical phase of the subject was stressed as a necessary supplement to the geometric. Mr. Lasley claimed for that subject an integral part in the training for teachers of elementary mathematics. It has been too long regarded as merely one of the avenues to graduate mathematics.

6. Professor Rankin defined culture as follows: Culture is an intelligent interest in the past, present, and future achievements of man. An intelligent interest in an achievement was explained to mean some knowledge of the history of the achievement, some knowledge of the underlying principles and laws which have assisted in making the achievement, and some knowledge of the value of the achievement. The definition of mathematics used was that of Benjamin Peirce —"Mathematics is the science which draws necessary conclusions." Professor Rankin based the claim for cultural value of mathematics on these two definitions and an intelligent interest in the following widely diversified list of achievements of man: Number system, language, process of thinking, art, architecture, music, protestant reformation, transportation and communication, banking and commerce, medicine, political economy and sociology. He showed that each of these achievements was in some way related to mathematics.

W. W. RANKIN, Jr., *Secretary-Treasurer.*

INTEGRAL INEQUALITIES WITH APPLICATIONS TO THE CALCULUS OF VARIATIONS.

By OTTO DUNKEL, Washington University.

1. Introduction. Two sets of integral inequalities will be developed in this paper which apply directly to the solution of certain problems of the calculus of variations. These problems belong, as might be expected, to the simplest cases which present themselves in that study. The examples in the calculus of variations which are considered are not the only ones which may be treated by these inequalities. They include, however, the two problems which have been treated before by the writer. When auxiliary conditions are introduced the inequalities do not appear to suffice except in certain special types to which the second set of inequalities apply. A more general case with auxiliary conditions is considered at the end of the paper. The method of treatment is uniform throughout the discussion. The first set of inequalities may be readily thrown into a form which exhibits the relation between the mean values of functions as expressed by definite integrals. The second set of inequalities give a necessary and sufficient condition for the linear independence of a set of functions of one variable. If the integrals are replaced by finite sums, corresponding inequalities and theorems may be obtained by the same methods used here.

All of the functions which are considered are assumed to be single-valued and continuous in a single independent variable in the intervals considered.

2. The First Set of Integral Inequalities. We shall consider first a known inequality theorem and derive from it a sequence of inequalities. The proof which will be given will show that the usual statement of this theorem, which follows, requires more than is necessary to establish the inequality.

THEOREM. *If $f_1(x)$ and $f_2(x)$ are two functions of x, neither of which is a constant, such that if either one of them experiences an increase in any finite interval included in the interval of integration the other experiences a decrease, and conversely, then*

$$\int_a^b f_1(x)dx \int_a^b f_2(x)dx > (b-a)\int_a^b f_1(x)f_2(x)dx. \tag{1}$$

If on the other hand the two functions experience increments in the same intervals and decrements in the same intervals, the inequality sign above is reversed.[1]

[1] The theorem as stated may be deduced directly from the identity

$$\left(\sum_1^n a_i\right)\left(\sum_1^n b_j\right) = n\sum_1^n a_ib_i + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_i - a_j)(b_j - b_i),$$

which gives

$$\int_a^b f_1(x)dx\int_a^b f_2(x)dx = (b-a)\int_a^b f_1(x)f_2(x)dx + \frac{1}{2}\int_a^b\int_a^b [f_1(x) - f_1(y)][f_2(y) - f_2(x)]dxdy.$$

From this result it will be seen that a proof without the use of the algebraic identity may be ob-

PROOF. Since $f_1(x)$ is continuous, we may set

$$\int_a^b f_1(x)dx = f_1(c)(b-a), \qquad a < c < b, \tag{2}$$

and then

$$f_1(x) = f_1(c) + \eta_1(x), \qquad f_2(x) = f_2(c) - \eta_2(x). \tag{3}$$

From (2) and (3) result

$$\int_a^b \eta_1(x)dx = 0, \qquad \int_a^b \eta_1(x)f_2(x)dx = -\int_a^b \eta_1(x)\eta_2(x)dx. \tag{4}$$

It now follows from (2), (3) and (4) that

$$\begin{aligned}\int_a^b f_1(x)dx \int_a^b f_2(x)dx &= (b-a)\int_a^b f_1(c)f_2(x)dx,\\ &= (b-a)\int_a^b f_1(x)f_2(x)dx + (b-a)\int_a^b \eta_1(x)\eta_2(x)dx. \qquad (5)\end{aligned}$$

By hypothesis $\eta_1(x)\eta_2(x)$ can never be negative. Since $f_1(x)$ is not a constant, $\eta_1(x)$ must be different from zero for some value of x, say $\bar{x}$, and by hypothesis $\eta_2(x)$ must also be different from zero and of the same sign. Hence $\eta_1(x)\eta_2(x)$ must be greater than zero in an interval about $\bar{x}$, since this product is continuous. It follows then that the last integral to the right in (5) is surely greater than zero, and therefore (1) must be true.

The second part of the theorem follows in the same way from (5), observing in this case that $\eta_1(x)\eta_2(x)$ is never positive, and that it must be less than zero in some interval within the interval of integration.

REMARKS. In this proof we need only the requirement that for one determination of c from either $f_1(x)$ or $f_2(x)$ the product $\eta_1(x)\eta_2(x)$ shall never be negative, in the case of the first part of the theorem; and for at least one value of x it shall be greater than zero. It is possible for two functions to satisfy this demand without satisfying the original requirement. Thus $f_1(x) = 1 - x$, $f_2(x) = 16x - x^2$, in the interval $0 \leqq x \leqq 9$, is such an example, where $c = 4.5$. Here $\eta_1(x)\eta_2(x) > 0$ if $x \neq 4.5$. From 0 to 8, $f_2(x)$ increases and $f_1(x)$ decreases, while from 8 to 9 both decrease.

Corresponding theorems in regard to double integrals may be easily stated and proved in the same way, where $f_1(x)$ and $f_2(x)$ are replaced by functions of two variables.

3. Special Inequalities of the First Set. From the above theorem follow special cases of the inequalities which will be used later and which we proceed to state.

---

tained by starting with the double integral on the right and proceeding as in Goursat-Hedrick's *Mathematical Analysis*, vol. 1, page 257. See also the proof of the special case where $b_i = a_i^{-1}$ in this journal [*1924*, 148–150].

THEOREM. *If $f_1(x), f_2(x), \cdots, f_n(x)$ are never negative and experience simultaneously increments, and decrements, then*

$$\int_a^b f_1(x)dx \int_a^b f_2(x)dx \cdots \int_a^b f_n(x)dx < (b-a)^{n-1}\int_a^b f_1(x)f_2(x)\cdots f_n(x)dx. \qquad (6)$$

In particular, if they are all equal,

$$\left[\int_a^b f(x)dx\right]^n < (b-a)^{n-1}\int_a^b [f(x)]^n dx. \qquad (7)$$

This follows at once from the second part of (1) since the product of any pair of these functions varies in the same manner as any one of them, and hence such a product may be used as one of the functions in (1).

REMARKS. If n is even, the requirement that $f(x)$ shall never be negative in (7) may be dropped. For the second part of (1) is true if $f_1(x) = f_2(x) = f(x)$ without any such restriction upon $f(x)$. Thus (7) is true in this case if $n = 2$. If n is even and more than 2 we have merely to combine the $n/2$ relations (7) in which $n = 2$ by a repeated application of the second part of (1).

THEOREM. *If $f(x) > 0$ and n is a positive integer,*

$$\left[\int_a^b \frac{dx}{f(x)}\right]^n \int_a^b [f(x)]^n dx > (b-a)^{n+1}. \qquad (8)$$

PROOF. Since $f(x) > 0$, the two functions $[f(x)]^n$ and $1/f(x)$ satisfy the conditions for the first part of (1) without further restrictions upon $f(x)$. Hence

$$\int_a^b [f(x)]^n dx \int_a^b \frac{dx}{f(x)} > (b-a)\int_a^b [f(x)]^{n-1}dx. \qquad (9)$$

If we multiply both sides by $\int_a^b [f(x)]^{-1}dx$ and apply to the resulting right side the above inequality (9) in which n is replaced by $n-1$, and continue in this way, we obtain finally the inequality (8).

REMARKS. In the inequalities (7) and (8) it has been assumed that $f(x)$ is not a constant. If it is a constant, then the equality sign must be used instead of the inequality sign.

In these results the exponent n is a positive integer, but these inequalities are also true within certain limitations when the exponent is any real number. This will be shown by considering next a general theorem.

4. The Generalized Inequality Theorem. Let $F(t)$ be a function of t which has a second derivative $F''(t)$ which does not change sign and which vanishes, if at all, only at isolated points of the interval $t_1 \leqq t \leqq t_2$, where $t_1 \leqq f(x) \leqq t_2$, $a \leqq x \leqq b$. We may state the following:

THEOREM.

$$\frac{\int_a^b F[f(x)]dx}{b-a} \gtreqless F\left[\frac{\int_a^b f(x)dx}{b-a}\right] \text{ according as } \begin{matrix} F''(t) \geqq 0, \\ \text{or} \\ F''(t) \leqq 0. \end{matrix} \qquad (10)$$

PROOF. Setting, as before in (2) and (3), $f(x) = m + \eta$, where for brevity we write $f(c) = m$ and $\eta(x) = \eta$, we have

$$F[f(x)] = F(m + \eta) = F(m) + \eta F'(m) + \frac{\eta^2}{2} F''[m + \theta\eta], \qquad 0 < \theta < 1. \quad (11)$$

If $\eta \neq 0$ for any value of x, then $F''[m + \theta\eta] \neq 0$; for if it is zero, then the function of t, $F[m + t] - F(m) - tF'(m)$ is zero for $t = 0$ and $t = \eta$. Hence $F'[m + t] - F'(m)$ must vanish for some value of t, say $\bar{t}$, between 0 and η. Thus $F'[m + \bar{t}] = F'(m)$ and this cannot be true since $F'[m + t]$ always increases or always decreases in the interval for t.

Integrating (11), and noting that the integral of $\eta\, F'(m)$ is zero, we have

$$\int_a^b F[f(x)]dx = (b - a)F(m) + \frac{1}{2}\int_a^b \eta^2 F''[m + \theta\eta]dx. \quad (12)$$

Since $\eta(x)$ is not identically zero, the integrand on the right is greater than zero for some values of x if $F''(t) \geqq 0$, less than zero if $F''(t) \leqq 0$. Hence the theorem follows.

REMARKS. This result may be put in a symmetric form. Let $\varphi(t)$ and $\psi(t)$ be functions of t which have first derivatives $\varphi'(t)$ and $\psi'(t)$ which do not change sign and which vanish at only isolated points. It will be convenient to suppose that $\psi'(t)$ is never negative. Denote the inverse of the two functions by $\varphi^{-1}(t)$ and $\psi^{-1}(t)$. Replace $f(x)$ in (10) by $\varphi[f(x)]$ and $F(t)$ by $\psi[\varphi^{-1}(t)]$. Then (10) becomes after a slight change

$$\psi^{-1}\left[\frac{\int_a^b \psi[f(x)]dx}{b - a}\right] \gtrless \varphi^{-1}\left[\frac{\int_a^b \varphi[f(x)]dx}{b - a}\right], \quad (10')$$

where the upper sign is used if the second derivative of $\psi[\varphi^{-1}(t)]$ is never negative, and the lower sign if it is never positive.[1]

It is now possible to extend the results in (7) and (8) to any real exponent. If in (10) we set $F(t) = t^\alpha$, $\alpha \neq 1$, then

$$(b - a)^{\alpha-1}\int_a^b [f(x)]^\alpha dx > \left[\int_a^b f(x)dx\right]^\alpha, \qquad \alpha = 2 + \frac{2j}{2i + 1}, \quad (13)$$

where i and j are integers greater than or equal to zero and where $f(x)$ may change sign.

If $f(x) > 0$, then

$$\begin{aligned}(b - a)^{\alpha-1}\int_a^b [f(x)]^\alpha dx &> \left[\int_a^b f(x)dx\right]^\alpha, \qquad \alpha < 0 \quad\text{or}\quad 1 < \alpha,\\ &< \left[\int_a^b f(x)dx\right]^\alpha, \qquad 0 < \alpha < 1.\end{aligned} \quad (14)$$

[1] This result is analogous to the theorem contained in the paper by the writer " Generalized Geometric Means and Algebraic Equations," *Annals of Mathematics*, 2d ser., vol. 11, 1909, page 26.

Except in the case of $\alpha < 0$, a simple consideration will show that (14) is true even when $f(x)$ is allowed to vanish in the interval of integration, but not becoming negative. [1]

If we replace $f(x)$ by $1/f(x)$ and α by $-\alpha$, the inequalities (14) may be written

$$\left[\int_a^b \frac{dx}{f(x)}\right]^\alpha \int_a^b [f(x)]^\alpha dx \begin{array}{ll} > (b-a)^{\alpha+1}, & \alpha > 0, \text{ or } \alpha < -1, \\ < (b-a)^{\alpha+1}, & -1 < \alpha < 0, \end{array} \tag{14'}$$

where $f(x) > 0$.

5. The Second Set of Integral Inequalities. This second set of integral inequalities is of a more complicated form, but it will be shown that it includes special cases of the first set. Here n functions, $f_1(x), f_2(x), \cdots, f_n(x)$, will be considered which are linearly independent in the interval $a \leqq x \leqq b$. This means that there do not exist n constants, $A_1, A_2, \cdots, A_n$, not all zero, such that

$$\sum_1^n A_i f_i(x) = 0, \qquad a \leqq x \leqq b.$$

If, however, there does exist such a set of constants, the n functions are said to be linearly dependent in the given interval. For brevity of writing we shall set

$$\int_a^b f_i(x) f_j(x) dx = (f_i f_j) = (f_j f_i),$$

and

$$\begin{vmatrix} (f_1f_1) & (f_1f_2) & \cdots & (f_1f_n) \\ (f_2f_1) & (f_2f_2) & \cdots & (f_2f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (f_nf_1) & (f_nf_2) & \cdots & (f_nf_n) \end{vmatrix} = |f_1 f_2 \cdots f_n|. \tag{15}$$

We shall prove the following

THEOREM. *If the n functions are linearly independent in the given interval, the determinant* (15) *is greater than zero. If they are linearly dependent in the interval, the determinant is zero.*

PROOF. The second part of the theorem is obvious. For in this case either one column consists of zeros, or is a linear combination of the other columns. We turn now to the first part of the theorem. Since the n functions are linearly independent, any smaller set selected from them must also be linearly independent. Suppose then that the theorem is true for $f_i(x), f_{i+1}(x), \cdots, f_n(x)$; it

[1] Certain known inequalities are contained in the above as also in (10′). For example, if $a = 0$, $b = 1$, $f(x) = x^q$, $\alpha = p/q$, $p > q > 0$, it follows from (14) that

$$(1+q)^{1/q} > (1+p)^{1/p}.$$

will then be possible to determine $n - i + 1$ constants, $A_i, A_{i+1}, \cdots, A_n$, such that

$$\begin{aligned}(f_i f_{i-1}) &= A_i(f_i f_i) + A_{i+1}(f_i f_{i+1}) + \cdots + A_n(f_i f_n),\\ (f_{i+1} f_{i-1}) &= A_i(f_{i+1} f_i) + A_{i+1}(f_{i+1} f_{i+1}) + \cdots + A_n(f_{i+1} f_n),\\ &\cdots\cdots\cdots\\ (f_n f_{i-1}) &= A_i(f_n f_i) + A_{i+1}(f_n f_{i+1}) + \cdots + A_n(f_n f_n),\end{aligned} \tag{16}$$

since the determinant $|f_i f_{i+1} \cdots f_n|$ of the equations is greater than zero by hypothesis. Set

$$f_{i-1}(x) = A_i f_i(x) + A_{i+1} f_{i+1}(x) + \cdots + A_n f_n(x) + \eta_{i-1}(x). \tag{17}$$

The function $\eta_{i-1}(x)$ cannot be identically zero in the given interval; for if it were, then the $n - i + 2$ functions would be linearly dependent, as also the whole set of n functions. Multiplying (17) by $f_i(x), f_{i+1}(x), \cdots, f_n(x)$ in turn and integrating each result, we find by (16)

$$(\eta_{i-1} f_j) = 0, \quad j = i, \quad i + 1, \quad \cdots, \quad n. \tag{18}$$

Multiplying (17) by $\eta_{i-1}(x)$ and integrating, we obtain by use of (18)

$$(f_{i-1} \eta_{i-1}) = (\eta_{i-1} \eta_{i-1}). \tag{18'}$$

In the same manner by multiplying (17) by $f_{i-1}(x)$, integrating, and then using (18′), we obtain

$$(f_{i-1} f_{i-1}) - (\eta_{i-1} \eta_{i-1}) = A_i(f_{i-1} f_i) + A_{i+1}(f_{i-1} f_{i+1}) + \cdots + A_n(f_{i-1} f_n). \tag{19}$$

Now this equation with (16) gives a set of $n - i + 2$ equations which admit the solution $-1, A_i, A_{i+1}, \cdots, A_n$. Hence their determinant must be zero. This determinant breaks up into two parts, and we obtain the equation

$$|f_{i-1} f_i f_{i+1} \cdots f_n| = (\eta_{i-1} \eta_{i-1}) |f_i f_{i+1} \cdots f_n|. \tag{20}$$

The factor $(\eta_{i-1} \eta_{i-1})$ is greater than zero; also $|f_i f_{i+1} \cdots f_n|$ is greater than zero by hypothesis; and hence the determinant on the left is also greater than zero. If $i = n$, $|f_n| = (f_n f_n)$ and (20) becomes

$$|f_{n-1} f_n| = (\eta_{n-1} \eta_{n-1})(f_n f_n).$$

Since the n functions are linearly independent no one of them can be identically zero and hence $(f_n f_n) > 0$. Then the proof above shows that $|f_{n-1} f_n|$ is greater than zero, and so on; and we find at last that the determinant in (15) is greater than zero.

REMARKS. If $f(x)$ is not a constant, then $f(x)$ and 1 are linearly independent. Hence

$$\begin{vmatrix}(ff) & (1f)\\ (f1) & (11)\end{vmatrix} > 0, \quad \text{or} \quad (b-a)\int_a^b [f(x)]^2 dx > \left[\int_a^b f(x)dx\right]^2.$$

If $f(x) > 0$ and is not a constant, then $\sqrt{f(x)}$ and $1/\sqrt{f(x)}$ are linearly independent and we find in the same way

$$\int_a^b \frac{dx}{f(x)} \int_a^b f(x)dx > (b-a)^2.$$

Thus this second set of inequalities leads to special cases of the first set. By considering powers of $f(x)$, other inequalities may be obtained.

If both sides of (17) are squared and integrated, there results by use of (18)

$$(f_{i-1}f_{i-1}) - (\eta_{i-1}\eta_{i-1}) = \int_a^b \Big[\sum_i^n A_j f_j(x) \Big]^2 dx. \tag{19'}$$

The integral on the right is zero only if $A_i = A_{i+1} = \cdots = A_n = 0$, and then by (16) $(f_{i-1}f_j) = 0$, $j = i, i+1, \cdots, n$, and conversely. In this case $f_{i-1}(x)$ is said to be orthogonal to the functions with the following subscripts. Hence from (20) there follows

$$|f_{i-1}f_i f_{i+1} \cdots f_n| \leqq (f_{i-1}f_{i-1})\,|f_i f_{i+1} \cdots f_n|, \tag{20'}$$

where the equality sign is used if $f_{i-1}(x)$ is orthogonal to each of the other functions appearing in this relation. By a repeated application of (20′) there results the

THEOREM.
$$|f_1 f_2 \cdots f_n| \leqq (f_1 f_1)(f_2 f_2) \cdots (f_n f_n),$$

where equality occurs only if every pair of different functions is an orthogonal pair.

6. The Minimum of Certain Definite Integrals. The previous results will now be applied to the determination of the minimum of certain types of definite integrals.

CASE 1. Let y be a function of x which takes on the values 0 and y_2, respectively, at the ends of the interval $0 \leqq x \leqq x_2$, and which possesses a derivative y' which is greater than zero in this interval. Also let $\varphi(y)$ be a function of y which is greater than zero for all the values of y which are considered. It is desired to find that function y of x which renders

$$\int_0^{x_2} \frac{\varphi(y)}{y'^{\alpha}}\,dx, \qquad \alpha > 0, \quad \text{or} \quad \alpha < -1, \tag{21}$$

a minimum. Using (14′) we have

$$\left[\int_0^{x_2} \frac{y'dx}{[\varphi(y)]^{1/\alpha}}\right]^{\alpha} \int_0^{x_2} \frac{\varphi(y)dx}{y'^{\alpha}} > x_2^{\alpha+1}. \tag{22}$$

The integral in the first factor on the left is a constant since it is equal to

$$\int_0^{y_2} \frac{dy}{[\varphi(y)]^{1/\alpha}}\,.$$

Since in (22) equality exists only when the integrand is a constant, the minimum of (21) is given[1] by

$$\frac{\varphi(y)}{y'^{\alpha}} = A, \qquad x_2^{\alpha} = A\left[\int_0^{y_2} \frac{dy}{[\varphi(y)]^{1/\alpha}}\right]^{\alpha}, \qquad x = A^{1/\alpha}\int_0^{y} \frac{dy}{[\varphi(y)]^{1/\alpha}}. \tag{23}$$

CASE 2. If the integral to be minimized is of the form

$$\int_0^{x_2} \frac{\varphi(x)dx}{y'^{\alpha}}, \qquad \alpha > 0, \quad \text{or} \quad \alpha < -1, \tag{24}$$

where $\varphi(x)$ is greater than zero, it will be convenient to write it

$$\int_0^{y_2} \varphi(x)\left[\frac{dx}{dy}\right]^{\alpha+1} dy, \tag{24'}$$

and to consider x as a function of y. Using (14) we have

$$\int_0^{y_2} \varphi(x)\left(\frac{dx}{dy}\right)^{\alpha+1} dy > \frac{1}{y_2^{\alpha}}\left[\int_0^{x_2} [\varphi(x)]^{1/\alpha+1}dx\right]^{\alpha+1}. \tag{25}$$

We have a minimum only when the integrand on the left is a constant. Hence

$$\varphi(x)\left(\frac{dx}{dy}\right)^{\alpha+1} = A, \qquad A^{1/\alpha+1}y = \int_0^{x} [\varphi(x)]^{1/\alpha+1}dx, \tag{26}$$

where A is determined by the fact that $y = y_2$ when $x = x_2$.

CASE 3. Let us consider now the case in which y is zero when x is zero, its derivative y' takes on the values p_1 and p_2, $-\infty < p_1 < p_2 < +\infty$, at $x = 0$ and $x = x_2$, respectively, and the second derivative y'' exists and is greater than zero in the interval for x. It is desired to determine the function y so as to render the integral

$$\int_0^{x_2} \frac{\varphi(y')dx}{y''^{\alpha}}, \qquad \alpha > 0, \quad \text{or} \quad \alpha < -1, \tag{27}$$

a minimum, where $\varphi(y') > 0$.

As before we have from (14′)

$$\left[\int_0^{x_2} \frac{y''dx}{[\varphi(y')]^{1/\alpha}}\right]^{\alpha} \int_0^{x_2} \frac{\varphi(y')dx}{y''^{\alpha}} > x_2^{\alpha+1},$$

where the integral in the first factor on the left is a constant, since it is equal to

$$\int_{p_1}^{p_2} \frac{dp}{[\varphi(p)]^{1/\alpha}}.$$

Hence the minimum is given by setting the integrand in (27) equal to a constant A. We have then

$$x = A^{1/\alpha}\int_{p_1}^{p} \frac{dp}{[\varphi(p)]^{1/\alpha}}, \qquad y = A^{1/\alpha}\int_{p_1}^{p} \frac{pdp}{[\varphi(p)]^{1/\alpha}}, \tag{28}$$

[1] The case in which $\varphi(y) = 1$ and $\alpha = -2$ is treated in this same manner in Goursat-Hedrick's *Mathematical Analysis*, vol. 1, pp. 257–258.

where A is determined by the fact that $x = x_2$ when $p = p_2$; the second integral determines the value of y when $x = x_2$.

REMARKS. If instead of fixing the value of x_2 as above, the value of y_2 is given and $y' = p_2$ when $y = y_2$ while the other conditions remain the same, the integral (27) may be written

$$\int_0^{y_2} \frac{\varphi(y')dy}{y''^{\alpha} y'}. \tag{27'}$$

Here $\varphi(y')/y'$ is to be continuous and greater than zero. Applying (14') we have

$$\left[\int_0^{y_2}\left[\frac{y'}{\varphi(y')}\right]^{1/\alpha} y''dy\right]^{\alpha}\int_0^{y_2}\frac{\varphi(y')dy}{y''^{\alpha}y'} > y_2^{\alpha_2+1}.$$

The integral in the first factor on the left is equal to

$$\int_{p_1}^{p_2}\left[\frac{p}{\varphi(p)}\right]^{1/\alpha} pdp,$$

and is therefore a constant. Hence the minimum of (27') is given by

$$\frac{\varphi(y')}{y''^{\alpha}} = By'. \tag{29}$$

These two problems are included in a more general one. Suppose that (27) is to be made a minimum by a function of x, $y = y(x)$, such that the curve defined by this equation passes through the origin with the slope p_1, and cuts the line $x \cos \gamma + y \sin \gamma = d$ at a given angle. By rotating the axes through the angle γ, this problem is reduced to one in which the integral is of the same form as (27). The conditions are to be made of the same nature as those given for that integral. The minimum is given by

$$\frac{\varphi(y')}{y''^{\alpha}} = C[\cos \gamma + y' \sin \gamma]. \tag{30}$$

If the values of x_2 and y_2 are both assigned, the preceding method does not seem to apply, but the results in the three cases considered would suggest that the solution is of the form

$$\frac{\varphi(y')}{y''^{\alpha}} = A + By', \tag{31}$$

where A and B are constants to be determined by the end conditions. If $\alpha = 1$, the second set of inequalities gives a proof of this, after changing the form of (27) and introducing auxiliary integral conditions. This case will now be considered.

7. Auxiliary Integral Conditions. The integral (27) will be written

$$\int_{p_1}^{p_2} \varphi(p)\delta^2 dp, \qquad \delta = \frac{dx}{dp}, \qquad \alpha = 1, \tag{32}$$

and we have the auxiliary conditions

$$x_2 = \int_{p_1}^{p_2} \delta dp, \qquad y_2 = \int_{p_1}^{p_2} \delta p dp. \tag{32'}$$

Here δ may be permitted to change sign. The two functions $1/\sqrt{\varphi(p)}$, $p/\sqrt{\varphi(p)}$ are continuous since we have assumed that $\varphi(p) > 0$, and they are linearly independent. Suppose that δ is any function of p which satisfies (32′) and that $\delta\sqrt{\varphi(p)}$ is linearly independent of the above two functions. Then by the theorem for linear independence, page 330, we have

$$\left|\delta\sqrt{\varphi(p)}, \quad \frac{1}{\sqrt{\varphi(p)}}, \quad \frac{p}{\sqrt{\varphi(p)}}\right| > 0, \tag{33}$$

where the expression on the left is defined in (15). In the determinant on the left of (33) we shall have the following integrals:

$$(\delta\sqrt{\varphi(p)}\ \delta\sqrt{\varphi(p)}) = \int_{p_1}^{p_2} \varphi(p)\delta^2 dp,$$

$$\left(\delta\sqrt{\varphi(p)}\,\frac{1}{\sqrt{\varphi(p)}}\right) = \int_{p_1}^{p_2} \delta dp = x_2, \qquad \left(\delta\sqrt{\varphi(p)}\,\frac{p}{\sqrt{\varphi(p)}}\right) = \int_{p_1}^{p_2} \delta p dp = y_2.$$

Inserting these in (33) the inequality may be written in the form

$$D\int_{p_1}^{p_2} \varphi(p)\delta^2 dp > -\begin{vmatrix} 0 & x_2 & y_2 \\ x_2 & \left(\frac{1}{\sqrt{\varphi(p)}}\frac{1}{\sqrt{\varphi(p)}}\right) & \left(\frac{1}{\sqrt{\varphi(p)}}\frac{p}{\sqrt{\varphi(p)}}\right) \\ y_2 & \left(\frac{p}{\sqrt{\varphi(p)}}\frac{1}{\sqrt{\varphi(p)}}\right) & \left(\frac{p}{\sqrt{\varphi(p)}}\frac{p}{\sqrt{\varphi(p)}}\right) \end{vmatrix} = -\Delta, \tag{34}$$

where D is the cofactor of the element 0 in the determinant on the right, and it is a constant greater than zero since $1/\sqrt{\varphi(p)}$ and $p/\sqrt{\varphi(p)}$ are linearly independent. Also the determinant on the right is a constant. Equality exists in (34) when and only when $\delta\sqrt{\varphi(p)}$ is linearly dependent upon the above two functions. Hence the minimum of (32) is given by

$$\delta = \frac{A + Bp}{\varphi(p)}. \tag{35}$$

It is easily verified that

$$\begin{vmatrix} \varphi(p)\delta & 1 & p \\ x_2 & \left(\frac{1}{\sqrt{\varphi(p)}}\frac{1}{\sqrt{\varphi(p)}}\right) & \left(\frac{1}{\sqrt{\varphi(p)}}\frac{p}{\sqrt{\varphi(p)}}\right) \\ y_2 & \left(\frac{p}{\sqrt{\varphi(p)}}\frac{1}{\sqrt{\varphi(p)}}\right) & \left(\frac{p}{\sqrt{\varphi(p)}}\frac{p}{\sqrt{\varphi(p)}}\right) \end{vmatrix} = 0. \tag{36}$$

The minimum value of the integral is $-\Delta/D = Ax_2 + By_2$. If the subscripts

2 in (32′) are omitted and the value of δ in (35) or (36) is inserted, the solution of this problem will be complete.[1]

Other independent conditions may be added to (32′); for example, it may be required that

$$l = \int_{p_1}^{p_2} \sqrt{1 + p^2}\delta dp,$$

where l is a given quantity. The solution may be obtained in the same way.

8. A More General Problem. Turning now to a more general case, suppose that it be required to determine δ as a function of p which renders

$$\int_{p_1}^{p_2} \varphi(p)F(\delta)dp, \tag{37}$$

a minimum, where $F(t)$ has a second derivative $F''(t)$ which is greater than zero, and where δ satisfies the conditions (32′). Let δ and $\delta + \eta$ be two different functions which satisfy the conditions (32′). Then

$$\int_{p_1}^{p_2} \eta dp = 0, \qquad \int_{p_1}^{p_2} \eta p dp = 0. \tag{38}$$

Also

$$\begin{aligned}\int_{p_1}^{p_2} \varphi(p)F(\delta + \eta)dp = \int_{p_1}^{p_2} \varphi(p)F(\delta)dp + \int_{p_1}^{p_2} \eta\varphi(p)F'(\delta)dp \\ + \frac{1}{2}\int_{p_1}^{p_2} \eta^2\varphi(p)F''(\delta + \theta\eta)dp, \qquad 0 < \theta < 1.\end{aligned} \tag{39}$$

Hence if δ is chosen so that

$$F'(\delta) = \frac{A + Bp}{\varphi(p)}, \tag{40}$$

where A and B are constants, then by use of (38) we shall have

$$\int_{p_1}^{p_2} \varphi(p)F(\delta + \eta)dp > \int_{p_1}^{p_2} \varphi(p)F(\delta)dp. \tag{39}$$

The constants A and B must be so chosen that the two equations in (32′) are satisfied. Since $F'(t)$ is an increasing function of t, the equation (40) admits a continuous solution for δ. In the case treated above where $F(t) = t^2$, there was no difficulty in the actual determination of the constants; but in other cases there may be difficulty not only in the actual determination of the constants but in knowing even if there exist values of A and B which make δ satisfy (32′). If there exists a set of values there can be only one set.

[1] For particular cases see the papers by the writer, "A Determination of the Curve Minimizing the Area Enclosed by it and its Evolute," in the MONTHLY [*1921*, 15–19]; "A Direct Determination of the Minimum Area between a Curve and its Caustic," *Annals of Mathematics*, 2 d ser., vol. 23, no. 2, pp. 135–140, Dec. 1921. Also the paper in the same journal by Rider, "On the Minimizing of a Class of Definite Integrals," 2d ser., vol. 24, no. 2, pp. 167–174, Dec. 1922.

REMARKS. If $F(t) = t^{\alpha+1}$, $\alpha = [2(i+j)+1]/(2i+1)$, where i and j are integers, $j \geqq 0$, $i \geqq 0$, then it may be shown that there exists a unique set of constants A and B of the kind desired.

A SYSTEM OF TRIANGLES RELATED TO A PORISTIC SYSTEM.

By J. H. WEAVER, Ohio State University.

1. Introduction. In the last half century considerable attention has been paid to poristic systems of triangles, *e.g.*, triangles which are inscribed in one circle and circumscribed to another. In this paper it is proposed to set forth some of the simpler properties of a set of triangles having a fixed circumcircle and a fixed nine-point circle.

In this discussion several terms will be used which may not be familiar to the general reader. We proceed to define these terms.

1. *Orthopole:* An orthopole is defined by J. Neuberg as follows. Let there be a triangle ABC and a line l in the plane of the triangle. From A, B, and C draw perpendiculars to l cutting it in P, Q, R respectively. From P draw a perpendicular to BC, from Q a perpendicular to AC and from R a perpendicular to AB. These three perpendiculars meet in a point S which is called the orthopole of l with respect to the triangle ABC. It seems wise to prove [1] the property of the orthopole used in theorem VII below.

LEMMA: To determine the orthopole of l geometrically.

Let AP meet the circle ABC again in K. Draw the chord $K\,K'K''$ perpendicular to BC, cutting BC in K' and the circle again in K''. Let BA and CA meet l in R' and R'' respectively.

Let l make angles θ_1, θ_2, θ_3 with BC, CA, AB respectively. We then have $BK = 2R \sin (BAK \text{ or } R'AP) = 2R \cos \theta_3$ (R = radius of circle ABC). Also $CK = 2R \cos \theta_2$.

Therefore $KK' = BK \cdot CK/2R = 2R \cos \theta_2 \cos \theta_3 = SP$. Hence S is found by drawing $K'S$, PS parallels to AP, KK' respectively.

THEOREM: *The Simson lines of the extremities of any chord TT' of the circle ABC pass through S the orthopole of TT'.*

If we draw TT_1 cutting BC in X and perpendicular to BC and draw KK_1 perpendicular to TT_1 and cutting it in K_1, we have $\angle\, TPK = \text{rt} \angle = \angle\, TK_1K$. Therefore TK_1PR are cyclic, and $\angle\, PK_1X$ or $PK_1T_1 = \angle\, TKP$, or $\angle\, TKA = \angle\, TT_1A$. Hence K_1P is parallel to AT_1. But $K_1X = KK' = SP$. Therefore XS is parallel to K_1P and to T_1A and is the Simson line of T. And similarly for T'.

2. *Antipedal triangle:* Let S be any point in the plane of the triangle ABC. Draw SA, SB, SC and through A, B, C draw lines perpendicular to SA, SB, SC. These perpendiculars determine a triangle called the antipedal triangle of S.

[1] This proof together with other interesting proofs of various properties of the orthopole are due to J. Neuberg and may be found in Gallatley, *Modern Geometry of the Triangle*, Chap. VI.

3. *Antimedial Triangle:* The antimedial triangle is a triangle formed by drawing through A, B, C lines parallel to the opposite sides.

4. *Nagel Point:* The Nagel point is the point of intersection of the lines AX_1, BY_2, CZ_3, where X_1 is the point of contact on BC of the ex-circle opposite A etc.

2. Theorems. THEOREM I: *If a variable triangle has a fixed nine-point circle and a fixed circumcircle, it envelops a conic.*

PROOF: Let (O) be the circumcircle, center O, and (N) the nine-point circle, center N. Let ABC be the variable triangle. Draw the altitude CF and extend it to K on (O). Draw KO cutting AB in L. Let H be the ortho-center of ABC. Draw LH.

Then $HF = FK$ and LF is perpendicular to HK. Therefore the triangle LHK is isosceles and $LH = KL$. Therefore $OL + HL = R$ (R is radius of (O)). Also AB bisects $\angle KLH$. Hence L is on an ellipse (E) with N as center and H and O as foci.

REMARKS: This proof assumes that the circles (O) and (N) do not intersect. If they intersect, the conic will be a hyperbola and the triangle ABC will be obtuse angled.

The circumcenter, orthocenter and nine-point center are fixed points in this system, but the in-center and the ex-centers are variable. We proceed to find the locus of these points.

THEOREM II: *The in-center and the ex-centers lie on a quartic curve consisting of two ovals each of which is the inverse of the other with respect to N as center.*

PROOF: Let I be the in-center and let $OI = d$, $ON = c$. Let O be the origin, ON the X-axis and $I = (x, y)$. Then

$$x^2 + y^2 = d^2 = R^2 - 2Rr, \text{ by Euler's theorem.} \tag{1}$$

Also since (N) and (I) are tangent,

$$(x - c)^2 + y^2 = (R/2 - r)^2. \tag{2}$$

Eliminating r from (1) and (2), we get

$$4R^2[(x - c)^2 + y^2] = (x^2 + y^2)^2. \tag{3}$$

In a similar manner we can show that the ex-centers lie on the quartic (3). Transferring the origin to N and writing (3) in polar coördinates, we obtain

$$\rho^2 - 2c\rho \cos\theta - 2R\rho + c^2 = 0. \tag{4}$$

In the third term of (4) only the negative sign is used, as the positive sign gives imaginary values for ρ. For any fixed θ there are two values for ρ in equation (4), such that their product is c^2. The curve (4), therefore, consists of two ovals, each of which is the inverse of the other with respect to N as center and c as radius.

In equation (3) the substitution of $x - c = iy$ gives a perfect square, hence the point $(c, O) = N$ is a focus of (3).

DEFINITION: Let a radius vector of (4) cut the curve in T and T', and let S be the mid point of TT'.

THEOREM III: *A line l through S perpendicular to TT' envelops a circle with center H and radius R.*

PROOF: From equation (4) we find that $NS = c \cos \theta + R$.

Now draw through H a line parallel to NS cutting l in S', then $HS' = R$ for any value of θ. Therefore l is tangent to a circle with H as center and radius R.

THEOREM IV: *Tangents to the curve* (4) *at T and T' are equally inclined to TT'.*

THEOREM V: *The first polar of O with respect to* (3) *is a circle on NH as diameter.*

THEOREM VI: *The polars of any point on* (O) *with respect to* (N) *and* (E) *intersect on the line NH.*

The proofs of the above three theorems are simple and are therefore omitted.

THEOREM VII: *The locus of the orthopole of any chord of* (O) *is a circle equal to* (N).

PROOF: Let TT' be a chord of (O). Then the orthopole of TT' is the intersection S of the Simson lines of T and T'. Moreover the angle between the Simson lines of T and T' is constant and measured by one half the arc TT'. Also the Simson lines of T and T' pass through two fixed points P and P' on (N). Therefore the locus of S is a circle of which PP' is a chord subtending an angle equal to the angle which it subtends in (N). Hence the locus of S is a circle equal to the circle (N).

REMARKS: If TT' is a diameter, the locus of S is (N) itself.

In the pedal triangle DEF of ABC, H is the in-center, N is the circumcenter and the circle (N) is fixed. Hence DEF has a fixed inscribed circle and generates a poristic system of triangles. Also the antipedal triangle of O generates a poristic system of triangles.

THEOREM VIII: *The Nagel Point of ABC traverses an oval of a quartic similar to the quartic* (3).

Consider the antimedial triangle $A_1B_1C_1$. This has G for centroid, H for circumcenter and O for nine-point center and these points are fixed. Moreover the triangles ABC and $A_1B_1C_1$ are homothetic. Therefore the in-center of $A_1B_1C_1$ traverses an oval similar to the oval which I traverses. But the in-center of $A_1B_1C_1$ is the Nagel point of ABC. Hence the theorem.

Also since the circumcenter of the triangle $I_1I_2I_3$ is the symmetric of I with respect to O, this point also traverses an oval of a quartic similar to (3).

Let def and $d'e'f'$ be the pedal triangles of H and H', the inverse of H with respect to the circle (O). Then the triangles def and $d'e'f'$ are inversely similar.

THEOREM IX: *The locus of the double point of the inversely similar triangles def and $d'e'f'$ is the circle* (N).

Let T and T' be the ends of the diameter OH, and let ω be the orthopole of TT'. Now consider the figures $ATOT'$ and $\omega XA'X'$ (X and X' are the feet of the perpendiculars from T and T' on BC, A' is the midpoint of BC). $\angle TAT' = \angle X\omega X' =$ rt. $\angle$.

Also O and A' are the midpoints of TT' and XX' and $\angle OTA = \angle A'X\omega$. Therefore the figures $ATOT'$ and $\omega XA'X'$ are similar.

In these two figures H and d are homologous points. Also d' is homologous to H'. But AT bisects $\angle HAH'$, therefore ωX bisects $\angle d\omega d'$ and we have

$$\omega d : \omega d' = Xd : Xd' = TH : TH' \text{ etc.}$$

Therefore

$$\omega d : \omega e : \omega f = \omega d' : \omega e' : \omega f'.$$

Hence ω is the double point of the triangles def and $d'e'f'$. But from Theorem VII, ω is on (N).[1]

QUESTIONS AND DISCUSSIONS.

Edited by C. F. Gummer, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

QUESTIONS.

The fourth of the discussions contained in this number suggests the following question:

52. Is there any simple treatment of the regular pentagon constructions the proof of which involves neither medial section nor trigonometric formulæ?

It is well known that simple constructions may be made based on the square root of five, and not unlike that given below; but works on formal geometry commonly follow Euclid's method.

DISCUSSIONS.

I. Geometrical Construction of Points on a Four-leaf Rose.

By H. H. Downing, University of Kentucky.

Consider a fixed circle of radius a and a fixed diameter with extremities O and A. Through O draw a chord cutting the circle in R. Locate P on OR so that $OP = RM$, where M is the foot of the perpendicular from R on OA. As R describes the semicircumference AO, P will trace out one loop of the four-leaf rose $r = a \sin 2\theta$. It will be at once evident how to trace the other loops.

Proof: Taking OA as initial line with O as the origin, $OP = r$, the angle $AOR = \theta$, we have

$$r = OP = RM = OR \sin\theta = OA \cos\theta \sin\theta = 2a \cos\theta \sin\theta = a \sin 2\theta.$$

[1] In this connection see Gallatley, *Modern Geometry of the Triangle*, p. 47.

II. Note on the Solution of a Set of Linear Equations.

By J. P. Ballantine, Columbia University.

Let x_1, x_2, x_3, x_4 satisfy the set of equations

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 &= a_5, \\ b_1x_1 + \cdots \qquad\qquad &= b_5, \\ c_1x_1 + \cdots \qquad\qquad &= c_5, \\ d_1x_1 + \cdots \qquad\qquad &= d_5, \end{aligned}$$

supposed independent.

Then[1]

$$\begin{vmatrix} a_1x_1 - a_5 & a_2 & a_3 & a_4 \\ b_1x_1 - b_5 & b_2 & b_3 & b_4 \\ c_1x_1 - c_5 & c_2 & c_3 & c_4 \\ d_1x_1 - d_5 & d_2 & d_3 & d_4 \end{vmatrix} = 0;$$

for, on multiplying the respective columns by 1, x_2, x_3, and x_4, and adding, a column of zeros is obtained. From the vanishing of the above determinant may be obtained the usual formula for the value of x_1.

Suppose it is desired to solve the set of equations for only one of the unknowns, and to check the result. The usual method affords no check on the value of x_1 until all of the other unknowns have been evaluated; except, perhaps, the consistency of the four equations in the other three unknowns when the value of x_1 is substituted. This consistency condition is precisely the vanishing of the above determinant, a fact which is not usually brought out in an elementary treatment.

III. Evaluation of the Determinant $|1/(r + s - 1)!|$.

By J. J. Nassau, Case School of Applied Science.

The evaluation of the determinant

$$D_n = \begin{vmatrix} \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{n!} \\ \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(n+1)!} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{n!} & \frac{1}{(n+1)!} & \cdots & \frac{1}{(2n-1)!} \end{vmatrix}$$

has been given recently by L. L. Dines (*1923*, 196–8). In this note I propose to reduce the elements of D_n to binomial coefficients and by so doing to facilitate its evaluation.

---

[1] Dickson, *Elementary Theory of Equations*, 1914, p. 145.

Multiplying the first column by $n!$, the second by $(n+1)!$ and so on and dividing the first row by $(n-1)!$, the second by $(n-2)!$ and so on up to the $(n-1)$th row, which is divided by $1!$, we obtain

$$D_n = \frac{(n-1)!(n-2)!\cdots 1!}{n!(n+1)!\cdots(2n-1)!}\begin{vmatrix} \frac{n!}{1!(n-1)!} & \frac{(n+1)!}{2!(n-1)!} & \cdots & \frac{(2n-1)!}{n!(n-1)!} \\ \frac{n!}{2!(n-2)!} & \frac{(n+1)!}{3!(n-2)!} & \cdots & \frac{(2n-1)!}{(n+1)!(n-2)!} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & & 1 \end{vmatrix}.$$

By writing the nth row of this determinant as the first row, the $(n-1)$th row as the second and so on, and replacing the fraction $n!/(n-r)!r!$ by n_r, we have

$$(-1)^{n(n-1)/2}\begin{vmatrix} n_0 & (n+1)_0 & \cdots & (2n-1)_0 \\ n_1 & (n+1)_1 & \cdots & (2n-1)_1 \\ \cdot & \cdot & \cdot & \cdot \\ n_{n-1} & (n+1)_{n-1} & \cdots & (2n-1)_{n-1} \end{vmatrix}.$$

Determinants of this type have been considered by Zeipel [1] and lately by Muir.[2] The above special case can easily be shown to be equal to 1.[3]

Therefore

$$D_n = (-1)^{n(n-1)/2}\cdot\frac{(n-1)!(n-2)!\cdots 1!}{n!(n+1)!\cdots(2n-1)!}.$$

IV. On the Division of a Circumference into Five Equal Parts.

By H. C. Bradley, Massachusetts Institute of Technology.

The following simple construction for dividing a circumference into 5 or 10 parts is, in the opinion of the author, new and differs considerably from methods commonly described in text-books of plane geometry. Instead of constructing the sides of regular inscribed decagon and pentagon, this method constructs arc sec $(\sqrt{5}-1)$ and arc sec $(\sqrt{5}+1)$ and assumes that it has been proved by trigonometry that these are, respectively, 36° and 72°. The details follow:

At a point A on the circumference of a circle of radius R and center O, draw the tangent AH. On AH lay off $AC = 2R$. On OC, towards O, lay off $CD = R$. With center O and radius OD draw an arc intersecting AH at F. Then $\angle FOA =$ arc sec $(\sqrt{5}-1) = 36°$.

On OC, away from O, lay off $CE = R$. With center O and radius OE draw an arc intersecting AH at G. Then $\angle GOA =$ arc sec $(\sqrt{5}+1) = 72°$.

[1] Om Determinanter, hvars elementer ars Binomialkoefficienter. *Lunds Univ. Arsskrift*, ii (1865), pp. 1–68. (Muir's *Theory of Determinants*, Vol. III.)

[2] *Transactions of the Royal Society of South Africa*, 1923, Vol. XI, pp. 191–196.

[3] Muir's *Theory of Determinants*, Vol. III, p. 448.

V. On the Definition of Determinants.

By A. A. Bennett, University of Texas.

This brief article will treat of but one feature of the definition of determinants, namely that of the algebraic sign to be prefixed before any one of the given products of n elements which constitute the terms of the expansion of the determinant. This important question is that of the parity (oddness or evenness) of a permutation, and except for the significance of this particular application belongs in the theory of permutation groups. Regarded abstractly, a set of elementsis given and a one-to-one reciprocal correspondence (called a permutation) of the set is also given, one asks whether the permutation is odd or even. Before this question can be answered in any given case it is essential that these terms be defined, and that we be convinced that each permutation must be either odd or even and can never be both.

Several methods of procedure are current but they usually agree by first establishing an arbitrary sequential order among the given elements, and even assigning labels to them as, for example, $e_1, e_2, e_3, \cdots, e_n$. A rearrangement of these is then tested for parity. This is ordinarily accomplished in one of two ways. One way is as follows. A set of n real distinct numbers, $x_1, x_2, x_3, \cdots, x_n$, is selected in one-to-one reciprocal correspondence with the e's, and hence with the given elements. The product, $X, = \Pi(x_i - x_j), i < j, i, j, = 1, 2, \cdots, n$, is constructed and it is noted that X can at most change sign if the quantities, x_i, be rearranged in any order. If X does not change sign under a given permutation, this permutation is said to be *even,* while if X does change sign, the permutation is *odd.* The other method is as follows. Let the elements, e_i, as rearranged by a permutation be denoted by $e_1', e_2', e_3', \cdots, e_n'$, where each e_i' is one of the original set, $(e_1, e_2, e_3, \cdots, e_n)$, under a different notation. A pair of distinct elements in the rearrangement, namely (e_j', e_k'), $j < k$, $j, k = 1, 2, \cdots, n$, defines a *permanence* if the subscript of e_j', when written as an e_i, is less than that of e_k' written as an e_i', otherwise the pair defines an *inversion.* In determining the number of permanences or of inversions, it is necessary to consider every pairing of distinct elements, and not merely pairs of successive elements. Thus the total set of permanences and of inversions in any given case will of course number exactly $n(n-1)/2$. The permutation is said to be *odd* or *even* according as the number of inversions is odd or even.

Both of these methods are open to the disadvantage that the definition involves a one-to-one reciprocal correspondence from the given set to the set of e's, $(e_1, e_2, e_3, \cdots, e_n)$. Until the contrary is established it is necessary to refer to a permutation as being odd or even *with respect to a given initial ordering.* The proof that the parity is independent of the ordering takes several steps and shows that the original definition introduced an extraneous item. The method of permanences and inversion might seem a trifle less artificial than the use of the algebraic function, X. On the other hand, the notion of permanence and that of inversion have only passing interest, and are merely incidental auxiliaries

to aid in the establishment of the concept of parity. The algebraic function, X, has further significance, being the square root of the discriminant, and plays a basic rôle in the theory of geometrical constructions, of alternating functions, of the algebraic solution of algebraic equations, and so forth. Despite these further uses of the same machinery, the objection stated still remains.

One might start with the real difficulty involved in the usual procedure, and define a permutation as odd or even according as it is the product of an odd or even number of transpositions. This leaves open the question as to whether the representation of a permutation as a product of transpositions is unique as to parity. It is nearly obvious that it is not unique as to the number or the sequence of the transpositions employed, so that one might reasonably question whether the parity is not dependent upon the particular analysis of the given permutation into transpositions.

It would seem desirable that texts should use a definition of parity not confused from the start by extraneous features or open to queries of this sort, even though such difficulties might be of short duration. If available, one would ask for a definition from which it is at once obvious that the parity is an inherent characteristic of the permutation. Whether or not the parity is related to the number of transpositions does not appear to be so essential in the first stages of the discussion. The parity having been once defined, the definition should be such as to make it readily seen that the product of any permutation by a transposition has a parity opposite to that of the given permutation. If one then proves that every permutation is the product in at least one way of a set of transpositions, the relation between parity and transpositions is completely explained. For then the identity, which is even, cannot be expressible as the result of an odd number of transpositions, so that if two distinct representations of a given permutation be at hand, each expressed as a continued product of transpositions, the product of the first representation by the inverse of the second would be the identity, and must involve an even number of transpositions, from which it is at once inferred that both of the given representations involved an odd number of transpositions or both involved an even number.

Fortunately one need not seek far for a characteristic property that might well serve as a definition, although, despite its familiarity, its advantages seem to have been generally ignored.

We shall assume that the notion of a permutation itself has been established. One may then study, as is the custom in some connections, the independent cycles generated in the set of elements by repetitions of the permutation. The facts that in a finite set the cycles may be listed in any order and that the elements in a cycle may be given any circular permutation without altering the defining permutation are almost self-evident and may be proved without mention of parity or the analysis of a permutation into the product of transpositions, except of course for such trivial examples as those in which no cycle is of greater order than 2. Furthermore there is no appeal to a notion of initial ordering. This is obvious from the fact that the cycles determine the permutation, and these

are subject to the rearrangements mentioned. The parity is now defined to be that of the total number of independent cycles each of which has an even number of elements. Since no extraneous correspondence is invoked, there is no occasion to prove that the parity is independent of a particular ordering. Incidentally the proof that a permutation is of like parity with its inverse becomes obvious without appeal to transpositions.

It seems surprising that so familiar a relation as that between the number of cycles of even order and the parity should be left so generally as a proposition to be proved upon the basis of an awkward definition of parity. This is perhaps due to one's tendency to regard any given finite set as having some initial sequential order as the basis of discussion, the fact that the order is immaterial being regarded as in no sense relieving us of thinking of the set as ordered. If, in teaching, one should use such symbols as "+", "−", "0", "*", or the names of several colors, or some other set for which no particular ordering at once suggests itself, there might be less temptation on the part of the student to think of the elements in a given discussion as inherently ordered. There is something more than merely mental discipline to be gained in thinking of sets as given without order, even in connection with the special applications of these notions to the initial problem of this article, namely, to the definition of determinants, which is the very topic in which so much energy is usually concentrated upon relations of order. Let A be a set of n symbols, and e_{ab}, where a and b are arbitrary elements of A, be a set of n^2 numbers. Consider a product of n symbols, e_{ab}, where the a's are distinct and the b's are also distinct. This product completely identifies a permutation in the set, A, as from the a's to the b's. This permutation depends upon the choice of the set of factors but is completely independent of the order in which they are written to form the product. If the permutation that is thus defined is odd, place a minus sign before this product, if even, place a plus sign. The algebraic sum of all such distinct signed products is defined as the determinant of the set of n^2 elements, e_{ab}. That an interchange of first and second subscripts throughout does not alter the determinant is proved with the greatest ease. In each term the new permutation is the inverse of the original one, and hence of the same parity. The new expression contains except for signs the terms necessary for the definition of a determinant, and we have just seen that the correct sign in the present case is the same as the sign of the terms from which each is secured. This may be stated as the proposition that a determinant is unaltered, if rows are changed into corresponding columns and columns into corresponding rows. In the case of several other familiar transformations which leave a determinant unaltered or at most change its sign, the proof is made appreciably simpler by using this definition of a determinant.

VI. Note on the Nature of the Correlation Coefficient.

By H. M. Roeser, Bureau of Standards.

The ideas brought out in Professor Dunham Jackson's paper "The Algebra of Correlation" in the Monthly (*1924*, 110) seem to give justification for the presentation of a brief note on the nature of the correlation coefficient which has not come to my attention in published form although my contact with literature on statistics has not been intimate during the past four or five years. While being of very elementary nature, it may be found useful in creating a working conception of the correlation coefficient in the minds of students of statistical theory.

As a starting point, consider section 9 of Professor Jackson's paper in which, after developing the equation for the line of regression, he further produces the familiar relation

$$\sigma_1^2 = \sigma^2(1 - r^2).$$

By solving for r the following relation is obtained,

$$r = \pm\sqrt{1 - \frac{\sigma_1^2}{\sigma^2}} = \pm\sqrt{1 - \frac{\Sigma v_k^2}{\Sigma y_k^2}}.$$

In this last expression the v's are what are termed "residuals" in the theory of least squares. In the correlation plot they are the vertical deviations of the y_k's from the line of regression. This being the case, a mental grasp of the correlation coefficient is immediately furnished especially if one has had previous experience with the method of least squares.

If the correlation coefficient is equal to unity, correlation is said to be perfect, that is; the values of y_k fall exactly upon the line of regression. This is directly apparent from the above equation because the coefficient cannot equal unity unless $\Sigma v_k^2 = 0$, which in turn requires that the values of y_k be exactly on the line.

If the correlation coefficient is zero, there is said to be no correlation, that is, the line of regression coincides with the x-axis. In this case each value of v is equal to the corresponding value of y_k, or each value of y_k is its own deviation from the line of regression. This is apparent in the above equation since it is the only condition under which the coefficient can be equal to zero.

It is further evident from the above equation that the correlation must vary from zero to unity in absolute value and the smaller the absolute value the poorer the agreement with the line of regression.

According to the preface only a most elementary knowledge of mathematics is needed for a proper study of the book but the reviewer feels that the introductions of such conceptions as limiting values (particularly that connected with the force of interest), convergency of series, the binomial theorem, etc., are not accompanied by explanations which will be clear to the student having only an elementary knowledge of mathematics.

On the whole, the book has been very carefully written and printed (no important errors were noted) and students who use the book properly are very apt to prove better drilled in the essentials of the mathematical theory of finance than those who use other current textbooks both because of the great variety and number of exercises and also because the treatment is restricted to essentials.

The book is divided into three parts. The first is entitled Annuities Certain; the second, Life Insurance; and the third, Auxiliary Subjects, including logarithms and progressions. Several topics are considered in an appendix. Answers to the exercises are issued in a separate booklet.

C. H. FORSYTH.

The Numerical Evaluation of the Incomplete B-*Function*, $\int_0^x x^{p-1}(1-x)^{q-1}dx$ *for Ranges of x between* 0 *and* 1. By H. E. SOPER. Cambridge University Press, 1921. 53 pages. Price 3s. 9d.

Table of the Logarithms of the Complete Γ-*Function* (*for Arguments* 2 *to* 1200, *i.e., beyond Legendre's Range*). By E. S. PEARSON. Cambridge University Press, 1922. 12 + 16 pages. Price 3s. 9d.

Log $\Gamma(x)$ *from* $x = 1$ *to* 50.9 *by Intervals of* .01. By JOHN BROWNLEE. Cambridge University Press, 1923. 23 pages. Price 3s. 9d.

These are respectively numbers VII, VIII and IX of Tracts for Computers edited by Karl Pearson and published by the Department of Applied Statistics of University of London, University College. We have already reviewed numbers I–IV (*1921*, 265–267) of these Tracts.

In Tract VII the following topics are considered: Integration in series attending index changes; Integration by polynomial approximations to the second factor, and other polynomial approximations; Trigonometric transformations and approximations; and Special cases of one index very large, both indices very large, and equal indices.

In many statistical problems this value of $\Gamma(p)$ is required for high values of p. It was to supply this need that the tables of Tract VIII for the new range of values of log $\Gamma(p)$, $p = 2$ to 1200, were published. From 2 to 5, p changes by tenths; from 5–10 by two-tenth intervals, and then by unit intervals to 1200. The table is to 10 places of decimals, and its use calls for a table of logarithms to 10 places such as the one by Duffield issued for the U. S. Coast and Geodetic Survey, Washington, D. C., in 1897.

Tract IX gives the values of log $\Gamma(p)$ to seven figures for values of p from 1 to 50.9 by intervals of .01 and thus supplements Tract VIII.

We have already given a bibliography of previously published tables of the gamma function (*1921*, 267).

R. C. ARCHIBALD.

Differential Equations in Applied Chemistry. By F. L. HITCHCOCK and C. S. ROBINSON. New York, John Wiley & Sons, 1923. vi + 110 pages. Price $1.50.

The purpose of the book, which is based on a recent course given to the chemical engineers at the Massachusetts Institute of Technology, is "not to teach Chemistry, but to teach Mathematics in a form readily assimilated by chemists and chemical engineers." More especially, the first object of this book is to help such students to think more readily "in terms of calculus." This, together with a general explanation of the fundamental concept of calculus and its value as a tool, is set out nicely in the preface and in a rather plastically written introductory chapter. As far as the reviewer can see, the latter is likely to attract chemists and to stimulate them to a neat and definite way of thinking and reasoning about problems.

The next two chapters are devoted to "processes of the first and of the second order," respectively, terms borrowed from the chemist's usual nomenclature. In plain mathematical language, these chapters treat the simple but all-important equations of the form

$$\frac{dx}{dt} = kx \tag{1}$$

and

$$\frac{dx}{dt} = k(a - x)(b - x), \tag{2}$$

whose integrals are described in a variety of ways and illustrated by over fifty practical examples.

In Chapter IV several types of simultaneous processes are treated, the corresponding mathematical problems being largely reducible to solutions of (1) and (2) and, in a few cases, to slightly more complicated equations, or rather evaluation of quadratures. The so-called "consecutive processes" offer a good opportunity for treating the linear equation

$$\frac{dx}{dt} + Px = Q,$$

where P, Q are functions of t. The subject is again illustrated by copious examples and exercises (problems, with hints).

Chapter V, equations of flow, acquaints the chemical reader with the one-dimensional case of Fourier's partial differential equation (heat conduction), *i.e.*,

$$\frac{\partial f}{\partial t} = a^2 \frac{\partial^2 f}{\partial x^2},$$

with a constant, and its integration by series. This chapter seems particularly promising with regard to its educative value.

The last chapter, VI, treats of the graphical evaluation of integral expressions, and is followed by an appendix containing forty-five miscellaneous problems.

The booklet, carefully printed and written in an agreeable style, can be warmly recommended not only to chemists, but to mathematical beginners of any other vocation.

L. SILBERSTEIN.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 30, no. 1–2, January–Febuary, 1924: " Note on stability à la Poisson " by F. H. Murray, 17–18; " Applicability with preservation of both curvatures " by W. C. Graustein, 19–23; " Complete sets of representations of two-element algebras " by B. A. Bernstein, 24–30; " Brouwer's contributions to the foundations of mathematics " by A. Dresden, 31–40; " A qualitative definition of the potential functions " by P. Franklin, 41–50; " On the location of the roots of polynomials " by J. L. Walsh, 51–62.

JOURNAL OF THE FRANKLIN INSTITUTE, volume 197, no. 4, April, 1924: " On stresses in a plate with a circular hole " by S. Timoshenko, 505–516.

JOURNAL OF MATHEMATICS AND PHYSICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, volume 3, no. 2, March, 1924: " In memory of Joseph Lipka " by N. Wiener, 63–65; " The coincident points of two algebraic transformations " by F. L. Hitchcock, 66–71; " The quadratic variation of a function and its Fourier coefficients " by N. Wiener, 72–94; " Note on operational calculus " by V. Bush, 95–107; " Note on quantization of non-conditioned-periodic systems " by M. S. Vallarta, 108–117; " A contribution to the generalization of a determinantal theorem of Frobenius " by L. H. Rice, 118–126.—No. 3, April, 1924: " The Dirichlet problem " by N. Wiener, 127–146; " On tautochronous motion " by S. D. Zelden, 147–161; " The theory of testimony " by J. S. Taylor, 162–173; " Notes on dynamical systems non-integrable by separation of variables and on the existence of ' unmechanical ' orbits in the atom " by M. S. Vallarta, 174–181; " Note on a property of rectilinear lines of principal stress " by P. Heymans, 182–185; " Note on the curl " by W. H. Ingram, 186–187; " Note on a method of evaluating the complex roots of a quartic equation " by W. V. Lyon, 188–190.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 10, no. 4, April, 1924: " Electrodynamics of the general relativity theory " by G. Y. Rainich, 124–127; " Principal directions in an affine-connected manifold of two dimensions " by J. L. Synge, 127–129; " A statistical discussion of sets of precise astronomical measurements" by E. B. Wilson and W. J. Luyten, 129–132; " The quantum theory of the Fraunhofer diffraction " by P. S. Epstein and P. Ehrenfest, 133–139.—No. 5, May, 1924: " Prime power substitution groups whose conjugate cycles are commutative " by G. A. Miller, 166–167; " An extension of the theorem that no countable point set is perfect " by R. L. Moore, 168–170; " Concerning the prime parts of certain continua which separate the plane " by R. L. Moore, 170–175; " Concerning the division of the plane by continua " by J. R. Kline, 176–177.

SCHOOL SCIENCE AND MATHEMATICS, volume 24, no. 204, April, 1924: " Classroom devices in teaching algebra and geometry " by J. A. Nyberg, 345–349; " High fifth mathematics " by J. L. Green, 366–369; " Objections to mathematics " by A. J. Cave, 376–381.—May, 1924: " Where shall I place that decimal point? " by B. C. Zimmerman, 507–508; " Historical note on the solution of equations " by G. A. Miller, 509–510.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 25, no. 4, October, 1923: " Generalized limits in general analysis. Second paper " by C. N. Moore, 459–468; " The equilong transformations of Euclidean space " by B. H. Brown, 469–484; " Invariant sets of equa-

tions in Riemann space" by P. Franklin, 485–500; "Some properties of spherical curves, with applications to the gyroscope" by O. D. Kellogg, 501–524; "The greatest and the least variate under general laws of error" by E. L. Dodd, 525–539; "The intersection numbers" by O. Veblen, 540–550; "The geometry of paths" by O. Veblen and T. Y. Thomas, 551–608.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

Phi Chi Mu of Washington and Jefferson College, Washington, Pennsylvania.

In January 1920, Phi Chi Mu was established as an honorary fraternity to promote interest and stimulate activity in the study of mathematics, physics, chemistry and biology. The idea originated with two senior students majoring in mathematics and these met with the hearty coöperation of Professor C. S. Atchison. The heads of the departments of these subjects are honorary members and act as counsellors. Instructors are also elected as honorary members. The student membership is limited to twelve honor men majoring in mathematics, physics, chemistry or biology.

The officers for the year 1923–24 were H. L. Dorwart '24, president, and D. H. Rosenberg '24, secretary-treasurer. Most of the club members are majoring in mathematics, and consequently most of the papers presented are on mathematical subjects. Recent papers by undergraduate members were

1. "The philosophy of mathematics" by C. E. Lowery '24.
2. "The theory of parallels" by Boyd C. Patterson '23.
3. "The derivation and meaning of the Lorentz transformation in relation to Einstein's relativity" by H. L. Dorwart '24.
4. "The value of research" by D. H. Rosenberg '24.
5. "The life and work of Archimedes" by R. A. Klieves '24.

(Report by Professor C. S. Atchison.)

Pi Mu Epsilon of the University of Alabama.

[*1923*, 144.]

About a year and a half ago the Newtonian Club of the University of Alabama became a chapter of the Pi Mu Epsilon mathematical fraternity. The officers for the year 1923–24 are: President, George Shelton '24; secretary-treasurer, Daniel Clark '25; faculty director, Professor Tomlinson Fort. Meetings were held the last Tuesday in each month.

The programs were as follows:

October: Organization and business.

November: "Mathematics taught in the colleges of engineering," a statistical report with letters from many engineers, Professor Fort, Professor Dahlene, Professor Lewis, and George Shelton '24.

December: "Calculation of logarithms," Edward Duffy '25, "Slide rule," Professor Dahlene, "Trilinear coördinates," W. S. Ernst '24.

January: "Egyptian and Babylonian mathematics," Lucille Ellenburg '24, "Mathematics in literature," Corinne Alexander, '25, "History of mathematics in America," Professor Fort.

February: Address by Professor H. E. Slaught of the University of Chicago.

March: "Cardan and Tartaglia," E. Dany '25, "Solution of the cubic," F. S. Gachet '24, "Trisection of the angle," O. A. Reed, '25, "Mathematics and music," C. E. Comeaux '25.

April: Solution of certain problems from the Monthly, F. S. Gachet '24, "The most pleasing rectangle," Esther Frank '25, "Graphical statics," George Shelton '24.

May: Business.

(Report by Professor Tomlinson Fort.)

THE MATHEMATICS CLUB OF COLUMBIA COLLEGE, New York City.
[*1922*, 77.]

The reorganization of the Mathematics Club of Columbia College, suggested by Walter F. Dantzscher '25, was intrusted by Professor Thomas Scott Fiske, executive officer of the department, to a student committee composed of Hector J. Battaglia '25, Walter F. Dantzscher '25, and Herbert A. Spurway '25. This committee with the approval of the membership continued to exercise its function as executive committee until the elections held in May. There were thirty students carried on the rolls as members. The average attendance was fifteen.

The programs were as follows:

December 17, 1923: Reorganization meeting. Introductory remarks concerning the reorganization of the club, Walter F. Dantzscher '25. "The Mathematics Club of Columbia College in the past," Albert E. Meder, Jr., assistant in mathematics and a former officer of the club. "The elementary theory of determinants," Herbert A. Spurway '25. Walter F. Dantzscher '25 presiding.

February 19, 1924: "Projective vector algebra," Hector J. Battaglia '25. Herbert A. Spurway '25 presiding.

March 4, 1924: "Numbers and the number concept," Walter F. Dantzscher '25. Hector J. Battaglia '25 presiding.

March 18, 1924: "Some curios of my collection," Professor D. E. Smith of Teachers College. Herbert A. Spurway '25 presiding.

April 29, 1924: "Some discontinuous functions," Professor Thomas Scott Fiske, executive officer. Walter F. Dantzscher '25 presiding.

May 13, 1924: Election of officers for the coming year. All elections were unanimous. President, Herbert A. Spurway '25; vice-president, Axel W. Berggren '25; secretary-treasurer, Joseph Ferony '26. A discussion of plans for the coming year also occupied a large part of the meeting, especially insofar as a constitution for the club was concerned.

Dr. G. A. Pfeiffer of the Department kindly acted as advisor to the Executive Committee.

(Report by Walter F. Dantzscher '25.)

THE MATHEMATICAL CLUB, NEW JERSEY COLLEGE FOR WOMEN, New Brunswick, N. J.

The following is the list of officers: President, Henrietta H. Dawson; vice-president and chairman of the program committee, Anne Dymock; secretary and treasurer, Dorothy McFarland.

During the year the following papers were read:

"Pythagoras, his life and works" by Miss Anne Dymock.

"Euclid, his life and works" by Miss Henrietta Dawson.

"Simson, his work in geometry" by Miss Margaret Dietrich.

"Measurement of time" by Professor W. E. Breazeale.

"Original theorems on Simson's line" by Mr. L. M. Paradiso.

"Linear perspective" by Professor A. A. Titsworth.

"The cyclic quadrilateral" by Professor R. Morris.

"Pseudo squares" by Miss Alna Ketterer.

"George Peacock, his life and works" by Miss Elizabeth Baier.

"De Moivre's theorem with applications" by Miss Ruth Thompson.

"Inscribed polygons" by Miss Anna Sebestyjak.

(Report by Professor Richard Morris.)

THE MATHEMATICS CLUB, RUTGERS COLLEGE, New Brunswick, N. J.

The following is the list of officers: President, N. Howard Ayers; vice-president, Robert M. Walter; secretary-treasurer, Wm. H. Mitchell, Jr.; faculty advisor, Professor Richard Morris.

During the year the following papers were read:

"Fixed points and circular loci," Professor Richard Morris.

"Simson's line and correlated theorems," Robert M. Walter.

Report on problems in the *Mathematics Teacher*, N. Howard Ayers.

Reports on various mathematical topics by members.

"Polygons and polygrams," Richard H. Cundy.

"De Moivre's theorem with applications," Addison Mallery.

"Application of trigonometric series," Simon Heimlich.

"Maxwell's law for the development of the distribution of the velocities of molecules in a gas," Charles J. Brasefield.

(Report by Professor Richard Morris.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3089. Proposed by NORMAN ANNING, University of Michigan.

Given four points, O, A, B, C, on a straight line, to construct, with straightedge only, the point P on the line such that OP shall be the harmonic mean of OA, OB, OC.

3090. Proposed by H. S. UHLER, Yale University.

Show that the volume of the (smallest) segment of a sphere (radius $= c$) cut out by two mutually perpendicular planes, the distances of which from the center are a and b respectively, may be expressed by the formula

$$\frac{2}{3}c^3\cos^{-1}\left[\left(\frac{a}{\sqrt{c^2-b^2}}\right)\left(\frac{b}{\sqrt{c^2-a^2}}\right)\right]-\frac{1}{3}b(3c^2-b^2)\cos^{-1}\left(\frac{a}{\sqrt{c^2-b^2}}\right)$$
$$-\frac{1}{3}a(3c^2-a^2)\cos^{-1}\left(\frac{b}{\sqrt{c^2-a^2}}\right)+\frac{2}{3}ab\sqrt{c^2-(a^2+b^2)}.$$

[*Note.* This formula may be used as major part of an alternative solution of problem **2947** [*1922*, 29] as given by J. B. Reynolds [*1923*, 209].]

3091. Proposed by PHILIP FITCH, Denver, Colorado.

The top of a grain hopper is in the form of a square whose side is 10 feet. The sides of the hopper are portions of right circular cylindrical surfaces, 20 feet in diameter. If the cylindrical surfaces meet the plane of the top at right angles and if the hole in the bottom of the hopper is one foot square, what is the volume of the hopper and how many bushels of grain will be taken out by lowering the level of the grain one foot?

3092. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

What must be the relation between the coefficients of a cubic equation in order that its roots, considered as lengths, shall form a triangle?

3093. Proposed by FRANK MORLEY, Johns Hopkins University.

Show that the equation

$$y=(a_0+a_1x+\cdots+a_nx^n)/(b_0+b_1x+\cdots+b_nx^n)$$

gives the differential equation

$$\begin{vmatrix} y_1, & y_2, & \cdots y_{n+1} \\ y_2, & y_3, & \cdots y_{n+2} \\ \cdots & \cdots & \cdots \\ y_{n+1}, & y_{n+2}, & \cdots y_{2n+1} \end{vmatrix}=0,$$

where $y_r=(D_x{}^r y)/r!$.

3094. Proposed by A. A. BENNETT, University of Texas.

Given the set of numbers, a_{nr}, of double subscripts, $n = 1, 2, \cdots, r = 0, 1, 2, \cdots, n$, where $a_{10} = 1$, $a_{11} = 1$, $a_{20} = 1$, $a_{21} = 2$, $a_{22} = 2$, and in general thereafter $a_{n+1,\,r} = a_{nn} + a_{n,\,n-1} + \cdots + a_{n,\,n-r}$, where $a_{n,\,-1}$ may be regarded as equal to zero, and therefore $a_{n+1,\,0} = a_{nn} = a_{n,\,n-1}$. Show that a_{nn} is alternately odd and even. Note that for the first few even n's, a_{nn} is divisible

by a power of 2 of exponent comparable with n, *e.g.*, $a_{44} = 2^4$, $a_{66} = 2^4(1 + 2^4)$, $a_{88} = 2^8(1 + 2 + 2^2 + 2^3 + 2^4)$, $a_{10,\,10} = 2^{10}$ times an odd number, $a_{12,\,12} = 2^{12}(1 + 2^2 + 2^4 + 2^6 + \cdots + 2^{12})$, $a_{16,\,16} = 2^{16}$ times an odd number. Determine if possible an expression for a_{nn} in finite form, for at least even values of n, or failing in this, determine an asymptotic expression for its magnitude. In any case give $a_{24,\,24}$ explicitly. These numbers arise in connection with a counting of types of polynomials.

SOLUTIONS.

2959 [1922, 129]. Proposed by J. H. M. WEDDERBURN, Princeton University.

Solve the functional equation $[g(x)]^2 = -2x + g(x^2)$.

NOTE BY THE EDITORS.

See the paper by the proposer in the *Annals of Mathematics*, Dec., 1922, pp. 121–140, entitled "The functional equation $g(x^2) = 2\alpha x + [g(x)]^2$."

3030 [1923, 275]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the envelope of the bisector of the angle that a given segment subtends at a variable point of a given line.

NOTE BY THE EDITORS: See solution of **3035** which follows. Other solutions of **3030, 3035** have been printed [1924, 312].

3035 [1924, 49]. Proposed by R. M. MATHEWS, Wesleyan University.

Generalize projectively and prove that the envelope of the bisectors of the angles between corresponding lines of two perspective pencils is a curve of the third class.

SOLUTION BY MABEL M. YOUNG, Wellesley College.

Let the corresponding rays of two perspective pencils P_1 and P_2 meet at Q on line q. Let the common ray be p. The external and internal bisectors of angles P_1QP_2 determine on p pairs of points of an involution of which P_1 and P_2 are double points. On the segments determined by these point-pairs as diameters construct circles. These form a coaxial system, S, and hence cut q also in pairs of points of an involution.

Let the bisectors of angle $P_1Q_1P_2$ meet p in B_1B_2. Since the bisectors are perpendicular, Q_1 lies on that circle of S which has diameter $\overline{B_1B_2}$. Let circle meet q again in Q_2. Since $\overline{Q_2B_1}$ and $\overline{Q_2B_2}$ are perpendicular and divide harmonically $\overline{Q_2P_1}$ and $\overline{Q_2P_2}$, they are bisectors of $P_1Q_2P_2$. Hence the bisectors of the angles with vertices at a pair of the involution on q pass by twos through a pair of the involution on p, and the four points lie on one circle of S. The involutions on p and q are then related in the following manner: To each of a pair of conjugate points on q corresponds one and the same pair of points in the involution on p. To each of a pair of conjugate points on p corresponds one and the same pair (real or imaginary) of the involution on q. Hence the two involutions are projective and corresponding pairs lie on the same circle of the system S. Since the circle on the point of intersection of p and q and its conjugate on p determines only one new point on q, this intersection is a self-corresponding point in the involutions.

Each point-pair in the involutions may be considered as a degenerate line conic. The two ranges in involution are then two projective pencils of line conics. The four lines joining the points of two corresponding pairs in the involutions become the common lines of corresponding conics of the pencils. By the theory[1] of the projective generation of curves these lines envelope a curve of class four with two double lines. Since however the intersection of the ranges is a self-corresponding point in the involutions, the four tangents which can be drawn to the curve from any point of the plane include three proper tangents, and a line through the self-corresponding point. The curve accordingly breaks into this point and a curve of class three to which the bases of the ranges are simple tangents.

We may then conclude that the external and internal bisectors of angles formed by corresponding rays of two perspective pencils determine on the axis of perspective and on the common ray corresponding point-pairs of two projective involutions with a self-corresponding element and hence envelope a curve of the third class.

3037 [1923, 337]. Proposed by E. O. BROWER, Chicago, Illinois.

The angles A, B, C of a triangle are given. A logarithmic spiral is tangent to AB at B and to AC at C; at what angle does the curve cut each radius vector?

---

[1] Fiedler, *Darstellende Geometrie*, vol. III, B, pp. 37–45.

Solution by C. K. Robbins, Purdue University.

The equation of the logarithmic spiral is $r = ke^{a\theta}$. If ψ is the angle between the radius vector for any point and the tangent at that point, then $\tan\psi = rd\theta/dr = 1/a$. Let the inclinations of the tangents at the two points $B(r_1, \theta_1)$ and $C(r_2, \theta_2)$ be φ_1 and φ_2, respectively; and suppose that the tangents intersect in A. Denote the angles of the triangle ABC by A, B, C.

Consider first the case in which $\theta_2 = \beta_2$, $\theta_1 = 2\pi + \beta_1$, where β_1 and β_2 are positive angles less than π. For simplicity suppose $\beta_1 < \pi/2$ and $\beta_2 > \pi/2$. We shall then have

$$\theta_1 - \theta_2 = 2\pi + \beta_1 - \beta_2, \qquad \beta_1 = \varphi_1 - \psi, \qquad \beta_2 = \varphi_2 - \psi + \pi, \qquad \varphi_1 - \varphi_2 = A.$$

Hence

$$\theta_1 - \theta_2 = \pi + A.$$

If the angles OBC, OCB, O being the pole, be denoted by α_1, α_2, respectively, then

$$\alpha_1 = \pi - \psi - B, \qquad \alpha_2 = \psi - C.$$

It now follows that

$$e^{a(\theta_1-\theta_2)} = \frac{r_1}{r_2} = \frac{\sin\alpha_2}{\sin\alpha_1}.$$

Inserting the above values, expanding the right-hand side and simplifying the result, we obtain

$$e^{a(\pi+A)} = \frac{\cos C - a\sin C}{\cos B + a\sin B}.$$

The value of a may be found by solving this equation by the usual approximation process when the values of A, B, C are given. An examination of the general case would show that multiples of π should be introduced in several of the above equations for the angles and also a number of $\pm$ signs should be placed before several of the angles. The resulting equation would be of the same type as the above with $\pm$ signs before the angles and with π replaced by $n\pi$ where n is an integer.

Also solved by W. B. Carver, William Hoover, and J. B. Reynolds.

3041 [1923, 402]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two fixed intersecting planes on a variable line passing through a fixed point.

I. Solution by Grace M. Bareis, Ohio State University.

(*a*) Let the two fixed planes and any plane through the fixed point U be chosen as the reference planes $x = 0$, $z = 0$, $y = 0$ for a system of oblique Cartesian coördinates. Let the coördinates of U be $(a, 0, c)$. Any line through U is

$$\frac{x-a}{l} = \frac{y}{m} = \frac{z-c}{n},$$

where l, m, n are the direction ratios of the line. This line intersects $x = 0$ in a point

$$S\left[0, -\frac{am}{l}, -\frac{an}{l} + c\right]$$

and it intersects $z = 0$ in a point

$$T\left[-\frac{cl}{n} + a, -\frac{cm}{n}, 0\right].$$

Let $P(x, y, z)$ be the mid-point of ST; then

$$2x = a - \frac{cl}{n}, \qquad 2y = -m\left(\frac{a}{l} + \frac{c}{n}\right), \qquad 2z = c - \frac{an}{l}.$$

Eliminating l, m, n, we have as the locus of P $2xz = cx + az$, a hyperbolic cylinder through U, having the intersection of the fixed planes as an element, and having $z = \frac{1}{2}c$ and $x = \frac{1}{2}a$ as asymptonic planes.

(*b*) This problem is a special case of the following more general theorem: *Given a quadric surface S, a fixed plane ρ, and a fixed point U that is not the pole of ρ with respect to S; then any line a through U meets ρ in a point A whose polar plane α passes through R the pole of ρ.* α meets a in a

point P conjugate to A with respect to S. Now it is evident that $U(a, b, c, \cdots) \,\overline{\wedge}\, \rho(A, B, C, \cdots) \,\overline{\wedge}\, R(\alpha, \beta, \gamma \cdots)$. Therefore, $U(a, b, c, \cdots) \,\overline{\wedge}\, R(\alpha, \beta, \gamma, \cdots)$ and therefore all points such as $a\alpha$, $b\beta$, $c\gamma$, $\cdots$ lie on a quadric surface, *i.e.*, the locus of P is a quadric surface. This locus passes through U and R, through the points common to ρ and S, and through the points of contact of tangents from U to S if such points exist.

If ρ is the plane at infinity, then P is the conjugate of an infinitely distant point and is therefore the mid-point of the chord through U, R is the center of S, and we have the theorem: *The locus of the mid-points of all chords of a quadric surface that pass through a fixed point is another quadric surface through the fixed point and the center of the given quadric.*

Again if ρ is the plane at infinity and S is a degenerate quadric, a pair of intersecting planes, we have the theorem: *The locus of the mid-point of the segment determined by two fixed intersecting planes on a variable line through a fixed point is a quadric surface through the fixed point and the line of intersection of the two planes, this line being the line of centers of the degenerate quadric S.* Moreover, this quadric passes through the points common to ρ and S, that is, through the lines at infinity in the two planes. The locus is a hyperbolic cylinder having its asymptotic planes parallel to the given planes. [v. Reye, *Geometrie der Lage* (1882), vol. I, p. 101, and vol. II, p. 35.]

II. Solution by Mabel M. Young, Wellesley College.

Let a pencil of planes π be passed through that line p on the fixed point P which is parallel to the intersection l of the fixed planes. Each plane π cuts the given planes in two lines parallel to l. A third parallel x midway between these two lines is the locus of the mid-points of the segments determined by the fixed planes on all the lines through P which lie in π. The infinitely distant line in π is the harmonic conjugate of x as to the intersections of π with the fixed planes. Hence the planes $\lambda(l, x)$ and $\pi(p, x)$ meet the infinitely distant plane in lines which are harmonic conjugates in an involution of parallel lines, and the two pencils of planes are accordingly projective. Since the axes of these projective pencils of planes are parallel, the required locus is a cylinder.

The section of the fixed planes by any other plane on P gives a triangle bounded by two fixed lines and the infinitely distant line. The mid-point of the segment determined on any line through P by the sides of the triangle in the finite part of the plane is the fourth harmonic of the infinitely distant point in which it meets the third side. By a known theorem, the locus of the fourth harmonic of the three points in which the sides of a triangle are met by a variable line on a point in its plane is a conic through the vertices of the triangle and the fixed point. In this case the conic has two distinct points at infinity. Hence, the section of the required locus by any plane on P not parallel to the intersection of the fixed planes is an hyperbola through P. The locus is, accordingly, a hyperbolic cylinder through the intersection of the fixed planes and the fixed point. The given planes determine the direction of the asymptotic planes of the cylinder.

Also solved by William Hoover, H. Halperin, R. A. Johnson, A. Pelletier, J. B. Reynolds, and C. K. Robbins.

Note by the Editors: See **2904** [*1923*, 338] for the problem in the plane corresponding to the case of ρ as the plane at infinity. There it is shown that the locus is similar to the original curve.

3042 [1923, 402]. Proposed by C. N. SCHMALL, New York City.

Given the bases and the sum of the areas of several triangles that have a common vertex; show that the locus of the vertex is a straight line.

Solution by L. R. Ford, The Rice Institute.

We shall interpret the problem to mean that all areas are to be considered positive and that no deduction is to be made for overlapping areas.

Let n be the number of triangles. The n straight lines on which the bases lie divide the plane into regions, not exceeding $\frac{1}{2}(n^2 + n + 2)$ in number, and the locus is different according as the common vertex lies in one region or another. Designate the regions by R_1, R_2, etc.

Let $d_1, d_2, \cdots, d_n$ be the lengths of the bases; and let

$$a_ix + b_iy + c_i = 0 \tag{1}$$

be the line on which the ith base lies. We shall suppose that this equation is in Hesse's normal

form, and that for a point (x, y) in one of the regions, R_1 say, the first member is positive. Then the first member is equal to the perpendicular distance from (x, y) to the line.

Let $P(x, y)$ be a point on the locus in R_1. Then the area of the ith triangle is

$$L_i(x, y) = \tfrac{1}{2}d_i(a_ix + b_iy + c_i), \tag{2}$$

and the equation of the locus is

$$L_1(x, y) + L_2(x, y) + \cdots + L_n(x, y) = A, \tag{3}$$

where A is the given constant area. Since this is linear in x and y, the locus is a straight line; and that portion of the line, if any, lying in R_1 is a part of the locus.

Pass now into an adjacent region R_2, by crossing the line of, say, the first base. In R_2 $L_1(x, y)$ is negative, and to keep the area positive we must change its sign. We have then another line

$$-L_1(x, y) + L_2(x, y) + \cdots + L_n(x, y) = A. \tag{4}$$

Since $L_1(x, y) = 0$ is a consequence of (3) and (4), it follows that these two lines meet on the common boundary of the two regions.

The locus for any region R_j is easily written down by merely changing the signs of certain terms in (3). If R_1 and R_j are on opposite sides of the line $L_i(x, y) = 0$, then the sign preceding $L_i(x, y)$ in (3) is changed from plus to minus. We get the equation of a straight line, and that part, if any, lying in R_j belongs to the locus.

At the boundary where a segment of the locus lying in one region terminates the segment lying in the adjacent region begins, so that we get for the locus a broken-line figure. We shall prove that *with the exceptions to be presently noted, these line segments form a closed convex polygon.*

The first exception arises in connection with an infinite region, where conceivably the locus may extend to infinity. However, as P moves along the line to infinity the area of each triangle increases without limit unless the locus is parallel to the base. Hence, this case can arise only if the bases lie on parallel lines.

A second exception occurs when the equation of the locus reduces to an identity; then all points of the region belong to the locus. The simplest example of this is the case in which the bases form the sides of a convex polygon and A is equal to the area of the polygon.

Putting aside these special cases we see that the locus consists of one or more closed broken-line figures. (The closure follows from the fact that each line terminates in others at its ends and that their number is finite.) That the locus consists of a single convex polygon will follow from the following proposition: *No three points on the locus which lie in different regions lie on a line.*

Suppose the contrary. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be collinear, P_3 lying on the segment P_1P_2. Choose the signs of the L's so that at P_3 all the terms of (3) are positive. Let $M_1(x, y)$ be the sum of the terms which are positive also at P_1 and P_2; let $M_2(x, y)$ be the sum of those which are positive at P_1 and negative at P_2; and let $M_3(x, y)$ be the sum of those which are positive at P_2 and negative at P_1. There are no terms negative at both P_1 and P_2, since no term can change sign twice as a point (x, y) moves along the line from P_1 to P_2. Substituting the coördinates of the points in the respective equations of the locus, we have

$$\left\{\begin{aligned} M_1(x_1, y_1) + M_2(x_1, y_1) - M_3(x_1, y_1) &= A, \\ M_1(x_2, y_2) - M_2(x_2, y_2) + M_3(x_2, y_2) &= A, \\ M_1(x_3, y_3) + M_2(x_3, y_3) + M_3(x_3, y_3) &= A, \end{aligned}\right. \tag{5}$$

where all the M's are positive. Now x_3 and y_3 can be written in the form

$$x_3 = lx_1 + mx_2, \qquad y_3 = ly_1 + my_2,$$

where l and m are positive and $l + m = 1$. Since each of the M's is linear we have

$$\begin{aligned} M(x_3, y_3) &= \alpha x_3 + \beta y_3 + \gamma \\ &= \alpha(lx_1 + mx_2) + \beta(ly_1 + my_2) + \gamma \\ &= lM(x_1, y_1) + mM(x_2, y_2). \end{aligned}$$

The third equation of (5) can then be written

$$l[M_1(x_1, y_1) + M_2(x_1, y_1) + M_3(x_1, y_1)] + m[M_1(x_2, y_2) + M_2(x_2, y_2) + M_3(x_2, y_2)] = A.$$

Multiplying the first equation of (5) by l and the second by m and subtracting both from the third, we have

$$2lM_3(x_1, y_1) + 2mM_2(x_2, y_2) = 0.$$

Since the terms in the first member are negative this equation is impossible,[1] and the proposition is established.

Two further facts may be noted. Parts of the locus lying in two regions on opposite sides of all the bases are parallel. For, the terms in the first members of the two equations all differ in sign and the coefficients of x and y in one case are therefore proportional to those in the other. By similar reasoning, the parts of the loci in a given region for various values of A are parallel.

The problem can be interpreted in mechanical terms. If each base represent a vector force so directed that its moment about a point in a given region is positive, then it is required that the sum of the moments of the forces about P be $2A$. Replacing the several forces by their resultant, we see that P must lie on a line parallel to the line of action of the resultant. The exceptional case in which the locus is an area will occur when the forces are equivalent to a couple, provided A has a suitable value.

Solutions of the problem in the sense intended by the PROPOSER were received from H. HALPERIN, WILLIAM HOOVER, R. A. JOHNSON, S. A. LUBKIN, A. PELLETIER and C. K. ROBBINS.

3043 [1923, 403]. Proposed by O. D. KELLOGG, Harvard University.

Let T denote an open continuum of the xy-plane, say the interior of a smooth simple closed curve. Then if U is continuous in T, and is such that

$$\iint_T U\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)dxdy = 0 \tag{1}$$

for all functions V with continuous second derivatives and vanishing on the boundary of T, U is harmonic, *i.e.*, satisfies Laplace's equation $(\partial^2U/\partial x^2)+(\partial^2U/\partial y^2)=0$. (It is understood that if U becomes infinite in the neighborhood of a boundary point of T, V is so further restricted that the integral shall have a sense.)

SOLUTION BY H. E. BRAY, Rice Institute.

Let P be any interior point of T. With center at P draw a small circle of radius r, C_r, entirely interior to T.

Let φ be a function defined as follows: $\varphi = 1$ inside C_r, $\varphi = 0$ otherwise.

For our function V, we shall take the approximating (average) function:

$$\varphi^{\nu\nu\nu}(x, y) = \frac{1}{(\pi\nu^2)^3}\iint_{C_\nu} d\xi''d\eta''\iint_{C_\nu} d\xi'd\eta'\iint_{C_\nu}\varphi(x+\xi+\xi'+\xi'', y+\eta+\eta'+\eta'')d\xi d\eta,$$

where C_ν represents the interior of the circle $\xi^2+\eta^2=\nu^2$.

φ being integrable and bounded, it follows that $\varphi^{\nu\nu\nu}$ is continuous with its second derivatives. For the function

$$\varphi^\nu(xy) = \frac{1}{\pi\nu^2}\iint_{C_\nu}\varphi(x+\xi, y+\eta)d\xi d\eta$$

is continuous and

$$\frac{\partial\varphi^{\nu\nu}}{\partial x} = \frac{1}{\pi\nu^2}\int_{C_\nu}\varphi^\nu dy, \qquad \frac{\partial\varphi^{\nu\nu\nu}}{\partial x} = \frac{1}{\pi\nu^2}\int_{C_\nu}\varphi^{\nu\nu}dy.$$

The continuity of $\partial\varphi^{\nu\nu}/\partial x$ follows from the first of these equations and that of $\partial^2\varphi^{\nu\nu\nu}/\partial x^2$ from the last.

Moreover $\varphi^{\nu\nu\nu}$ is identically zero outside of a circle with center at P and radius equal to $r+3\nu$; and $\lim_{\nu=0}\varphi^{\nu\nu\nu}=\varphi$ inside and outside C_r; on C_r $\lim\varphi^{\nu\nu\nu}=\frac{1}{2}$. Thus $\varphi^{\nu\nu\nu}$ satisfies the conditions required of V provided ν is sufficiently small.

[1] $M_3(x_1, y_1)$ and $M_2(x_2, y_2)$ are zero if and only if P_1 and P_2 lie on the boundary of the region in which P_3 lies and at the termini of the segment of the locus through P_3. The three points could be considered as lying in the same region; so obviously this case should be ruled out.

Instead of U consider the approximating function $U^{\mu\mu}$.

$$U^{\mu\mu}(xy) = \frac{1}{(\pi\mu^2)^2}\iint_{C_\mu} d\xi' d\eta' \iint_{C_\mu} U(x+\xi+\xi', y+\eta+\eta')d\xi d\eta.$$

Thus $U^{\mu\mu}$ is continuous with its second derivatives. Now

$$\iint_T U^{\mu\mu}\nabla^2\varphi^{\nu\nu\nu}dxdy = \frac{1}{(\pi\mu^2)^2}\iint_{C_\mu} d\xi' d\eta' \iint_{C_\mu} d\xi d\eta \iint_T U(x+\xi+\xi', y+\eta+\eta')\nabla^2\varphi^{\nu\nu\nu}dxdy,$$

and since $\nabla^2\varphi^{\nu\nu\nu}$ vanishes identically outside the circle with center P and radius $r+3\nu$ it follows, from our hypothesis, that if μ is sufficiently small,

$$\iint_T U(x+\xi+\xi', y+\eta+\eta')\nabla^2\varphi^{\nu\nu\nu}dxdy = 0,$$

for all (ξ, η), (ξ', η') such that $\xi^2+\eta^2 \leqq \mu^2$, $\xi'^2+\eta'^2 \leqq \mu^2$.
Hence,

$$\iint_T U^{\mu\mu}\nabla^2\varphi^{\nu\nu\nu}dxdy = 0.$$

We can now apply Green's Theorem to $U^{\mu\mu}$ and $\varphi^{\nu\nu\nu}$. Since $\varphi^{\nu\nu\nu}$ vanishes with all its derivatives everywhere outside the circle with center at P and radius equal to $r+3\nu$, we obtain

$$\iint_T \nabla^2 U^{\mu\mu}\varphi^{\nu\nu\nu}dxdy = 0.$$

Now let ν approach zero; $\varphi^{\nu\nu\nu}$ approaches φ and

$$\lim \iint_T \nabla^2 U^{\mu\mu}\varphi^{\nu\nu\nu}dxdy = \iint_{C_r} \nabla^2 U^{\mu\mu}dxdy = 0,$$

and since C_r is a circle of arbitrarily small radius

$$\nabla^2 U^{\mu\mu} = 0$$

at P, *i.e.*, $U^{\mu\mu}$ is harmonic at every interior point of T.

Apply the average value theorem to $U^{\mu\mu}$. For any circle C_a of small radius about any point P of T as center

$$U^{\mu\mu}(P) = \frac{1}{2\pi a}\int_{C_a} U^{\mu\mu}ds.$$

Let μ approach zero. Since U is continuous $U^{\mu\mu}$ approaches U uniformly on C_a. Hence

$$U(P) = \frac{1}{2\pi a}\int_{C_a} Uds,$$

i.e., the average value of U on the circumference of any sufficiently small circle inside T is equal to the value at the center. Hence by the definition of $U^{\mu\mu}$ we see that $U^{\mu\mu}$ and U are equal if 2μ is less than the distance of the point P from the boundary of T. U is therefore harmonic.

N. B. It can be proved similarly, by considering the function $U^{\mu\mu\mu}$ instead of $U^{\mu\mu}$, that the theorem remains true, with a certain reservation, if we assume that U is summable in the Lebesgue sense without being continuous. In fact U may then have removable discontinuities; but when these are removed, by redefinition at points of a set of measure zero, U becomes harmonic.

Also solved by NORBERT WIENER and the PROPOSER. Their solutions are different from the above, but agree with each other in using Koebe's converse of Gauss' theorem, to the effect that if U is the arithmetic mean, at each point, P, of T, of its values on the circumference of every circle with center at P and lying in T, then U is harmonic. The desired result is then obtained by what amounts to the use of the DuBois-Reymond form of the fundamental lemma of the Calculus of Variations.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, 22 East 38th St., New York City.

Miss KATHERINE S. ARNOLD, for the past three years professor of mathematics at Constantinople Woman's College, has been appointed a member of the administrative staff of the American Association of University Women at the National Headquarters, Washington, D. C. Miss Arnold will assume her new duties in September, 1924.

Adjunct Professor P. M. BATCHELDER, of the University of Texas, has been appointed acting assistant professor of mathematics at Brown University for the academic year 1924–1925.

Mr. D. H. MACPHERSON, of Brown University, has been appointed instructor of mathematics at the Brooklyn Polytechnic Institute.

At Cornell University, Assistant Professor W. A. HURWITZ has been promoted to a full professorship of mathematics. Professor C. F. CRAIG was on leave of absence for the second semester of 1923–1924, and Professor D. C. GILLESPIE has been granted leave of absence for the academic year 1924–1925.

Dr. D. S. MORSE, of Cornell University, has been appointed assistant professor of mathematics at Union College.

Assistant Professor W. L. G. WILLIAMS, of Cornell University, has been appointed assistant professor of mathematics at McGill University.

At Wells College, Professor T. R. HOLLCROFT has been granted leave of absence for the first semester of 1924–1925, and will go to Italy to study at the University of Rome. Miss EVELYN T. CARROLL has been promoted to an assistant professorship and will be acting head of the department during the absence of Professor Hollcroft. Miss FRANCES THOMAS has been appointed instructor of mathematics for the year 1924–1925.

Dr. R. F. BORDEN, of Brown University, has been appointed assistant professor of mathematics at George Washington University, Washington, D. C.

Miss NELLIE C. STOKES, of Brown University, has been appointed instructor of mathematics and English at Coker College, Hartsville, S. C.

Miss LESLIE GAYLORD, of Agnes Scott College, Decatur, Georgia, has been promoted to an assistant professorship of mathematics.

At Shorter College, Rome, Georgia, Professor RUBY U. HIGHTOWER has been granted leave of absence for the academic year 1924–1925, which she will spend in study at the University of Missouri. Miss EDNA ROBINSON, of the University of Missouri, has been appointed to fill the vacancy for the year.

Miss JULIA L. HAWKINS, associate professor of mathematics at the Oklahoma College for Women, has been appointed Dean of Women.

Professor J. M. HOWIE of Alma College, formerly of the Peru State Normal School, will teach in the summer session at Greeley, Colorado, and in September will assume the headship of the department of mathematics at Nebraska Wesleyan University. He has spent the past year in graduate work at Columbia University.

Mr. E. P. MARTINSON, a graduate of the University of Nebraska, has been appointed instructor in mathematics in the Colorado School of Mines.

Mr. A. E. ANDERSON, a graduate of the University of Nebraska, has been appointed instructor in mathematics in the University of Oklahoma.

The following 28 doctorates with mathematics as major subject were conferred by American universities in the calendar year 1923; the university and the title of the dissertation are given with each name. E. F. Allen, Missouri, "A revision of certain types of the Lie theory;" Constance R. Ballantine, Chicago, "Modular invariants of a binary group with composite modulus;" J. P. Ballantine, Chicago, "A postulational introduction to the four color problem;" A. D. Campbell, Cornell, "The classification of linear systems of conics in various domains;" W. E. Cleland, Princeton, "Permutable transformations F and transformations W;" M. M. Feldstein, Chicago, "Invariants of the linear group, modulo p^r;" C. A. Garabedian, Harvard, "A method of series in elasticity, with applications (1) to circular plates of constant or variable thickness, and (2) to rods of constant or variable circular cross-section;" R. E. Gleason, Princeton, "On a calculus of average value functions;" B. Z. Linfield, Harvard, "On the theory of discrete variates;" N. B. MacLean, Chicago, "On certain surfaces related covariantly to a given ruled surface;" D. S. Morse, Cornell, "Relative inclusiveness of certain definitions of summability;" F. H. Murray, Harvard, "The real solutions of certain systems of differential equations;" H. L. Olson, Chicago, "Congruences with constant absolute invariants;" J. O. Osborn, Cornell, "A study of the rational involutorial transformations in space which leave a web of sextic surfaces invariant;" G. E. Raynor, Princeton, "Dirichlet's Problem;" Emeterio Roa, Michigan, "A number of new generating functions, with applications to statistics;" J. B. Scarborough, Johns Hopkins, "The aerodynamics of a warped airplane wing;" G. E. F. Sherwood, Chicago, "Equivalence of triples of bilinear forms;" C. A. Shook, Johns Hopkins, "The distribution of lift over thin wing sections;" J. M. Thomas, Pennsylvania, "Congruences of circles studied with reference to the surface of centers;" T. Y. Thomas, Princeton, "Geometries of paths admitting first integrals;" C. E. Van Horn, Chicago, "A system of relative existential propositions connected with the relation of class membership;" F. M. Weida, Iowa, "The valuation of life annuities with refund of an arbitrarily assigned part of the purchase price;" Louis Weisner, Columbia, "Groups whose maximum cyclic subgroups are independent;" Miss A. M. Whelan, Johns Hopkins, "The theory of the binary octavic;" R. L. Wilder, Texas, "Concerning continuous curves;" R. E. Wilson, Chicago, "Representations of certain functions of two variables by Stieltjes integrals;" B. F. Yanney, Chicago, "Modular invariants of the binary quartic."

As is known to MONTHLY readers, the NATIONAL RESEARCH COUNCIL is now offering research fellowships in mathematics as well as in physics and chemistry. These fellowships are intended for men of the greatest promise in mathematical research. The Fellowship Board meets four times a year, so that it is possible to make application at any time during the year. The mathematical representatives of the Board which awards the fellowships are Professor G. A. BLISS of the University of Chicago and Professor OSWALD VEBLEN of Princeton University. Applications should be sent directly to Professor W. E. TISDALE, Executive Secretary, Research Fellowship Board, Washington, D. C.

This Board met on April 24, and the following appointments in Mathematics were made: L. M. GRAVES, Ph.D., June, 1924, Chicago; H. LEVY, Ph.D., June, 1924, Princeton; J. H. TAYLOR, Ph.D., June, 1924, Chicago; J. M. THOMAS, Ph.D., 1923, Pennsylvania. Also the following re-appointments in Mathematical Physics: H. ZANSTRA, Ph.D., 1923, Minnesota; T. Y. THOMAS, Ph.D., 1923, Princeton.

Attention is again called to the coöperative relationship between the Association and the *Annals of Mathematics*. This arrangement enables members of the Association to subscribe for the *Annals* at one half the regular subscription rate, namely, $1.50 instead of $3.00. It has long been the hope of the editors of the MONTHLY that financial conditions might eventually warrant the enlargement of the MONTHLY volumes so as to give space for more expository papers. In lieu of this desirable condition, the Association has for several years given a subsidy to the *Annals* in return for which the *Annals* has published numerous and extensive expository papers and has also given to Association members the special half-price subscription privilege. It is soon to publish an eighty page exposition of the Lie Theory of Differential Equations by Professor L. E. DICKSON. Reprints of such expository papers are for sale by the *Annals*. The cost for this particular one will doubtless be well toward that of a subscription for an entire year at the special rate to Association members. Subscriptions should be sent directly to the *Annals of Mathematics*, Princeton, N. J., the volume beginning with the issue for Septembes of each year.

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to D. C. GILLESPIE, Cayuga Heights, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

The following are dates of Section meetings of the Association in 1924 (unless otherwise specified):

ILLINOIS, Elgin, May 2–3

IOWA, Iowa State College, Ames, May 2–3

KANSAS, Topeka, February 2

KENTUCKY, Center College, April

MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA Annapolis, December 8, 1923

MICHIGAN, Ann Arbor, April 3

MINNESOTA, Hamline University, St. Paul, May 24

MISSOURI, Kansas City, November or December

OHIO, Ohio State University, Columbus, April 4–5

ROCKY MOUNTAIN, Laramie, April, 1925

SOUTHEASTERN, University of Georgia, Athens March 7–8

TEXAS, San Antonio, November 29–30

The American Mathematical Monthly

No reprints of articles are furnished to contributors free of charge. Two or three copies of an issue are, however, sent to contributors of longer articles. Orders for reprints, at rates indicated below should be made when returning galley proof to the Editor-in-Chief.

	4pp.	8pp.	12pp.	16pp.		4pp.	8pp.	12pp.	16pp.
25 copies...	$1.45	$2.50	$3.75	$4.55	100 copies..	$2.35	$4.05	$5.70	$6.80
50 " ...	1.70	3.00	4.45	5.35	150 " ..	2.90	4.95	7.05	8.25
75 " ...	2.05	3.55	5.15	6.00	200 " ..	3.20	5.40	7.70	9.00

Covers extra, on regular stock: 25 copies $1.75, and 1½c for each additional copy.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924
NUMBER 8, OCTOBER

PUBLISHED BY THE ASSOCIATION,
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others

THE MAY MEETING OF THE ILLINOIS SECTION.

The fifth annual meeting of the Illinois Section of the Mathematical Association of America was held in the high school building at Elgin, Illinois, on May second and third, 1924, in conjunction with the Illinois State Academy of Science. There were three sessions. On Friday afternoon, Chairman Comstock called the meeting to order at 2:00. The Friday evening session was a joint meeting of the Academy and of the Illinois Section. Professor Comstock presided at the Saturday morning session, beginning at 9:30.

The attendance was thirty, including the following seventeen members of the Association: A. B. Coble, C. E. Comstock, M. W. Coultrap, D. R. Curtiss, C. D. Garlough, Marie M. Johnson, E. P. Lane, Martha P. McGavock, Bessie I. Miller, E. J. Moulton, Mary W. Newson, C. I. Palmer, H. P. Pettit, Marie Plapp, G. H. Scott, G. T. Sellew, F. E. Wood.

The following officers were elected: E. J. MOULTON, Chairman; E. B. LYTLE, Vice-Chairman; BESSIE I. MILLER, Secretary-Treasurer. By a unanimous vote it was decided to leave the time and place of the next meeting of the Section to the decision of a special committee consisting of the executive committee and Professors Slaught and Scott, with the understanding that the meeting will be separate from that of the Academy of Science.

After the reading of the minutes of the 1923 meeting, the following papers were presented:

Friday afternoon.

(1) "College geometry for teachers" by Professor F. E. WOOD, Northwestern University.

(2) "New proofs in the theory of functions" by Professor D. R. CURTISS, Northwestern University.

(3) "Report on algebraic geometry" by Professor A. B. COBLE, University of Illinois.

Saturday morning.

(4) "Courses in mathematics for freshmen" by Miss MARY BEATY, Illinois College.

(5) General discussion on the paper, opened by Professor A. E. GAULT, Bradley Polytechnic Institute.

Abstracts of papers, numbered as in the above list, follow:

1. In this paper it is argued that a course of mathematical type designed for teachers should

(1) be specific, yet have general applications,
(2) deal with high school algebra and geometry, at least indirectly,
(3) differ from high school courses in content,
(4) coördinate other courses and fill in gaps.

Professor N. A. Court of the University of Oklahoma has developed a course

called college geometry, synthetic in character with mostly euclidean, but some projective theorems. The mimeographed sheets of his course were used as a basis for part of a teachers' course at Northwestern University last year. The latter part of this paper dealt with the lack of training in synthetic geometry in many schools and outlined the course developed by Professor Court.

2. Professor Curtiss' paper suggested definitions and proofs for the integration of functions of a complex variable which, if not new, differ in some respects from the conventional ones. Some demonstrations were simplified by the use of the indefinite integral. Cauchy's integral formula was especially considered. For polynomials this formula is a direct corollary of the divisibility of $f(t) - f(z)$ by $(t - z)$. This proof was extended to the general case without the use of infinite series, and examples were given tending to throw light on the meaning and applicability of the formula. Cauchy's inequality and Liouville's theorem were discussed and the latter was deduced from the former.

3. The National Research Council has authorized the preparation and publication of certain reports on mathematics. Professor Coble's paper was devoted to a discussion of some of the problems considered by a committee in connection with such a report in the field of algebraic geometry.

4. Miss Beaty read a paper which served as an introduction for further discussion on the advantages and disadvantages of unified courses in freshman work. Unified courses seem to present the field of mathematics in a way which develops a thorough understanding, a broader vision and a more comprehensive knowledge than do the separate systems. In her paper she cited as authorities, various statements made by professors throughout the country and all seem to favor the idea of unification especially for students who take only the required amount of mathematics in their college work.

G. H. SCOTT, *Secretary-Treasurer.*

EIGHTH ANNUAL MEETING OF THE KENTUCKY SECTION.

The eighth annual meeting of the Kentucky Section was held at Centre College, Danville, Ky., on Saturday, May 17, 1924. The chairman, Professor A. R. Fehn, of Centre College, presided. Between the morning and afternoon sessions Centre College entertained at a luncheon those in attendance. At the close of the luncheon Dr. Ames Montgomery, president of Centre College, made a few brief remarks welcoming the Section.

Among the twenty-five in attendance were the following six members of the Association: P. P. Boyd, J. M. Davis, H. H. Downing, A. R. Fehn, F. Elizabeth Le Stourgeon, E. L. Rees.

Professor J. M. DAVIS, University of Kentucky, and Professor A. R. FEHN, Centre College, were elected chairman and secretary-treasurer, respectively, for 1924–25. The meeting in 1925 will be held at the University of Kentucky.

The following papers were read:

(1) "Archimedes' solution derived from mechanics" by Dean P. P. BOYD, University of Kentucky.

(2) "Distribution of the zeros of Bessel's functions" by Professor GUY STEVENSON, Georgetown College (by invitation).

(3) "A vector proof of the theorem of Coriolis" by Professor E. L. REES, University of Kentucky.

(4) "An investigation concerning mathematics" by Professor HENRY LLOYD, Transylvania College (by invitation).

(5) "Manipulation versus understanding in mathematics" by Professor W. B. HUGHES, Kentucky Wesleyan College (by invitation).

(6) "Gauss' rule for Easter" by Professor H. H. DOWNING, University of Kentucky.

Abstracts of the papers follow:

1. Professor Boyd's talk was based on the article by Archimedes discovered by Dr. J. L. Heiberg in Constantinople in 1906, and which was published in the *Monist* of April, 1909. This article, with introduction by Professor D. E. Smith, was reprinted by the Open Court Company. An account of Archimedes' mathematical work was given by the speaker and examples of his solutions by means of mechanics were presented. A characteristic demonstration of Cavalieri's was added to illustrate the early origin of those concepts that immediately preceded the introduction of the calculus.

2. Professor Stevenson gave a brief history of Bessel's differential equation and of the Bessel functions. His paper included a short digest of the theorems of Bôcher on the relative positions of the zeros of J_n and J_{n+k}, for $k = 1$ and $k = 2$ and for $2p < k \leqq 2p + 2$. A few graphs of J_n were shown for some special values of the parameter n.

3. Professor Rees gave a new vector proof of the theorem of Coriolis, using the Gibbs' notation. This proof affords a good example of the economy of the vector notation and of the simplicity and brevity of vector methods in the treatment of kinematics.

4. Professor Lloyd gave some results of a questionnaire that he had sent to numerous colleges and universities. The questions related to the student's preparation upon entering college, his interest in his work while in college, the teacher's attitude, and the various features involved. Professor Lloyd has received many answers and in his review of these many interesting things were brought out.

5. Professor Hughes emphasized the importance of mathematics for the training of the faculty of reason. The formulas, rules and tables are the mere tools of the subject and the rapid use of these in manipulation should not be mistaken for an understanding of the underlying principles. If more time were spent in the grades and high school in getting a clear insight into the fundamentals of the subject, students would enter college with a very different attitude. The college teacher can assist by going back and helping the student to understand fundamentals passed over in the lower grades and he can point out to

prospective teachers the importance of securing a correct understanding of these on the part of the pupils.

6. Professor Downing employed Gauss' rule in the determination of the dates of Easter for certain years. The reasons for using the remainders obtained by dividing the year number, the century number, and certain other quantities by various divisors, were given. The two exceptional cases due to the interval between two successive full moons were mentioned and the modifications to the usual form of the rule given.

H. H. DOWNING, *Acting Secretary.*

THE APRIL MEETING OF THE MICHIGAN SECTION.

The first meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on April 3, 1924, in conjunction with the Michigan Academy of Science, with Chairman T. H. Hildebrandt presiding.

The attendance was fifty-seven, including the following thirty-six members of the Association:

N. H. Anning, J. F. Barnhill, H. Blair, P. N. Blessing, C. C. Craig, A. Darnell, B. F. Dostal, W. W. Denton, L. C. Emmons, J. P. Everett, Florence E. Field, P. Field, W. B. Ford, E. F. Gee, J. W. Glover, V. G. Grove, L. A. Hopkins, T. H. Hildebrandt, L. C. Karpinski, D. Kazarinoff, T. Lindquist, C. E. Love, S. W. Mullen, A. N. Nelson, H. L. Olson, W. H. Pearce, O. J. Peterson, L. C. Plant, V. C. Poor, C. Reid, C. H. Richardson, L. J. Rouse, T. R. Running, R. C. Shellenbarger, E. R. Sleight, W. J. Thome.

After three papers of the program had been presented a business session resulted in the following transactions:

The minutes of the organization meeting of the Section were read and approved. The secretary reported a section membership of fifty out of a possible seventy-four members of the Mathematical Association from Michigan.

Chairman Hildebrandt presented an invitation from the executive officers of the Michigan Academy of Science for this section to hold its meetings regularly under the auspices of the Academy. The following motion was unanimously adopted:

Resolved that the Michigan Section of the Mathematical Association of America approve of the action of the Executive Committee in accepting the invitation of the Michigan Academy of Sciences to meet with it at this time; that this Section expresses itself as favoring the organization of a Mathematics Section of the Academy, the understanding being that the chairman of the section of the Association be automatically the chairman of the section of the Academy, and that the two bodies meet as one, whenever the meetings of the Association coincide in time and place with those of the Academy.

To comply with the constitutional provisions of the Academy of Science in recognizing new sections, a petition asking for such recognition was drawn up,

signed by about twenty members of the Section, and left in the hands of the chairman to be transmitted to the Academy.

A motion was carried to the effect that the chairman appoint a nominating committee to present names of officers for the ensuing year. The chairman appointed as this committee, W. B. FORD, H. BLAIR, and A. DARNELL. A little later the committee reported, recommending as officers the following persons: Chairman, E. R. SLEIGHT, Albion College, Albion; Secretary-Treasurer, J. P. EVERETT, Western Normal College, Kalamazoo; Member of the Executive Committee, N. H. ANNING, University of Michigan, Ann Arbor. Upon motion, which carried, the secretary was instructed to cast the unanimous ballot of the Section in accordance with the report of the nominating committee, and the persons named therein were declared elected.

The Section adjourned soon after noon and enjoyed a joint luncheon with the Mathematics Section of the Michigan Schoolmasters' Club.

The following papers were presented:

(1) "The theory of sampling" by Professor H. C. CARVER, University of Michigan (by invitation).

(2) "Some configurations associated with a net on a surface" by Professor V. G. GROVE, Michigan Agricultural College.

(3) "On the hatchet planimeter" by Mr. D. KAZARINOFF, University of Michigan.

(4) "Early American arithmetics" by Professor L. C. KARPINSKI, University of Michigan.

(5) "Some problems in yield in land contracts" by Professor L. C. EMMONS, Michigan Agricultural College.

(6) "A theorem on means" by Professor N. H. ANNING, University of Michigan.

Abstracts of papers, numbered as in the above list, follow:

(1) Professor Carver outlined the theory of sampling, of which the theory of least squares is a special case. The development showed that the formula for the probable error of the mean should have the factor $n - 1$ in the denominator instead of n, and that the choice of $n - 1$ was not arbitrary. Moreover, the probable error of the mean of a series of observations that are not symmetrically arranged about their mean is quite as significant as though they lacked skewness. He mentioned the fact that the correct formula for the probable error of the mean is necessarily an infinite series, and has as yet never been developed. This arises from the fact that the numerator of the formula generally employed contains as a factor the so-called dispersion or measure of precision, and that this dispersion is not the dispersion of the sample but rather that of the entire population from which the sample is drawn. In other words, the probable error of the mean is a function of a "second moment," which is itself subject to a probable error. This probable error is a function of higher "moments," and so on ad infinitum. The resulting infinite series should be investigated. Until it can be shown that the series is convergent, we have no exact idea of the nature of

probable error. In fact, if the population from which the sample be drawn is strictly "normal," the distribution of higher moments must be significantly skew, and the problem is almost as difficult as though no restrictions as to skewness were made.

(2) Professor Grove defined the associate conjugate and the reflected associate nets associated with a general net. The usual definition of the ray (axis) of a point with respect to a net was given, thus giving rise to three rays (axes), one for each of the three nets. If any two of these rays (axes) coincide, all three do. There exists one and only one pair of Green reciprocal lines in Green's relation R with respect to the general net. If two of the rays (axes) coincide with the plane (space) component of this pair, then the net is self-dual.

(3) Mr. Kazarinoff explained the mechanical construction and general theory of the hatchet planimeter and showed that it might be employed with a relative error as small as one per cent.

(4) Professor Karpinski illustrated his paper with projections of photographic reproductions taken from early American arithmetics. The first arithmetical work published in the new world was written in Spanish and published at Mexico City by Juan Diez Freyle in 1556. The first Colonial arithmetic appeared in 1705; it was entitled, "*The Secretary's Guide & Young Man's Companion,*" published in New York by William Bradford. About one hundred twenty-five works on arithmetic were published in America before 1800, of which the most popular was written by Thomas Dilworth and reprinted many times from 1770 to 1825. The next in popularity was another English work, "*The Young Man's Companion,*" written by George Fisher, of which probably twenty editions appeared before 1800. Professor Karpinski also showed a number of slides portraying Maya and Aztec methods of representing numbers.

(5) Professor Emmons discussed the construction of tables for determining maturity dates of land contracts upon which fixed monthly payments are made. He showed how these tables might be used to solve a variety of problems, including the following: to determine the payment necessary to pay off a contract in any given time; to find the interest yield on a contract that is sold at a discount; to find the yield on a similar contract where a straight mortgage is involved; and to find the yield when a mortgage requiring fixed payments at equal intervals is involved.

(6) A comprehensive definition of a mean was set up by Professor Anning. With its aid the following theorem was proved:

THEOREM: *The mean of n quantities is the same as the mean of the* ${}_nC_k$ *means formed by taking* $(n-k)$ *of the quantities in all possible ways,* $(1 \leq k \leq n-1)$.

Some elementary applications were mentioned.

J. P. EVERETT, *Secretary-Treasurer.*

THE MAY MEETING OF THE MINNESOTA SECTION.

The regular spring meeting of the Minnesota Section was held at Hamline University, St. Paul, on Saturday, May 24, 1924. There was an afternoon session followed by dinner in the Hamline University Refectory. Professor H. H. Dalaker, chairman of the Section, presided.

The attendance was thirty-five including the following thirteen members of the Association: W. O. Beal, R. W. Brink, W. H. Bussey, H. H. Dalaker, O. C. Edwards, Clara L. Hancock, C. A. Herrick, D. Jackson, R. A. Johnson, E. L. Mickelson, A. L. Underhill, W. R. Warne, H. B. Wilcox.

The following officers were elected for the coming year: Chairman, W. H. KIRCHNER, College of Engineering, University of Minnesota; Secretary, A. L. UNDERHILL, College of Science, Literature and the Arts, University of Minnesota; Executive Committee, R. A. JOHNSON, Hamline University, CLARA L. HANCOCK, Virginia Junior College, G. C. PRIESTER, University of Minnesota. A motion was passed, expressing the appreciation of the section for the hospitality of Hamline University in entertaining the Section.

The following six papers were read:

(1) "The numbers of Fibonacci and a problem in algebraic equations" by Professor R. A. JOHNSON, Hamline University.

(2) "On the pressure due to wave action" by Professor G. C. PRIESTER, University of Minnesota.

(3) "An explosion transformation" by Mr. C. A. RUPP, Hamline University.

(4) "The trigonometry of correlation" by Professor DUNHAM JACKSON, University of Minnesota.

(5) "An example of numerical differentiation" by Professor W. O. BEAL, University of Minnesota.

(6) "Parametric representations of algebraic curves" by Professor H. H. DALAKER, University of Minnesota.

Abstracts of the papers follow:

(1) The sequence of numbers 1, 1, 2, 3, 5, 8, $\cdots$ defined by the relation $u_n = u_{n-1} + u_{n-2}$, $u_0 = 0$, $u_1 = 1$, has a number of interesting and amusing properties. Mr. Johnson discussed some of the more obvious of these, and showed their connection with the solution of the simultaneous equations $x_1x_2 = x_3$, $x_2x_3 = x_4$, $\cdots$ $x_{n-2}x_{n-1} = x_n$, $x_{n-1}x_n = x_1$, $x_nx_1 = x_2$.

(2) Professor Priester's paper gives a method of determining the hydrostatic head at the crest and trough of a trochoidal wave. The fundamental theory is based on Gerstner's equations for trochoidal waves and by means of the usual dynamical equations of motion the equation of pressure is developed and expressed in logarithmic form rather than the common exponential form. By plotting certain functions of this equation it is possible to determine the hydrostatic head for any length and height of wave and at any depth below the free surface directly instead of by the usual method of trial. The necessary graphs and the solution of a specific problem are incorporated in the paper.

(3) The explosion transformation discussed by Mr. Rupp consists in exploding the points of an arbitrary plane curve into circles in that plane having the exploded point as center, and an arbitrary point function as radius. Several cases were discussed. If the radius is constant, we have a dilation. If the radius is proportional to the distance of the point from a fixed pole, the transform of a straight line is a conic whose eccentricity is the constant of proportionality. If the radius is equal to the distance of the exploded point from the pole, so that all the circles pass through a common point, the transform is the podoid of the original curve. Several applications were given.

(4) Professor Jackson's paper appeared in full in the June number of the MONTHLY.

(5) In 1888, G. W. Hill obtained by mechanical quadrature a numerical tabular solution of the differential equations of motion of Hyperion considered as attracted by both Saturn and its satellite Titan, the latter being assumed to move in a circular orbit. Mr. Beal has computed the derivatives of the coördinates of Hyperion for each of the arguments in the table, and exhibited how well the differential equations of motion and their Jacobian integral are satisfied. The first derivatives of the coördinates were also shown to satisfy the differential "equations of variation" arising when Titan is assumed to move in an ellipse.

(6) In his paper Professor Dalaker gave a brief discussion of the parametric representations of algebraic curves, together with examples of the different classes of functions necessary for the uniformization of curves of genus *zero, one* and *two.*

R. W. BRINK, *Secretary-Treasurer.*

ORGANIZATION MEETING OF THE NEBRASKA SECTION.

A meeting for the purpose of organizing a Nebraska Section of the Association was held at Lincoln March 14, 1924. Miss Emma Hanthorn of the State Teachers College at Kearney presided. After a brief discussion by Dr. W. C. Brenke of the State University outlining the history and purposes of the Association, and the policy of the MONTHLY, an executive committee was chosen to draw up the by-laws.

The newly formed section was called to meet jointly with the Mathematics-Physics Section of the Nebraska Academy of Sciences which met at Creighton College, Omaha, May 2, 1924. The Executive Committee presented the by-laws which were adopted and signed by the following charter members:

Harriet Anderson, Grand Island College; W. C. Brenke, University of Nebraska; A. L. Candy, University of Nebraska; A. R. Congdon, University of Nebraska; M. G. Gaba, University of Nebraska; Emma E. Hanthorn, Kearney Teachers College; Mary F. Jackson, Lincoln High School; Nellie M. Johnston, Gibbon High School; Stella B. Kirker, Lincoln High School; E. P.

Martinson, University of Nebraska; R. M. McDill, Hastings College; J. E. Opp, University of Nebraska; T. A. Pierce, University of Nebraska; T. I. Porter, University of Omaha; W. T. Rigge, Creighton University; C. R. Sherer, University of Nebraska.

The following officers were elected for the ensuing year: Chairman—W. C. BRENKE; Secretary-Treasurer—EMMA E. HANTHORN; Member Executive Committee—R. M. McDILL.

Papers read at the first annual meeting of the Nebraska Section of the Mathematical Association of America, meeting jointly with the Mathematics-Physics Section of the Academy of Sciences, with W. C. Brenke presiding, were as follows:

(1) "A function approximating the least root of an equation," Prof. T. A. PIERCE, University of Nebraska. (By title.)

(2) "The expansion of the cube root of an integer into a continued fraction," Mr. A. E. ANDERSON, University of Nebraska.

(3) "Some problems in geometry," Prof. M. G. GABA, University of Nebraska.

(4) "Two problems," Prof. R. M. McDILL, Hastings College.

(5) "A compound harmonic motion machine" (illustrated), Prof. W. H. RIGGE, Creighton University.

(6) "An angle connected with the mean anomaly in elliptic orbits," Prof. W. C. BRENKE, University of Nebraska.

(7) "An explicit solution for a general congruence with prime modulus," Prof. PIERCE. (By title.)

(8) "A space transformation," Mr. E. P. MARTINSON, University of Nebraska.

(9) "An example of the value of diophantine analysis to the teacher in the secondary school," Mr. A. E. CAMPBELL, Omaha Technical High School.

EMMA E. HANTHORN, *Secretary-Treasurer.*

A THEOREM ON ISOGONAL TETRAHEDRA.

By B. H. BROWN, Dartmouth College.

1. Introduction. It has often been remarked how greatly our present knowledge of the geometry of the triangle exceeds that of the tetrahedron, and various excellent reasons may be cited therefor. I think this superiority likely to continue, since new methods for the study of the tetrahedron may well make even more important contributions to the geometry of the triangle. Such was the case with the space locus for the fourth vertex of an involutory tetrahedron of fixed base, discussed by Neuberg in his remarkable *Mémoire sur le tétraèdre.*[1] From this locus he deduces the theorem *in plano,* that for the general triangle the following 21 notable points lie on a circular cubic:

[1] Published as *Supplément V* to *Mathesis,* 1885, also *Mémoires couronnés,* etc., Belgium, 1886, pp. 1–72.

(*a*) the 3 vertices A_1, A_2, A_3,
(*b*) the 3 points symmetric to the vertices in the opposite sides a_1, a_2, a_3,
(*c*) the 6 vertices of the equilateral triangles constructed on a_1, a_2, a_3,
(*d*) the 3 points symmetric to A_1, A_2, A_3 in the lines $q_{23}q_{32}$, $q_{31}q_{13}$, $q_{12}q_{21}$ respectively, where q_{ns} is the point where the side a_n meets the perpendicular bisector of a_s,
(*e*) the circumcenter,
(*f*) the orthocenter,
(*g*) the 2 isodynamic centers,[1]
(*h*) the 2 isogonal centers.[2]

The development of this theorem is long but not difficult. Still, the chain of theorems with which we are concerned, although entirely elementary, cannot be casually referred to as "well-known." It seems desirable to summarize the pertinent portion of the memoir.

Definition. If the six planes are constructed perpendicular to the mid-points of the six edges of a tetrahedron, and if the six points in which these planes meet the opposite edges are coplanar, the tetrahedron is said to be involutory.

The necessary and sufficient (N. S.) condition for an involutory tetrahedron is

$$(a_1^2 - a_6^2)(a_2^2 - a_4^2)(a_3^2 - a_5^2) = (a_1^2 - a_5^2)(a_2^2 - a_6^2)(a_3^2 - a_4^2), \qquad (1)$$

or

$$\begin{vmatrix} 1 & a_1^2 + a_4^2 & a_1^2a_4^2 \\ 1 & a_2^2 + a_5^2 & a_2^2a_5^2 \\ 1 & a_3^2 + a_6^2 & a_3^2a_6^2 \end{vmatrix} = 0; \qquad (2)$$

where a_1, a_2, a_3 are the lengths of the three edges of a base, and a_4, a_5, a_6 the lengths of the edges opposite a_1, a_2, a_3 respectively.

If the base of a tetrahedron be fixed, and if the fourth vertex vary so that the tetrahedron remain involutory, this fourth vertex has in general two degrees of freedom; its locus is consequently a surface, which Neuberg has shown to be a circular cubic surface. The circular cubic curve in which the plane of the base triangle is met by this surface is the cubic previously referred to. Neuberg finds various notable one-parameter families of involutory tetrahedra with this base, finds the space locus (a curve) for the fourth vertex; the intersection of this curve with the plane of the base gives a point or points on the cubic. We shall limit ourselves to the most interesting cases (*f*), (*g*), and (*h*), and approach these from the following standpoint.

In general, in a tetrahedron each of the following quadruples of lines belongs to one set of rulings of a hyperboloid: (*f*) the four altitudes, (*g*) the four lines

[1] The two points W such that

$$A_1A_2 \cdot A_3W = A_1A_3 \cdot A_2W = A_1W \cdot A_2A_3.$$

Each of A_1, A_2, A_3, W is an isodynamic center of the other three.

[2] The two points Z from which the sides a_1, a_2, a_3 subtend angles of 60° or of 120°. *Cf.* Lufkin, MONTHLY (*1923*, 127–131).

from the vertices to the centers of the circles inscribed in the opposite faces, (*h*) the four lines from the vertices to the points of contact of the opposite faces with the inscribed sphere. A tetrahedron in which any such quadruple are concurrent is called: (*f*) orthogonal, (*g*) isodynamic, (*h*) isogonal.

(*f*) The N.S. condition for an orthogonal tetrahedron is

$$a_1^2 + a_4^2 = a_2^2 + a_5^2 = a_3^2 + a_6^2, \tag{3}$$

hence from (2) Neuberg observes that every orthogonal tetrahedron is involutory. The space locus here is a line perpendicular to the plane of the base of the tetrahedron at the orthocenter of the base triangle, which is therefore a notable point on the cubic.

(*g*) The N.S. condition for an isodynamic tetrahedron is

$$a_1a_4 = a_2a_5 = a_3a_6, \tag{4}$$

hence from (2) Neuberg observes that every isodynamic tetrahedron is involutory. The space locus here is a circle perpendicular to the plane of the base triangle with the isodynamic centers as diametral points, consequently the isodynamic centers are notable points on the cubic.

(*h*) Here we find a striking lacuna in Neuberg's treatment. He does indeed show that the space locus for the fourth vertex of isogonal tetrahedra with fixed base is a pair of hyperbolas each with vertices at an isodynamic and at an isogonal center of the triangle. But the evidence is conclusive that he never even suspected that isogonal tetrahedra were involutory. Yet from other considerations he evidently thought it conceivable that the isogonal centers are notable points on the cubic, and he proves they are, the proof being long and circuitous.

2. The Missing Theorem. After this tedious but necessary introduction we may state the main purpose of this paper which is simply to prove the

THEOREM: *Every isogonal tetrahedron is involutory.*

Proof. It is a characteristic property of an isogonal tetrahedron that lines from any point of tangency of the inscribed sphere to the vertices in that face make angles of 120° with each other.[1] Hence by the cosine law

$$\begin{aligned}
a_1^2 &= \rho_2^2 + \rho_3^2 + \rho_2\rho_3 \\
a_2^2 &= \rho_3^2 + \rho_1^2 + \rho_3\rho_1 \\
a_3^2 &= \rho_1^2 + \rho_2^2 + \rho_1\rho_2 \\
a_4^2 &= \rho_1^2 + \rho_4^2 + \rho_1\rho_4 \\
a_5^2 &= \rho_2^2 + \rho_4^2 + \rho_2\rho_4 \\
a_6^2 &= \rho_3^2 + \rho_4^2 + \rho_3\rho_4,
\end{aligned} \tag{5}$$

[1] In every tetrahedron the three angles at the points of tangency are the same in each face, a theorem apparently due to Bang, *Tidsskrift for Math.*, 1897, p. 48; solved by Gehrke, *ibid.*, p. 84; extended by Meyer, *Jahresbericht*, 1903, vol. 12, p. 137. In the isogonal tetrahedron the Bang angles are equal.

where ρ_i is the tangential distance from A_i. Neuberg was "content with proposing"[1] the elimination of the ρ's. Now

$$\begin{cases} a_1^2 - a_6^2 = (\rho_2 - \rho_4)(\rho_2 + \rho_3 + \rho_4) \\ a_2^2 - a_4^2 = (\rho_3 - \rho_4)(\rho_3 + \rho_4 + \rho_1) \\ a_3^2 - a_5^2 = (\rho_1 - \rho_4)(\rho_4 + \rho_1 + \rho_2) \end{cases}$$
$$\begin{cases} a_1^2 - a_5^2 = (\rho_3 - \rho_4)(\rho_2 + \rho_3 + \rho_4) \\ a_2^2 - a_6^2 = (\rho_1 - \rho_4)(\rho_3 + \rho_4 + \rho_1) \\ a_3^2 - a_4^2 = (\rho_2 - \rho_4)(\rho_4 + \rho_1 + \rho_2), \end{cases}$$

and we see immediately that (1) is a first eliminant of (5).[2]

From Neuberg's results we may therefore conclude immediately:

THEOREM: *The isogonal centers are notable points on the cubic.*

3. Degenerate Case. It is of interest to note the degeneration when the base triangle is equilateral. Those of the 21 points which are determinate reduce to 7, to which we may adjoin the circular points at infinity. However, there is no unique cubic herewith associated, for this turns out to be one of those groups of 9 points through which pass a pencil of cubics. This could have been foreseen in two different ways. First, as Lufkin (*1923*, 128) pointed out, in an equilateral triangle one isogonal point may be anywhere on the circumcircle of the triangle. Second, every tetrahedron with an equilateral base is involutory. Consequently the space locus of the fourth vertex is all finite space except for this plane, and hence every finite point of the plane is continuously accessible.

4. A Second Eliminant. From a suggestion of my colleague, Professor Wilder, I have succeeded in finding a second eliminant of the six equations (5). The following equations are easily verified:

$$\begin{bmatrix} - a_4^4a_3^2 - a_5^4a_3^2 + a_4^4a_2^2 + a_5^4a_1^2 - a_4^2a_2^4 \\ - a_5^2a_2^4 - a_3^2a_2^4 - a_4^2a_1^4 - a_5^2a_1^4 - a_1^4a_3^2 \\ - a_4^2a_5^2a_2^2 - a_4^2a_5^2a_1^2 - 2a_5^2a_2^2a_3^2 + 2a_5^2a_1^2a_3^2 \\ + 2a_4^2a_5^2a_3^2 + 2a_4^2a_2^2a_3^2 - 2a_4^2a_1^2a_3^2 \\ + 2a_4^2a_2^2a_1^2 + 2a_5^2a_1^2a_2^2 + 2a_1^2a_2^2a_3^2 \end{bmatrix} \tag{6}$$
$$= (\rho_4 - \rho_3)(\rho_1 - \rho_2)^2(\rho_4^2 + \rho_1^2 + \rho_2^2 - \rho_4\rho_1 - \rho_4\rho_2 - \rho_1\rho_2)(\rho_1 + \rho_2 + \rho_3),$$

$$a_4^2 - a_5^2 = (\rho_1 - \rho_2)(\rho_1 + \rho_2 + \rho_4), \tag{7}$$

$$\begin{pmatrix} a_2^2a_5^2 - a_1^2a_4^2 - a_2^2a_3^2 \\ + a_1^2a_3^2 + a_4^2a_3^2 - a_5^2a_3^2 \end{pmatrix} \tag{8}$$
$$= (\rho_1 + \rho_2 + \rho_3)(\rho_1 - \rho_2)(\rho_4 - \rho_3)(\rho_4 + \rho_1 + \rho_2),$$

[1] *Mémoire sur le tétraèdre,* p. 52 in the Belgian memoirs; see also *Jahresbericht,* 1907, vol. 16, p. 358. The reader will also find another unsolved algebraic problem relating to isogonal tetrahedra in these articles.

[2] If a tetrahedron is isogonal to the inscribed sphere it is isogonal to the seven escribed spheres; that is, the corresponding quadruples of lines are concurrent. The proofs that (1) is an eliminant for escribed isogonality are unnecessary, but the reader will find them interesting variants on the proof given.

$$(a_4^4 - a_4^2a_5^2 + a_5^4 - a_3^2a_5^2 - a_4^2a_5^2 + a_3^4) \\ = (\rho_4 + \rho_1 + \rho_2)^2(\rho_4^2 + \rho_1^2 + \rho_2^2 - \rho_4\rho_1 - \rho_4\rho_2 - \rho_1\rho_2), \tag{9}$$

$$(a_2^2 - a_1^2)^2 = (\rho_1 - \rho_2)^2(\rho_1 + \rho_2 + \rho_3)^2, \tag{10}$$

$$(a_1^2 + a_4^2 - a_2^2 - a_5^2)^2 = (\rho_1 - \rho_2)^2(\rho_4 - \rho_3)^2. \tag{11}$$

Now the product of the right-hand members of (6), (7), and (8) is equal to the product of the right-hand members of (9), (10), and (11). Hence equating the product of the left-hand members of the first group to that of the second group, we can easily verify that we actually have an eliminant, and that it is independent of (1) since it does not contain a_6.

5. Associated Unsolved Problems. The other problem proposed by Neuberg, reference to which was made in a foot-note, is this. If f_i denote the area of the face of an isogonal tetrahedron opposite A_i, and if t_{ij} denote the area of either of the two equal small triangles bounded by an edge a_{ij} and by lines from vertices A_i and A_j to the points of tangency of the inscribed sphere with the faces opposite A_k or A_l, then

$$\begin{cases} t_{12} + t_{23} + t_{31} = f_4 \\ t_{23} + t_{34} + t_{42} = f_1 \\ t_{34} + t_{41} + t_{13} = f_2 \\ t_{41} + t_{12} + t_{24} = f_3 \\ t_{23}t_{41} = t_{31}t_{42} = t_{12}t_{43}, \end{cases}$$

the last series of equalities characterizing isogonal tetrahedra. Solve for the t's in terms of the f's. The reader will note that an equivalent problem is to find the edges of an isodynamic tetrahedron in terms of the perimeters of the faces.

Finally, what geometric significance attaches to this second eliminant? Can the eliminants be simplified or symmetrized? What is the geometric significance of the eccentricities of the hyperbolas mentioned above? What is the significance of the real infinite point on the cubic with respect to the base triangle? Are there not other notable points to be found from other types of involutory tetrahedra?

UNIQUENESS OF THE LORENTZ TRANSFORMATION.

By ALONZO CHURCH, Princeton University.

1. The object of this inquiry is to obtain a set of logically independent postulates which uniquely determine the Lorentz transformation for one dimension. For this purpose we propose the following set:

1. The required transformation expresses $\bar{x}$ and $\bar{t}$ as functions of x, t, and v, which have continuous partial derivatives with respect to x and t, where $\bar{x}$, $\bar{t}$, x, and t are real variables, and v is a parameter which may have any value less than 1 and greater than -1.

2. For no value of v is the partial derivative of $\bar{t}$ with respect to t negative for every value of x and t.

3. $dx/dt = 1$ implies that $d\bar{x}/d\bar{t} = 1$.

4. $dx/dt = v$ implies that $d\bar{x}/d\bar{t} = 0$.

5. The inverse transformation, which expresses x and t in terms of $\bar{x}$ and $\bar{t}$, is obtained from the direct transformation by replacing $\bar{x}$ by x, and $\bar{t}$ by t, and x by $\bar{x}$, and t by $\bar{t}$, and v by $-v$.

6. The transformation is unchanged if $\bar{x}$ be replaced by $-\bar{x}$, and x by $-x$, and v by $-v$.

If we suppose our units of measurement so chosen that the velocity of light is 1, the physical meaning of the last five of these postulates is as follows:

2. The transformed time, $\bar{t}$, does not flow backwards with respect to t.

3. The velocity of light is invariant.

4. The origin of the transformed coördinate system is moving along the x-axis with velocity v in the original coördinate system.

5. The resultant of two velocities with the same numerical value and with opposite directions is zero.

6. The form of the transformation is independent of our choice of a positive direction on the x-axis.

2. Let the required transformation have the form:

$$\bar{x} = \varphi(x, t, v), \qquad \bar{t} = \psi(x, t, v).$$

Then

$$\left(\psi_x \frac{dx}{dt} + \psi_t\right)\frac{d\bar{x}}{d\bar{t}} = \varphi_x \frac{dx}{dt} + \varphi_t. \tag{1}$$

Setting $dx/dt = v$ in (1), it follows from postulate 4 that

$$v\varphi_x + \varphi_t = 0. \tag{2}$$

It is a consequence of postulates 4 and 5 that $dx/dt = 0$ implies that $d\bar{x}/d\bar{t} = -v$. Therefore, setting $dx/dt = 0$ in (1), it follows that

$$\varphi_t + v\psi_t = 0. \tag{3}$$

Integrating this with respect to t, we obtain

$$\varphi + v\psi = X, \tag{4}$$

where X is independent of t.

Setting $dx/dt = 1$ in (1), it follows from postulate 3 that

$$\psi_x + \psi_t = \varphi_x + \varphi_t. \tag{5}$$

From (2) and (3) it follows that

$$\varphi_x = \psi_t \tag{6}$$

and therefore by (5)

$$\psi_x = \varphi_t \tag{7}$$

Substituting in (2) this gives

$$v\varphi_x + \psi_x = 0$$

and therefore integrating with respect to x,

$$v\varphi + \psi = T, \tag{8}$$

where T is independent of x.

For values of v between $+1$ and -1 equations (4) and (8) can be solved simultaneously. In this way we obtain:

$$\varphi = X_1 + T_1, \qquad \psi = X_2 + T_2,$$

where X_1 and X_2 are independent of t, and T_1 and T_2 are independent of x.

From equations (6) and (7) it now follows that

$$\frac{dX_1}{dx} = \frac{dT_2}{dt}, \qquad \frac{dX_2}{dx} = \frac{dT_1}{dt}.$$

These conditions, however, can be satisfied only if dX_1/dx, dX_2/dx, dT_1/dt, and dT_2/dt are all independent of both x and t. Therefore X_1, X_2, T_1, and T_2 are of the first degree in x and t. Therefore φ and ψ are of the first degree in x and t. Let us accordingly write the transformation in the form,

$$\bar{x} = p_1x + q_1t + r_1, \qquad \bar{t} = p_2x + q_2t + r_2,$$

where p_1, p_2, q_1, q_2, r_1, and r_2 are functions of v.

Then, by (6)

$$p_1 = q_2,$$

by (7)

$$p_2 = q_1,$$

and by (3)

$$q_1 = -vq_2.$$

Therefore

$$p_2 = -vq_2.$$

The required transformation can therefore be written in the form

$$\bar{x} = \beta(x - vt) + r_1, \qquad \bar{t} = \beta(t - vx) + r_2, \tag{9}$$

where β, r_1, and r_2 are functions of v. In order that postulate 6 be satisfied it is necessary that β and r_2 be even functions of v, and that r_1 be an odd function.

By solving (9) for x and t we obtain

$$x = \frac{\bar{x} + v\bar{t}}{(1 - v^2)\beta} - \frac{r_1 + vr_2}{(1 - v^2)\beta}, \qquad t = \frac{\bar{t} + v\bar{x}}{(1 - v^2)\beta} - \frac{r_2 + vr_1}{(1 - v^2)\beta}.$$

Consequently postulate 5 requires that

$$r_1 = \frac{r_1 + vr_2}{(1 - v^2)\beta}, \qquad r_2 = -\frac{r_2 + vr_1}{(1 - v^2)\beta}, \qquad \beta = \frac{1}{(1 - v^2)\beta}.$$

Solving these equations simultaneously for β, r_1, and r_2, we find that

$$r_1 = 0, \qquad r_2 = 0, \qquad \beta = \pm \frac{1}{\sqrt{1 - v^2}}.$$

The expression obtained for β represents, not two but indefinitely many determinations of β as a function of v, since it is possible to take the upper sign for some values of v and the lower sign for the remaining values, in any arbitrary manner which makes β an even function of v. It is for this reason that postulate 2 is required.

In accordance with postulate 2, there is only one possible determination of β, namely,

$$\beta = \frac{1}{\sqrt{1 - v^2}}.$$

The required transformation, therefore, has the form:

$$\bar{x} = \frac{x - vt}{\sqrt{1 - v^2}}, \qquad \bar{t} = \frac{t - vx}{\sqrt{1 - v^2}},$$

which is the Lorentz transformation for one dimension.

3. We shall give independence proofs for the last five of our postulates. There is no immediately evident independence proof for the first postulate. It is quite possible that it is not necessary to assume the existence of the partial derivatives, φ_x, ψ_x, φ_t, ψ_t.

As an independence proof for postulate 2 we may cite the transformation:

$$\bar{x} = f(v)\frac{x - vt}{\sqrt{1 - v^2}}, \qquad \bar{t} = f(v)\frac{t - vx}{\sqrt{1 - v^2}},$$

where $f(v)$ is a function of v which is restricted to have always one of the two values $+1$ and -1, in such a way that $f(v) = f(-v)$ for all values of v, and $f(v)$ is not equal to $+1$ for every value of v.

An independence proof for postulate 3 is the transformation:

$$\bar{x} = x - vt, \qquad \bar{t} = t.$$

The simplest independence proof for postulate 4 is:

$$\bar{x} = x, \qquad \bar{t} = t,$$

but we may also take the transformation:

$$\bar{x} = \frac{x - kvt}{\sqrt{1 - k^2v^2}}, \qquad \bar{t} = \frac{t - kvx}{\sqrt{1 - k^2v^2}},$$

where k is any constant which is real and less than 1 in numerical value.

An independence proof for postulate 5 is:

$$\bar{x} = x - vt, \qquad \bar{t} = t - vx.$$

And, finally, for postulate 6 we have:

$$\bar{x} = \frac{(1 - v)^n}{(1 + v)^{n+1}}(x - vt), \qquad \bar{t} = \frac{(1 - v)^n}{(1 + v)^{n+1}}(t - vx), \tag{10}$$

where n is any real number different from $-\frac{1}{2}$.

4. The transformations (10) have all the essential properties of the Lorentz transformation except the property of symmetry expressed by postulate 6.

The transformation corresponding to $n = -\frac{1}{2}$ is the Lorentz transformation. The transformations corresponding to other values of n are arranged symmetrically about the value $-\frac{1}{2}$ in the sense that the transformation corresponding to $n = a$ and that corresponding to $n = -(a - 1)$ apply to the same space with a different choice of the positive direction on the x-axis.

In a space in which one of these transformations is valid there is an effect on the apparent length of a moving body analogous to that given by the Lorentz transformation, but this effect may be either a lengthening or a shortening, and, in accordance with the asymmetric character of the transformations, it depends not only on the speed of the motion but also on the direction.

The invariant of the transformations is:

$$\frac{(t - x)^{n+1}}{(t + x)^n}.$$

The proper time is given by

$$d\tau = \frac{(dt - dx)^{n+1}}{(dt + dx)^n}.$$

The transformation obtained by setting $n = 0$ in (10) is:

$$\bar{x} = \frac{x - vt}{1 + v}, \qquad \bar{t} = \frac{t - vx}{1 + v}.$$

This transformation has the properties just enumerated. The proper time given by it is $d\tau = dt - dx$. Since $\int (dt - dx)$ is independent of the path of integration, the absolute character of time is preserved in a way in which it is not by the Lorentz transformation. This means that there are no geodesics in the two-dimensional space-time to which this transformation applies; and that a unique proper time can be attached to every event, independently of a moving particle to which the time is referred.

5. We shall prove that the transformations (10) are the only transformations which satisfy postulates 1 to 5 above and the following three in addition:

7. $x = 0$, $t = 0$ implies that $\bar{x} = 0$, $\bar{t} = 0$.

8. The transformations corresponding to the various possible values of v constitute a group.

9. $\bar{x}$ and $\bar{t}$ are continuous as functions of v.

A transformation which satisfies these postulates must be of the form, (9), above. By postulate 7 we have $r_1 = r_2 = 0$. Therefore the transformation has the form:

$$\bar{x} = \beta(v)(x - vt), \qquad \bar{t} = \beta(v)(t - vx).$$

A second transformation of the set may have the form:

$$\bar{\bar{x}} = \beta(v')(\bar{x} - v'\bar{t}), \qquad \bar{\bar{t}} = \beta(v')(\bar{t} - v'\bar{x}).$$

Combining these two transformations, we obtain:

$$\bar{\bar{x}} = \beta(v)\beta(v')(1 + vv')\left(x - \frac{v + v'}{1 + vv'}t\right),$$

$$\bar{\bar{t}} = \beta(v)\beta(v')(1 + vv')\left(t - \frac{v + v'}{1 + vv'}x\right).$$

Therefore, in order that postulate 8 be satisfied, it is necessary that

$$\beta(v)\beta(v')(1 + vv') = \beta\left(\frac{v + v'}{1 + vv'}\right).$$

Let

$$\beta(v) = \frac{\varphi(v)}{1 + v}.$$

Then

$$\varphi(v)\varphi(v') = \varphi\left(\frac{v + v'}{1 + vv'}\right). \tag{11}$$

Now we know, by postulate 2, that $\varphi(v)$ is positive for some value of v between $+1$ and -1. Therefore, since it follows from postulate 9 that $\varphi(v)$ is continuous, there is some interval between $+1$ and -1 throughout which $\varphi(v)$ is positive. In order, therefore, to find the form of $\varphi(v)$ in this interval, we are justified in taking the logarithm of both sides of (11). It will turn out that the form of $\varphi(v)$ so obtained cannot vanish for any value of v between $+1$ and -1, from which it will follow that $\varphi(v)$ has this form throughout the interval from -1 to $+1$.

Taking the logarithm of each side of (11), we obtain:

$$\log \varphi(v) + \log \varphi(v') = \log \varphi\left(\frac{v+v'}{1+vv'}\right).$$

Since v and v' lie between $+1$ and -1, we may make the following substitutions without going outside the real number system:

$$v = \tanh u$$
$$v' = \tanh w$$
$$\log \varphi(\tanh u) = F(u).$$

Making these substitutions, we obtain:

$$F(u) + F(w) = F(u+w). \tag{12}$$

Since u and w are independent variables, we may assign any relation between them that we please without affecting the validity of (12). Setting $u = w$, we have:

$$F(2w) = 2F(w).$$

Similarly, setting $u = 2w$,

$$F(3w) = 3F(w)$$

and so on. It is clear that we may prove by induction that

$$F(aw) = aF(w). \tag{13}$$

for every positive integer, a.

Setting $u = 0$ in (12), we obtain:

$$F(w) + F(0) = F(w).$$

Since $F(w)$ is not infinite for all values of w, it follows that

$$F(0) = 0.$$

Therefore, setting $u = -w$ in (12),

$$F(-w) = -F(w),$$

from which it follows that (13) is also true when a is a negative integer.

Setting $w = w'/a$ in (13), we obtain:

$$F\left(\frac{w'}{a}\right) = \frac{1}{a}F(w').$$

Therefore if p/q is any rational number,

$$F\left(\frac{p}{q}w\right) = \frac{p}{q}F(w),$$

and therefore, since $F(w)$ is continuous,

$$F(aw) = aF(w)$$

for every real number, a. And, dividing by aw,

$$\frac{F(w)}{w} = \frac{F(aw)}{aw}.$$

Therefore

$$\frac{F(w)}{w} = b,$$

or

$$F(w) = bw,$$

where b is an arbitrary constant. In order to make sure that b is actually arbitrary, we must, of course, substitute this expression for $F(w)$ in equation (12).

We now have

$$\varphi(v) = e^{b \tanh^{-1} v}$$

and therefore

$$\varphi(v) = \left(\frac{1 - v}{1 + v}\right)^n,$$

where n is an arbitrary constant. This gives

$$\beta(v) = \frac{(1 - v)^n}{(1 + v)^{n+1}},$$

from which it follows that the required transformation is in the form (10).

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

I. INFINITE AND IMAGINARY ELEMENTS IN ALGEBRA AND GEOMETRY: REPLY TO CRITICISMS.

By R. M. WINGER, University of Washington.

Two criticisms of my article [1] have been printed in the MONTHLY, one by Professor Tomlinson Fort,[2] the other by Professor W. L. G. Williams.[3] Both these gentlemen express emphatic disagreement with my suggestions, partly on the grounds of pedagogy, but chiefly as it seems to me because they regard half of the program as mathematically unsound. Professor Williams indeed by a detailed analysis of several hypothetical cases attempts to prove the mathematics faulty. But his proofs, as I shall point out, are based on a misinterpretation of my paper. For he quotes me as *defining* a number infinity by the relation $k/0 = \infty$, where k cannot be zero.

To answer his argument, I must review a part of my original statement. I first recall briefly the service of infinite elements in modern geometry and, assuming the geometry as known, suggest that the advantages be carried over into algebra. I say that perhaps the best approach is through homogeneous equations but that we might begin by postulating an "improper number" ∞ which should play in algebra a rôle analogous to that played in geometry by the "improper point" at infinity on the line. The next few lines involve the heart of the matter at issue: "This number is now clothed with properties to conform to those of its geometric counterpart. Thus the x-intercept of the line

$$y = mx - k \tag{4}$$

is

$$x_0 = k/m. \tag{5}$$

Now if $m = 0$, the line is parallel to the x-axis and cuts it therefore at infinity. Accordingly, in virtue of (5) we attribute to ∞ the property

$$k/0 = \infty, \qquad k \neq 0, \tag{6}$$

for if $k = 0$ at the same time, the line (4) coincides with the x-axis and intersects it at every point."

I cannot see how Professors Fort and Williams find in this a proposal to define the symbol ∞ by the single relation (6).[4] For, attributing a property to a symbol

[1] This MONTHLY, September, 1922, p. 290.

[2] *Ibid.*, July-August, 1923, p. 255.

[3] *Ibid.*, November, 1923, p. 384.

[4] Professor Williams says that I propose "to create a number system in which there shall exist a single infinite number ∞ defined by the relation (6)" while Professor Fort states: "It

is quite a different thing from defining it. I gave in fact no formal definition at all. I only postulated the existence of the symbol ∞, attributed to it one property and stated a theorem about infinite roots of equations—for this was all I had occasion to use in the illustrations considered in my paper. Other properties than (6) are however implied by the context, especially by the phrase "*clothed with properties to conform to those of its geometric counterpart.*" Further I remark in a footnote that other properties may be assigned by considering the intercepts of the parabola $y = ax^2 + bx + c$.

Indeed relation (6) does not even say that k cannot be zero, a restriction that Professor Williams uses at least five times.[1] What it does say is that $k/0 = \infty$, *if* $k \neq 0$. If $k = 0$ the expression is indeterminate and might have any value including ∞, since the line (4) then coincides with the x-axis and meets it at each of its points, including the point at infinity. In other words $0 \cdot \infty$ is also indeterminate and might have the value 0, contrary to Professor Williams' assertion. This disposes of the first "contradiction" that arises when he seeks to show that $0 \cdot x + 0 = 0$ cannot have an infinite root.

Again when he finds by formal algebraic manipulation that there is doubt whether a certain line contains a point at infinity, a result in conflict with the geometry, what has he learned? Merely that the alleged "definition" (6) is inadequate. But why does Professor Williams ignore the qualification in my paper that the algebra *must conform to the geometry?* He might have found, I should think, quite as much mental recreation in ascertaining what additional property must be ascribed to the symbol ∞ in this case to make it fit the geometry as in speculating on what certain expressions mean in order to catch me in a logical inconsistency.

Since Professor Williams objects that my little sketch falls short of the treatment of number systems in books on the theory of functions, it will be instructive to see how an authority of his own choosing handles the question under discussion. Turning to Burkhardt,[2] § 12, I find this statement: "*In addition to the complex numbers and their symbols already introduced we introduce now a new one, 'infinity,' with the symbol ∞, which is to be regarded as the result of the division* 1/0." This he parallels with a geometrical definition of a point at infinity.[3] He then goes on to qualify the symbol by several other conventions and points out that $\infty \pm \infty$, $0 \cdot \infty$, and ∞/∞ as well as $0/0$ are indeterminate forms. A few lines farther on he says: "According to conventions of this kind, certain words and symbols previously defined are assigned a wider meaning. That this procedure is permissible we have repeatedly stated in the first chapter; that it is useful is justified by results." Still later he reconciles this view with

---

is with a kind of horror that I read where the author advocates the postulation of 'The Number Infinity' defined by (6)."

[1] In such statements as : "for $k \neq 0$, since by definition $k \neq 0$, we do not know the value of k except that it is not 0, which is untrue since $k \neq 0$ by hypothesis."

[2] *Theory of Functions of a Complex Variable*, Rasor Translation.

[3] He uses the convention of a point instead of a line at infinity in the plane, but the *principle* of introducing corresponding concepts into analysis and geometry is the same as I advocated.

kinds of geometry, the synthetic geometry, which considers the figures themselves, and the analytic geometry which builds up its system essentially with the aid of analysis. Besides these two kinds of geometry, we may construct still a third kind which in a sense is the inverse of the two and which will form the subject of the present course of lectures. Thus whereas we ordinarily apply analysis to geometry, it shall be the purpose of this exposition inversely to apply geometry to analysis, to get acquainted with analytic relations in a geometric manner, or somewhat more precisely expressed, *with the aid of geometry to gain an insight into the theory of functions of several variables.*" If the dualism here indicated between algebra and geometry is to be without exception, we shall require on the one hand imaginary points in geometry to correspond to imaginary solutions of equations while on the other we shall need a symbol ∞ to correspond in the non-homogeneous coördinate system to the point at infinity on every line. In accordance with a cardinal principle of mathematical development, I advocated both steps to the end that algebra and geometry appear as merely different aspects of the same abstract truth. This concept of algebra and geometry as dualistic or isomorphic in my view leads to a more adequate picture of analytic geometry and is at the same time a distinct advantage in teaching.

II. Early History of Division by Zero.

By H. G. Romig, University of California.

In the development of algebra the question was bound to arise whether zero could be used in division. It was easily seen that $a + 0 = a$, $a - 0 = a$, and $a \cdot 0 = 0$, but the quotient resulting from the division by zero was questionable.

In 628 A.D. Brahmagupta [1] first speaks of the division by cipher, but gives no quotient. Bhâscara,[2] in 1152, terms such a quotient infinite. John Wallis [3] first declares in 1657 that zero is no number but he introduces the form $1/0 = \infty$, being the first to use the symbol ∞ for infinity. He states further [4] that $1/3 < 1/2 < 1/1 < 1/0 < 1/(-1)$, etc., where he considers negative fractions greater than the infinite.

Isaac Newton [5] speaks of the quotient resulting from a division by zero, obtained by integrating dx/x, viz., $x^0/0$, as the infinite area under a hyperbola. In 1716 John Craig [6] declares that zero must be an infinitesimal for if absolutely of no value it cannot be used as a divisor, but his ideas are not altogether clear, for he repeats Wallis' mistake of considering negative fractions greater than

[1] Henry T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhâscara*, Chap. XVIII, Sec. II, "Algorithm," London, 1817, § 35–36, pp. 339–340.

[2] Henry T. Colebrooke, *idem*, Chap. II, Sec. IV, "Cipher," § 44–45, and footnote 5, p. 19.

[3] John Wallis, *Opera* I, London, 1695, Chap. IV, p. 27; Prop. LXIV, p. 395; Prop. CIV, p. 409.

[4] Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. II, Leipzig, 1913, p. 902.

[5] Isaac Newton, *Opuscula* (ed. Johann von Castillon), vol. I, Lausannæ and Genevæ, 1744, p. 4.

[6] George Cheyne, *Philosophical Principles of Religion*, Part II, London, 1716, pp. 167–169.

kinds of geometry, the synthetic geometry, which considers the figures themselves, and the analytic geometry which builds up its system essentially with the aid of analysis. Besides these two kinds of geometry, we may construct still a third kind which in a sense is the inverse of the two and which will form the subject of the present course of lectures. Thus whereas we ordinarily apply analysis to geometry, it shall be the purpose of this exposition inversely to apply geometry to analysis, to get acquainted with analytic relations in a geometric manner, or somewhat more precisely expressed, *with the aid of geometry to gain an insight into the theory of functions of several variables.*" If the dualism here indicated between algebra and geometry is to be without exception, we shall require on the one hand imaginary points in geometry to correspond to imaginary solutions of equations while on the other we shall need a symbol ∞ to correspond in the non-homogeneous coördinate system to the point at infinity on every line. In accordance with a cardinal principle of mathematical development, I advocated both steps to the end that algebra and geometry appear as merely different aspects of the same abstract truth. This concept of algebra and geometry as dualistic or isomorphic in my view leads to a more adequate picture of analytic geometry and is at the same time a distinct advantage in teaching.

II. Early History of Division by Zero.

By H. G. Romig, University of California.

In the development of algebra the question was bound to arise whether zero could be used in division. It was easily seen that $a + 0 = a$, $a - 0 = a$, and $a \cdot 0 = 0$, but the quotient resulting from the division by zero was questionable.

In 628 A.D. Brahmagupta [1] first speaks of the division by cipher, but gives no quotient. Bháscara,[2] in 1152, terms such a quotient infinite. John Wallis [3] first declares in 1657 that zero is no number but he introduces the form $1/0 = \infty$, being the first to use the symbol ∞ for infinity. He states further [4] that $1/3 < 1/2 < 1/1 < 1/0 < 1/(-1)$, etc., where he considers negative fractions greater than the infinite.

Isaac Newton [5] speaks of the quotient resulting from a division by zero, obtained by integrating dx/x, viz., $x^0/0$, as the infinite area under a hyperbola. In 1716 John Craig [6] declares that zero must be an infinitesimal for if absolutely of no value it cannot be used as a divisor, but his ideas are not altogether clear, for he repeats Wallis' mistake of considering negative fractions greater than

[1] Henry T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bháscara*, Chap. XVIII, Sec. II, "Algorithm," London, 1817, § 35–36, pp. 339–340.

[2] Henry T. Colebrooke, *idem*, Chap. II, Sec. IV, "Cipher," § 44–45, and footnote 5, p. 19.

[3] John Wallis, *Opera* I, London, 1695, Chap. IV, p. 27; Prop. LXIV, p. 395; Prop. CIV, p. 409.

[4] Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. II, Leipzig, 1913, p. 902.

[5] Isaac Newton, *Opuscula* (ed. Johann von Castillon), vol. I, Lausannæ and Genevæ, 1744, p. 4.

[6] George Cheyne, *Philosophical Principles of Religion*, Part II, London, 1716, pp. 167–169.

Rudolf Lipschitz[20] explains in 1877 that in dividing F/m by F_1/m_1, where $F_m \neq 0$, it is not permissible to let $F_1/m_1 = 0$, for the reason that there exists no fraction which, when multiplied by zero, gives F/m. The class concept was stated by Axel Harnack of Dresden in 1881, who used rational numbers in his calculus and from his definition of class he says[21] that the use of zero as a divisor is impossible. Four years later Stolz elaborates[22] upon this definition of number class and excludes zero as a divisor. The students of elementary mathematics accepted this exclusion principle and developed it in their textbooks using the definition of division but failed to use the definition of number class.

III. Dominical Letter and Perpetual Calendars.

By W. K. Nelson, University of Colorado.

The article in the November–December, 1922, issue of the Monthly on Uncle Zadock's rule for obtaining the dominical letter for any year by Mr. Vail was of interest to the writer and perhaps the following discussion of a similar rule and of two perpetual calendars will be of interest to others.

In Rietz and Crathorne's *College Algebra*, page 29, appears the following formula:[1]

$$D = \left\{ \frac{P + 2q + \left[\frac{3(q+1)}{5}\right] + N + \left[\frac{N}{4}\right] - \left[\frac{N}{100}\right] + \left[\frac{N}{400}\right] + 2}{7} \right\}_r,$$

where D is the day of the week counting Sunday the first, P is the day of the month, q is the number of the month in the year, counting January and February as the 13th and 14th months of the preceding year; and N the year. The brackets indicate that the largest integer contained is to be used and the braces with the subscript r indicate that the remainder is to be taken.

The dominical or Sunday letter for any common year is determined by 9 minus the day of the week upon which January 1st falls. By substituting in the above formula $P = 1$, $q = 13$, $N = 100C + T$, we obtain

$$L = 8 - \left\{ \frac{5C + \left[\frac{C}{4}\right] + T + \left[\frac{T}{4}\right]}{7} \right\}_r,$$

where L is the number indicating the dominical letter, C the centurial number and T the odd years. To avoid using the preceding year for the months of January

[20] Rudolf Lipschitz, *Lehrbuch der Analysis*, Bonn, 1877, pp. 27–28.

[21] Axel Harnack, *Die Elemente der Differential- und Integral-Rechnung*, Leipzig, 1881, p. 5; Eng. Ed. by Cathcart, 1891, p. 4.

[22] Otto Stolz, *Vorlesungen über Allgemeine Arithmetik nach den Neueren Ansichten*, Part I, Leipzig, 1885, p. 52.

[1] Quoted from C. Zeller, *Acta Mathematica*, vol. 9 (1887), pp. 131–6. Editor.

Rudolf Lipschitz[20] explains in 1877 that in dividing F/m by F_1/m_1, where $F_m \neq 0$, it is not permissible to let $F_1/m_1 = 0$, for the reason that there exists no fraction which, when multiplied by zero, gives F/m. The class concept was stated by Axel Harnack of Dresden in 1881, who used rational numbers in his calculus and from his definition of class he says[21] that the use of zero as a divisor is impossible. Four years later Stolz elaborates[22] upon this definition of number class and excludes zero as a divisor. The students of elementary mathematics accepted this exclusion principle and developed it in their textbooks using the definition of division but failed to use the definition of number class.

III. Dominical Letter and Perpetual Calendars.

By W. K. Nelson, University of Colorado.

The article in the November–December, 1922, issue of the Monthly on Uncle Zadock's rule for obtaining the dominical letter for any year by Mr. Vail was of interest to the writer and perhaps the following discussion of a similar rule and of two perpetual calendars will be of interest to others.

In Rietz and Crathorne's *College Algebra*, page 29, appears the following formula:[1]

$$D = \left\{ \frac{P + 2q + \left[\frac{3(q+1)}{5}\right] + N + \left[\frac{N}{4}\right] - \left[\frac{N}{100}\right] + \left[\frac{N}{400}\right] + 2}{7} \right\}_r,$$

where D is the day of the week counting Sunday the first, P is the day of the month, q is the number of the month in the year, counting January and February as the 13th and 14th months of the preceding year; and N the year. The brackets indicate that the largest integer contained is to be used and the braces with the subscript r indicate that the remainder is to be taken.

The dominical or Sunday letter for any common year is determined by 9 minus the day of the week upon which January 1st falls. By substituting in the above formula $P = 1$, $q = 13$, $N = 100C + T$, we obtain

$$L = 8 - \left\{ \frac{5C + \left[\frac{C}{4}\right] + T + \left[\frac{T}{4}\right]}{7} \right\}_r,$$

where L is the number indicating the dominical letter, C the centurial number and T the odd years. To avoid using the preceding year for the months of January

[20] Rudolf Lipschitz, *Lehrbuch der Analysis*, Bonn, 1877, pp. 27–28.

[21] Axel Harnack, *Die Elemente der Differential- und Integral-Rechnung*, Leipzig, 1881, p. 5; Eng. Ed. by Cathcart, 1891, p. 4.

[22] Otto Stolz, *Vorlesungen über Allgemeine Arithmetik nach den Neueren Ansichten*, Part I, Leipzig, 1885, p. 52.

[1] Quoted from C. Zeller, *Acta Mathematica*, vol. 9 (1887), pp. 131–6. Editor.

PERPETUAL CALENDAR

	★	★	★	★	★	★	★
JULIAN	6	5	4	3	2	1	0
CALENDAR	13	12	11	10	9	8	7
				17	16	15	14
GREGORIAN		16	15		22		21
CALENDAR		20	19		18		17

DIRECTIONS: Find the hundreds part, or first two figures of year, in above table and last two figures of year in bottom table. Connect by a straight line the stars at the tops of the columns containing these figures. This line will cross a star at the head of a column in the middle table. The letter in this column opposite the required month indicates the proper calendar at the left of this page.

Use the first of the two letters opposite January and February if year is a common year and second if year is a leap year. Leap years are in Italics, and are marked +.

	No. days in Month	★	★	★	★	★	★	★	★	★	★	★	★	★
JAN.	31	BC	AB	GA	FG	EF	DE	CD	BC	AB	GA	FG	EF	DE
FEB.	28–29	FG	EF	DE	CD	BC	AB	GA	FG	EF	DE	CD	BC	AB
MAR.	31	F	E	D	C	B	A	G	F	E	D	C	B	A
APR.	30	C	B	A	G	F	E	D	C	B	A	G	F	E
MAY	31	A	G	F	E	D	C	B	A	G	F	E	D	C
JUN.	30	E	D	C	B	A	G	F	E	D	C	B	A	G
JUL.	31	C	B	A	G	F	E	D	C	B	A	G	F	E
AUG.	31	G	F	E	D	C	B	A	G	F	E	D	C	B
SEPT.	30	D	C	B	A	G	F	E	D	C	B	A	G	F
OCT.	31	B	A	G	F	E	D	C	B	A	G	F	E	D
NOV.	30	F	E	D	C	B	A	G	F	E	D	C	B	A
DEC.	31	D	C	B	A	G	F	E	D	C	B	A	G	F

The letters of the top row are dominical or Sunday letters. The letter to the left is to be used, except for the first two months of leap year, when the second one is used.

★	★	★	★	★	★	★
00	01	02	03		*04+*	05
06	07		*08+*	09	10	11
	12+	13	14	15		*16+*
17	18	19		*20+*	21	22
23		*24+*	25	26	27	
28+	29	30	31		*32+*	33
34	35		*36+*	37	38	39
	40+	41	42	43		*44+*
45	46	47		*48+*	49	50
51		*52+*	53	54	55	
56+	57	58	59		*60+*	61
62	63		*64+*	65	66	67
	68+	69	70	71		*72+*
73	74	75		*76+*	77	78
79		*80+*	81	82	83	
84+	85	86	87		*88+*	89
90	91		*92+*	93	94	95
	96+	97	98	99		

The years 1700, 1800, 1900, 2100, 2200 in Gregorian Calendar are not leap years.

A

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

B

S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23/30	24/31	25	26	27	28	29

C

S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24/31	25	26	27	28	29	30

D

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

E

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

F

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

G

S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

EXAMPLE

Find day of week of Armistice Day, Nov. 11, 1918. The line drawn determines the calendar for 1918, November's calendar being C. Then Nov. 11th falls on Monday.

FIG. 1.

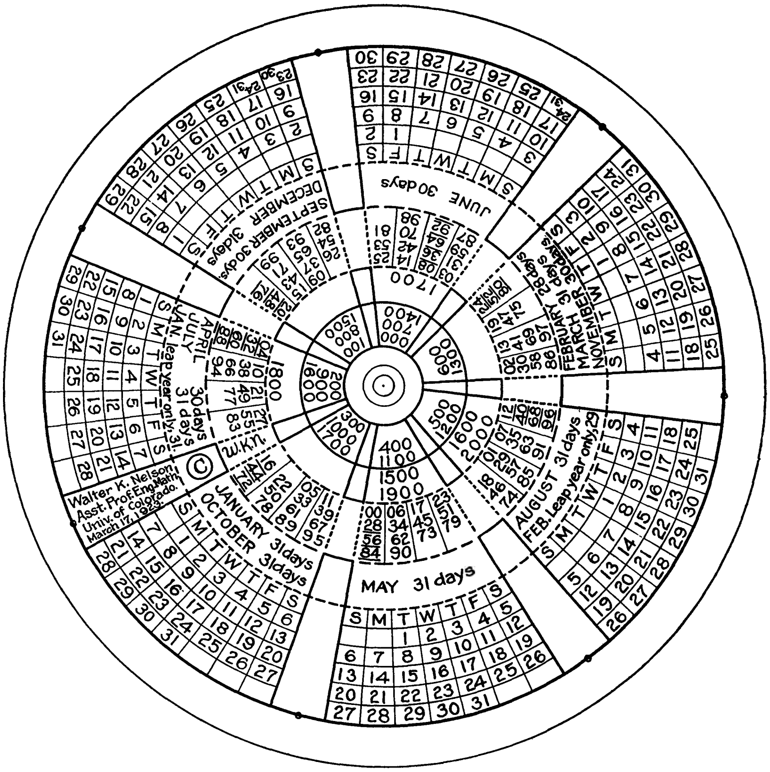


Fig. 2.

the names of the months are opposite the correct monthly calendars. (Fig. 2 shows the discs in this position.) This setting also shows all the years since the beginning of the Christian era which had this same calendar. The numbers in the inner ring are centurial numbers of the Julian calendar, while centurial numbers of the Gregorian calendar are in the adjacent ring.

IV. Some Results Involving π.

By R. S. Underwood, Alabama Polytechnic Institute.

Let

$$f_0(x) = \log(a + x).$$

$$f_1(x) = \int_0^x f_0(x)dx,$$

$$f_2(x) = \int_0^x f_1(x)dx,$$

and generally

$$f_n(x) = \int_0^x f_{n-1}(x)dx,$$

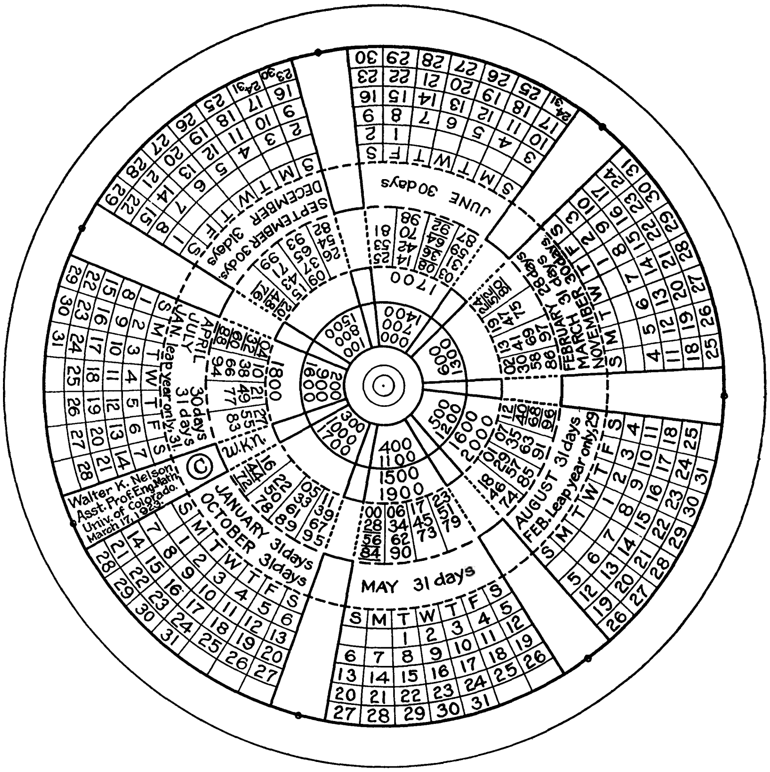


Fig. 2.

the names of the months are opposite the correct monthly calendars. (Fig. 2 shows the discs in this position.) This setting also shows all the years since the beginning of the Christian era which had this same calendar. The numbers in the inner ring are centurial numbers of the Julian calendar, while centurial numbers of the Gregorian calendar are in the adjacent ring.

IV. Some Results Involving π.

By R. S. Underwood, Alabama Polytechnic Institute.

Let

$$f_0(x) = \log (a + x).$$

$$f_1(x) = \int_0^x f_0(x)dx,$$

$$f_2(x) = \int_0^x f_1(x)dx,$$

and generally

$$f_n(x) = \int_0^x f_{n-1}(x)dx,$$

where a is any number, real or complex, except one that is negative or zero, and where the principal logarithm is used, and the integrations are made over the real interval $(0, x)$.

By integration,

$$f_1(x) = (x + a)[\log (x + a) - 1] - a \log a + a,$$

and finally, as may be shown by induction,

$$n!\, f_n(x) = (x + a)^n[\log (x + a) - s_n] + \log a[x^n - (x + a)^n] + \sum_{r=1}^{n} s^r \binom{n}{r} a^r x^{n-r}, \quad (1)$$

where $s_r = 1 + 1/2 + 1/3 + \cdots + 1/r$.

Suppose further that

$$g_0(x) = \tan^{-1}x, \qquad g_n(x) = \int_0^x g_{n-1}(x)dx.$$

Now $\tan^{-1} x = \pi/2 - (i/2) \log (x - i) + (i/2) \log (x + i)$.

Hence, by applying (1), and making some reductions,[1]

$$n!\, g_n(x) = \left\{ x^n - \binom{n}{2} x^{n-2} + \binom{n}{4} x^{n-4} - \cdots \right\} \tan^{-1} x - \left\{ \binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} + \cdots \right\} \{\log \sqrt{(1 + x^2)} - s_n\} - \left\{ \binom{n}{1} s_1 x^{n-1} - \binom{n}{3} s_3 x^{n-3} + \cdots \right\}. \quad (2)$$

Hence, on setting $x = 1$, and using Demoivre's Theorem,

$$n!\, g_n(1) = 2^{n/2}\{\cos (n\pi/4)\}\pi/4 - 2^{n/2}\{\sin (n\pi/4)\} \left(\tfrac{1}{2} \log 2 - s_n \right) - \left\{ \binom{n}{1} s_1 - \binom{n}{3} s_3 + \cdots \right\}. \quad (3)$$

Again, starting from

$$\tan^{-1} x = x - x^3/3 + x^5/5 - \cdots, \quad (-1 < x \leqq 1),$$

we get (using term by term integration of uniformly convergent series)

$$g_n(1) = 0!/(n + 1)! - 2!/(n + 3)! + 4!/(n + 5)! - \cdots. \quad (4)$$

On replacing n by $4n$ and $4n - 2$ respectively in (3), and comparing with (4), we get

[1] Formula (2) may also be established directly by induction, without the use of either logarithms or imaginaries. EDITOR.

$$\pi = M + (-1)^n(4n)!\,2^{-2n+2}\{0!/(4n+1)! - 2!/(4n+3)! + \cdots\}, \tag{5}$$

$$\log 2 = N + (-1)^n(4n-2)!\,2^{-2n+2}\{0!/(4n-1)! - 2!/(4n+1)! + \cdots\}, \tag{6}$$

where

$$M = (-1)^n 2^{-2n+2}\left\{\binom{4n}{1}s_1 - \binom{4n}{3}s_3 + \cdots \pm 4ns_{4n-1}\right\}$$

and

$$N = 2s_{4n-2} + (-1)^n 2^{-2n+2}\left\{\binom{4n-2}{1}s_1 - \binom{4n-2}{3}s_3 + \cdots \pm (4n-2)s_{4n-3}\right\};$$

and hence both M and N are rational.

Also

$$\pi = \lim_{n\to\infty} M \text{ and } \log 2 = \lim_{n\to\infty} N.$$

From (5) we get the result that the sum of the infinite series

$$0!/(4n+1)! - 2!/(4n+3)! + \cdots$$

is a transcendental number for every positive integer n.

Specific series obtained by this method are

$$\pi = 0 + 4[0!/1! - 2!/3! + \cdots],$$

$$\pi/2 - \log 2 = 0 + 2[0!/2! - 2!/4! + \cdots],$$

$$\log 2 = 1 - 2[0!/3! - 2!/5! + \cdots],$$

$$\pi/2 + \log 2 = 5/2 - 3![0!/4! - 2!/6! + \cdots],$$

$$\pi = 10/3 - 4![0!/5! - \cdots],$$

$$\log 2 = \frac{79}{120} + 6!/2^2[0!/7! - \cdots],$$

$$\pi = 109/35 + 8!/2^2[0!/9! - \cdots],$$

$$\pi = 87217/27720 - 12!/2^4[0!/13! - \cdots].$$

RECENT PUBLICATIONS.

EDITED BY D. C. GILLESPIE, Cornell University, Ithaca, N. Y., to whom communications should be sent.

REVIEWS.

Relativity, a Systematic Treatment of Einstein's Theory. By J. RICE. London, Longmans, Green & Co., 1923. 389 pages.

Of works written originally in English and with the purpose of giving a *detailed treatment* of the general theory of relativity with *requisite proofs* this is, so far as the reviewer is aware, the third. The other two works are those of Eddington [1] and Birkhoff.[2] We do not hesitate to recommend the work of Professor Rice in preference to these two, to any reader who is but slightly acquainted with the general theory or is not well versed in the higher parts of mathematics *and* mathematical physics. In fact, our opinion of this book is so high that the only other work of its kind in any language that we would care to place with it is the *Relativitätstheorie* of M. v. Laue [3] which we believe has not been done into English.

Professor Rice's book is not large. Its 389 pages of text fall roughly into three parts. After an interesting introduction of 30 pages, the author devotes 130 pages to special relativity and 130 more to the general theory. The remainder of the book deals with world geometry.

The Einstein theory is a composite of physics and four-dimensional non-Euclidean geometry, and seldom is one well versed in both these fields. To the physicist the mathematics is very difficult and abstruse, to the mathematician the finer physical theories are all but incomprehensible. The task of any author who does not appeal to the trained specialist is thus one of extreme difficulty. We have only praise for the way Professor Rice has developed his material.

Few readers will find difficulty in the first part of the book which, as stated, treats of the restricted theory. The author makes haste slowly and the reader has time to orient himself in the new world. The author has perhaps given too much attention to the four-dimensional vector analysis; to some it may seem that he has missed the opportunity of introducing the reader to the tensor analysis by replacing these vectors by tensors based on the metric of the restricted relativity theory.

Leaving the restricted relativity we enter in Part II into the real gist of the matter and here difficulties abound in plenty. The author's first task is to develop the tensor theory based on the general quadratic form $g_{ij}dx_idx_j$. This involves the introduction of Christoffel's symbols [4] $[\alpha\beta, \gamma]$ and $\{\alpha\beta, \gamma\}$, covariant differentiation, the Riemannian curvature tensor $R_{ijk}{}^l$ and the equations of a geodesic. The notion of parallel displacement has been deferred till a late

[1] A. S. Eddington, *Mathematical Theory of Relativity,* Cambridge University Press, 1923.

[2] G. D. Birkhoff, *Relativity and Modern Physics,* Harvard University Press, 1923.

[3] Two volumes, Braunschweig, Vieweg und Sohn, 1921.

[4] See L. P. Eisenhart, *Differential Geometry,* Boston, Ginn & Co., 1909.

chapter and the reader may well wonder how one ever arrived at the complicated expressions defining the $R_{ijk}{}^l$ and why these tensors play such a dominant part in Einstein's theory. Einstein's thesis is this:—the presence of gravitating matter affects the geometry of space. Denoting the momentum energy tensor by $T_{ij} = \rho(dx_i/ds)(dx_j/ds)$, he assumes the geometrical effect is proportional to T_{ij}. The most natural geometrical tensor to take is the contracted curvature tensor R_{ij}. This is not allowable because the conservation of energy and momentum requires the divergence of T_{ij} to be 0. As the divergence of R_{ij} is not 0 while that of $R_{ij} - \frac{1}{2}g_{ij}R$ is, Einstein takes

$$R_{ij} - \tfrac{1}{2}g_{ij}R = - uT_{ij}, \tag{1}$$

where R is the invariant curvature tensor obtained by contracting R_{ij}. Writing these equations in the equivalent form

$$R_{ij} = - u(T_{ij} - \tfrac{1}{2}g_{ij}T),$$

we see that in space devoid of matter $R_{ij} = 0$. The equation div $T_{ij} = 0$ leads at once to the geodesics

$$\ddot{x}_\lambda + \{\alpha\beta, \lambda\}\dot{x}_\alpha\dot{x}_\beta = 0$$

as the path of a free particle in the field of gravitation.

In chapter XIV we have a simple treatment of the three problems that have excited endless discussion in the newspapers and vain attempts to make the man in the street understand relativity. Alas there is no royal road to relativity. We refer of course to the baffling motion of the perihelion of Mercury (43″ per century), the prediction of the deflection of a ray of light in passing a strong gravitating body (eclipse phenomenon), and the shift of the spectral lines toward the red. The last is still a matter of controversy and investigation.

Part III is devoted to world geometry and can be appreciated only by the advanced student. Einstein's spherical cylindrical world is studied in detail and some account of de Sitter's spherical space-time is given. The reviewer would have welcomed a fuller account of the latter. It not only explains the systematic shift of the spectral lines of distant stars towards the red but as Silberstein has recently shown [1] it also affords a means of determining the size of the universe by measuring the parallax and radial velocities of stellar clusters. The result gives a value of $R = 6.10^{12}$ astronomical units, roughly, and so agrees well with the result obtained by Einstein based on Kapteyn's estimate of the number of suns in a cube of 10 parsecs about our sun.

On page 348 we read "Of course in elliptical space even with Einstein's hypothesis of cylindrical time, the existence of anti-suns, etc., would no longer follow." This, we believe, rests on a misapprehension. The time it would take is πR divided by the velocity of light. In de Sitter's theory [2] the velocity slows down to 0 as $r \doteq R$, but this is not so in Einstein's theory.

---

[1] *Nature*, 1924, pp. 350 and 602.

[2] W. de Sitter, On Einstein's theory of gravitation, *Monthly Notices, Royal Astronomical Society*, vol. 78 (1916–17), p. 3.

$|R_{kl}|$ being the determinant formed of the R_{kl}. We quote the closing words of Einstein's paper: "The foregoing investigation shows that Eddington's general idea in connection with Hamilton's principle leads to a theory which is almost free from arbitrary hypotheses, which is in accord with our knowledge of gravitation and electricity and which unites these two theories in a really complete manner,"—a graceful tribute indeed to the great English astronomer.

Descartes urged that a novel theory should be made so transcendentally clear that the man in the street could understand it. We fear that the mythical man in the street must still be left in darkness but we believe that what is humanly possible to make the obscure clear, our author has achieved.

JAMES PIERPONT.

NOTES ON RECENT PUBLICATIONS.

The library of the Mathematical Association has received from P. WIJDENES, Amsterdam, the editor of *Nieuw Tijdschrift voor Wiskunde,* the following texts published by P. Noordhoff, Groningen: P. Wijdenes, *Nieuwe School-Algebra* (Deel I, II, III), and *Grafieken-Schrift;* Molenbroek and Wijdenes, *Planimetrie* (Deel I, II); Wijdenes, *Vraagstrukken over Hoogere Algebra en Rekenkunde;* Wijdenes, *Tien Jaargangen van het Nieuw Tijd. v. Wisk.* (Deel II); Versluys, *Tafel H* (logarithms); Molenbroek, *Leerboek der Stereometrie.* These examples of current Dutch texts, like other books in the Association library, are available to the members of the Association for inspection, on application to the Secretary.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

AMERICAN JOURNAL OF MATHEMATICS, volume 46, no. 2, April, 1924: "Two-dimensional tensor analysis without coördinates" by G. Y. Rainich, 71–94; "Functional operations as applied to a class of Volterra integral equations" by H. T. Davis, 95–109; "Representation of three-element algebras" by B. A. Bernstein, 110–116; "The Riemann adjoints of completely integrable systems of partial differential equations" by C. A. Nelson, 117–130; "Further types of involutorial transformations which leave each cubic surface of a web invariant" by V. Snyder, 131–140.

ANNALS OF MATHEMATICS, second series, volume 25, no. 1, September, 1923: "The history of notations of the calculus" by F. Cajori, 1–46; "New applications of a fundamental theorem of substitution groups" by G. A. Miller, 47–52; "Geodesic representation between Riemann spaces" by H. Levy and A. Bramley, 53–56; "On the residues of figurate numbers" by O. E. Glenn, 57–70; "On symmetric forms in N variables. II" by A. Dresden, 71–84; "On an infinite system of non-abelian groups of order nm^{n-1}" by W. E. Edington, 85–90.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 30, nos. 3–4, March–April, 1924: "Problems in involutorial transformations in space" by V. Snyder, 101–124; "A generalization of the syllogism" by B. A. Bernstein, 125–127; "On certain quinary quadratic forms" by E. T. Bell, 127–130; "The invariants of forms under the binary linear homogeneous group G_6 modulo 2" by O. E. Glenn, 131–139; "On the application of the theory of ideals to Diophantine analysis" by G. E. Wahlin, 140–154.

Journal of Mathematics and Physics, Massachusetts Institute of Technology, volume 3, no. 4, May, 1924: "On dynamic stresses in pseudo-continuous media" by P. Heymans, 237–252.

School Science and Mathematics, volume 24, no. 6, June, 1924: "Checking the results of classification in nine first year algebra classes by means of the Holz algebra scales" by E. W. Schreiber, 614–622; "Mistakes in the computation of standard deviations" by W. C. Eells, 623–626.

The Scientific Monthly, volume 18, no. 6, June, 1924: "The origin, nature and influence of relativity" by G. D. Birkhoff, 616–624.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. Ettlinger, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

The Pascal Circle, Trinity College, Washington, D. C. [*1922*, 419.]

The officers for the year 1922–1923 were: honorary president, Professor Marie Cecilia Mangold; president, Margaret Kelly '23; vice-president, Margaret McAuliffe '23; secretary, Agnes Perrot '24; treasurer, Anne Foley '25.

The officers for the year 1923–1924 were: honorary president, Professor Marie Cecilia Mangold; president, Helen Keller '24; vice-president, Marguerite Dwyer '24; secretary, Helena Crowley '25; treasurer, Orillia Hollis '26.

The programs for the years 1922–1923 and 1923–1924 were the following:

October 23, 1922. Business meeting. It was decided that membership should be limited to those who could maintain an average grade of 80 per cent. in mathematics.

November 6, 1922. Discussion.

November 13, 1922. "The interest of mathematics" by Margaret Kelly '23. Geometrical fallacies.

December 6, 1922. Annual social.

February 26, 1923. Geometrical fallacies.

May 23, 1923. Election for the year 1923–1924.

October 23, 1923. "History of the Pascal Circle" by Helen Keller '24. "Life of Pascal" by Helen McMahon '24.

November 13, 1923. "Mathematics and psychology" by Agnes Perrot '24. Article on the seismograph from the *Scientific Monthly* by Elizabeth Frank '26.

November 18, 1923. "The seismograph" by Rev. Francis Tondorff, S.J., Director of the Seismological Laboratory of Washington, D. C.

December 18, 1924. "Women in mathematics" by Ruth Eileen Lynch '26. Problem presented by Helena Crowley '25.

February 3, 1924. An account of the prize award to a Chicago mathematician for work in science by Helen Keller '24. "The history of mathematics" by Helen McMahon '24. Problem presented by Blanche Brunini '25.

March 26, 1924. Discussion on "The value of mathematics in training the mind" introduced by Helen Sheehan. Trick with cards by Rose O'Donnell '25.

May meeting. Election for the ensuing year.

(Report by Professor Mangold.)

The Mathematics Club of Hood College, Frederick, Md.

The club had a membership of twenty-five during the year 1923–1924. The meetings were held monthly. The following were the officers for the year: president, Helen Goodfellow '24;

Journal of Mathematics and Physics, Massachusetts Institute of Technology, volume 3, no. 4, May, 1924: "On dynamic stresses in pseudo-continuous media" by P. Heymans, 237–252.

School Science and Mathematics, volume 24, no. 6, June, 1924: "Checking the results of classification in nine first year algebra classes by means of the Holz algebra scales" by E. W. Schreiber, 614–622; "Mistakes in the computation of standard deviations" by W. C. Eells, 623–626.

The Scientific Monthly, volume 18, no. 6, June, 1924: "The origin, nature and influence of relativity" by G. D. Birkhoff, 616–624.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. Ettlinger, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

The Pascal Circle, Trinity College, Washington, D. C. [*1922*, 419.]

The officers for the year 1922–1923 were: honorary president, Professor Marie Cecilia Mangold; president, Margaret Kelly '23; vice-president, Margaret McAuliffe '23; secretary, Agnes Perrot '24; treasurer, Anne Foley '25.

The officers for the year 1923–1924 were: honorary president, Professor Marie Cecilia Mangold; president, Helen Keller '24; vice-president, Marguerite Dwyer '24; secretary, Helena Crowley '25; treasurer, Orillia Hollis '26.

The programs for the years 1922–1923 and 1923–1924 were the following:

October 23, 1922. Business meeting. It was decided that membership should be limited to those who could maintain an average grade of 80 per cent. in mathematics.

November 6, 1922. Discussion.

November 13, 1922. "The interest of mathematics" by Margaret Kelly '23. Geometrical fallacies.

December 6, 1922. Annual social.

February 26, 1923. Geometrical fallacies.

May 23, 1923. Election for the year 1923–1924.

October 23, 1923. "History of the Pascal Circle" by Helen Keller '24. "Life of Pascal" by Helen McMahon '24.

November 13, 1923. "Mathematics and psychology" by Agnes Perrot '24. Article on the seismograph from the *Scientific Monthly* by Elizabeth Frank '26.

November 18, 1923. "The seismograph" by Rev. Francis Tondorff, S.J., Director of the Seismological Laboratory of Washington, D. C.

December 18, 1924. "Women in mathematics" by Ruth Eileen Lynch '26. Problem presented by Helena Crowley '25.

February 3, 1924. An account of the prize award to a Chicago mathematician for work in science by Helen Keller '24. "The history of mathematics" by Helen McMahon '24. Problem presented by Blanche Brunini '25.

March 26, 1924. Discussion on "The value of mathematics in training the mind" introduced by Helen Sheehan. Trick with cards by Rose O'Donnell '25.

May meeting. Election for the ensuing year.

(Report by Professor Mangold.)

The Mathematics Club of Hood College, Frederick, Md.

The club had a membership of twenty-five during the year 1923–1924. The meetings were held monthly. The following were the officers for the year: president, Helen Goodfellow '24;

May 2, 1924. "Indeterminate forms" by Miss Helen Nugent '23. "Equations of wave motion" by Mr. H. A. Hoover, Gr. Election of officers for the year 1923–1924 resulted as follows: director, Mr. A. H. Eschebach, Gr.; vice-director, Mr. H. P. Doole, Gr.; secretary, Mr. M. E. Nordberg '24; treasurer, Miss Marion Miller '24; librarian, Mr. William Nielsen '25.

November 5, 1924. Business meeting. Director Eschebach presented his resignation and Mr. Doole was elected to the office, Mr. Roger I. Wilkinson being made vice-director.

November 27, 1924. Initiation and banquet. Vice-director Wilkinson gave a toast of welcome to which Miss Ruth Dewey responded. Dr. E. R. Smith spoke of the history and advantages of honorary fraternities.

December 13, 1924. Dr. Turner gave an interesting talk on "Early texts and manuscripts of mathematics" and showed his own collection of old books.

January 11, 1924. "The planimeter" by Mr. Roger I. Wilkinson '24. "The game of Nim" by Mr. W. Lee Harris, Gr. Report of the national convention of Pi Mu Epsilon by Dr. E. R. Smith.

February 1, 1924. Business meeting.

February 14, 1924. "Conic sections" by Dr. E. S. Allen

April 10, 1924. After the initiation of six new members, the officers for the coming year were elected: director, Professor Marian E. Daniells; vice-director, Mr. W. Lee Harris, Gr.; secretary, Miss Leora Porter '25; Miss Helen Smith, instructor.

On the second of May the fraternity gave a dinner for the visiting mathematicians in attendance at the annual meeting of the Iowa Section of the Mathematical Association of America. The following program of toasts was given:

Hall of Mathematics.

Guide	Dr. E. R. Smith,
Lower Floors	Mr. H. C. Tingleff '24,
Recreation Room	Pi Mu Epsilon Sax-tet,
Upper Floor	Dr. H. L. Rietz, University of Iowa,
Rotunda	Dean Maria M. Roberts,
Roof	Professor F. M. McGaw, Cornell College.

(Report by Miss Leora Porter, Secretary.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3095. Proposed by GEORGE RUTLEDGE, Massachusetts Institute of Technology.

Establish the following n identities involving the binomial coefficients of any even order:

$$1!\left\{\frac{1}{1}\binom{1}{1}\binom{2n}{n-1}-\frac{1}{2}\binom{2}{1}\binom{2n}{n-2}+\frac{1}{3}\binom{3}{1}\binom{2n}{n-3}-\cdots+(-1)^{n-1}\frac{1}{n}\binom{n}{1}\binom{2n}{0}\right\}=\frac{1}{2}\binom{2n}{n}.$$

$$3!\left\{\frac{1}{2}\binom{3}{3}\binom{2n}{n-2}\sum^{2}\binom{n-1}{1}-\frac{1}{3}\binom{4}{3}\binom{2n}{n-3}\sum^{3}\binom{n-1}{1}+\cdots+(-1)^{n-2}\frac{1}{n}\binom{n+1}{3}\right.$$
$$\left.\times\binom{2n}{0}\sum^{n}\binom{n-1}{1}\right\}=\frac{1}{2}\binom{2n}{n}.$$

$$5!\left\{\frac{1}{3}\binom{5}{5}\binom{2n}{n-3}\overset{3}{\Sigma}_{\binom{n-1}{2}}+\cdots(-1)^{n-3}\frac{1}{n}\binom{n+2}{5}\binom{2n}{0}\overset{(n)}{\Sigma}_{\binom{n-1}{2}}\right\}=\frac{1}{2}\binom{2n}{n}.$$

. . . .

$$(2n-1)!\left\{\frac{1}{n}\binom{2n-1}{2n-1}\binom{2n}{0}\overset{(n)}{\Sigma}_{\binom{n-1}{n-1}}\right\}=\frac{1}{2}\binom{2n}{n}.$$

The notation $\overset{(j)}{\Sigma}_{\binom{n-1}{i-1}}$ is used to indicate the sum of the $\binom{n-1}{i-1}$ products of the squared reciprocals of the first n integers excepting j, taken $i-1$ at a time.

The first of these identities is well known in the form

$$\binom{2n}{0}-\binom{2n}{1}+\binom{2n}{2}-\cdots+\binom{2n}{2n}=0$$

and the last one is obvious.

3096. Proposed by W. J. SIDIS, New York City.

In a scale of numeration whose radix is prime,

(1) There cannot be four distinct digits whose cubes all end in the same digit.

(2) If the cubes of two distinct digits end alike (that is, in the same digit), there will always be a third such digit.

(3) Under the conditions of (2), given any digit except 0, there will be two digits whose cubes have the same last figure as the cube of the given digit.

An extension of (2) and (3) is possible. If n is an odd prime, then, if the nth powers of two distinct digits end alike, there will be a group of n distinct digits whose nth powers end alike (that is, in the same figure), and such a group may be made to include any given digit not 0.

As an extension of (1), we may say that, under the original conditions, there cannot be $n+1$ digits whose nth powers end alike.

3097. Proposed by A. A. BENNETT, University of Texas.

Show that there are five distinct types of sets admitting associative multiplication and containing but two elements. Determine the number of types of sets admitting associative multiplication and containing three elements.

SOLUTIONS.

3044 [1923, 449]. Proposed by EUGENE M. BERRY, West Lafayette, Indiana.

1. Express $\prod_{k=1}^{k=n}\cos a_k$ as a summation of cosines, each term to be of the form $C\cos(\pm a_1 \pm a_2 \pm a_3 \pm \cdots \pm a_n)$.

2. Express $\prod_{k=1}^{k=n}\sin a_k$ as a summation of sines or cosines according as n is odd or even.

3. Express $\prod_{k=1}^{k=n}\sin a_k\cdot\prod_{j=1}^{j=m}\cos b_j$ as a summation of sines or cosines according as n is odd or even.

Solution by the Proposer.

We will take the third one first as the first two are special cases of it.

$$\begin{aligned}2^{n+m}i^n\prod_{k=1}^{n}\sin a_k\cdot\prod_{j=1}^{m}\cos b_j &= 2^{n+m}i^n\prod_{k=1}^{n}\frac{e^{ia_k}-e^{-ia_k}}{2i}\cdot\prod_{j=1}^{m}\frac{e^{ib_j}+e^{-ib_j}}{2}\\ &= \prod_{k=1}^{n}(e^{ia_k}-e^{-ia_k})\cdot\prod_{j=1}^{m}(e^{ib_j}+e^{-ib_j}).\end{aligned} \tag{1}$$

Expressed as a sum of exponential terms any term t will be of the form $\pm e^{i(\pm a_1\pm a_2\pm\cdots\pm a_n\pm b_1\pm b_2\pm\cdots\pm b_m)}$. Since in (1) e^{-ia_k} carries with it a negative sign before the e, the sign before any term will be $+$

or $-$ according as the number of negative a's is even or odd, or if we let p_t be the number of negative a's in any term t the sign of that term will be $(-1)^{p_t}$. The signs of the a's and the b's can be taken in all possible ways, so the number of terms is 2^{n+m} and we can write it as follows:

$$2^{n+m}i^n \prod_{k=1}^{n} \sin a_k \cdot \prod_{j=1}^{m} \cos b_j = \sum_{t=1}^{2^{n+m}} (-1)^{p_t} e^{i(\pm a_1 \pm a_2 \pm \cdots \pm a_n \pm b_1 \pm b_2 \pm \cdots \pm b_m)}. \qquad (2)$$

Case 1. *n odd.* For every term e^{iu} there is the term $-e^{-iu}$, for changing the sign of the exponent changes the signs of all the a's, hence changes the number of negative a's from odd to even or from even to odd. But

$$e^{iu} - e^{-iu} = i[\sin u - \sin(-u)];$$

hence we get

$$2^{n+m}i^{n-1} \prod_{k=1}^{n} \sin a_k \cdot \prod_{j=1}^{m} \cos b_j = \sum_{t=1}^{2^{n+m}} (-1)^{p_t} \sin(\pm a_1 \pm a_2 \pm \cdots \pm a_n \pm b_1 \pm \cdots \pm b_m) \qquad (3)$$

where the summation means the same as before.

Case 2. *n even.* For every term e^{iu} there is the term e^{-iu} and we have

$$e^{iu} + e^{-iu} = \cos u + \cos(-u);$$

hence we get

$$2^{n+m}i^{n} \prod_{k=1}^{n} \sin a_k \cdot \prod_{j=1}^{m} \cos b_j = \sum_{t=1}^{2^{n+m}} (-1)^{p_t} \cos(\pm a_1 \pm a_2 \pm \cdots \pm a_n \pm b_1 \pm \cdots \pm b_m). \qquad (4)$$

Since $-\sin(-u) = \sin u$ and $\cos(-u) = \cos u$, we have twice as many terms as we need in both (3) and (4). In either (3) or (4), we could take one of the a's (or one of the b's) as always positive and then divide the left member of the equation by two. In (3), we could exclude all the negative terms and then divide the left member of the equation by two. It also should be noticed that in (3) and (4) the power of i is such as to equal either plus or minus one.

As a special case of (3), we have

$$2^n i^{n-1} \prod_{k=1}^{n} \sin a_k = \sum_{t=1}^{2^n} (-1)^{p_t} \sin(\pm a_1 \pm a_2 \pm \cdots \pm a_n)$$

where n is odd. Two special cases of (4) are

$$2^n i^n \prod_{k=1}^{n} \sin a_k = \sum_{t=1}^{2^n} (-1)^{p_t} \cos(\pm a_1 \pm a_2 \pm \cdots \pm a_n)$$

where n is even and

$$2^m \prod_{j=1}^{m} \cos b_j = \sum \cos(\pm b_1 \pm b_2 \cdots \pm b_m).$$

The summation in each is to be taken the same as before and as in (3) and (4) contains twice as many terms as necessary. This can be taken care of in the same manner as in (3) and (4).

3046 [1923, 449]. Proposed by A. L. WECHSLER, New York City.

What is the probability that there will be at least r consecutive heads out of n tosses of a coin?

SOLUTION BY OTTO DUNKEL, Washington University.

If $P(n)$ is the number of different ways at least r consecutive heads may turn up in n tosses of a coin, then the probability that at least r consecutive heads will turn up is $p(n) = P(n)/2^n$.

If the coin is tossed $n + 1$ times, then, since the $(n + 1)$th toss may give a head or tail, $P(n + 1)$ is at least as great as $2P(n)$. There are, however, a certain number of unfavorable cases in n tosses which become favorable if the $(n + 1)$th toss is a head. These are the cases in which less than r consecutive heads turn up in the first $n - r$ tosses, then a tail, and at last

$r-1$ consecutive heads. The number of such cases is $2^{n-r} - P(n-r)$. We have then for the total number of favorable cases

$$P(n+1) = 2P(n) - P(n-r) + 2^{n-r}, \tag{1}$$

and then by dividing by 2^{n+1}, we have

$$p(n+1) = p(n) - \frac{p(n-r)}{2^{r+1}} + \frac{1}{2^{r+1}}. \tag{2}$$

This last equation is to be solved with the conditions that $p(n) = 0$ when n is less than r, and $p(r) = 1/2^r$. If we neglect the term in (2) with the minus sign, then we easily find by setting in (2) $n = r, r+1, \cdots,$

$$p(n) = \frac{n-r+2}{2^{r+1}}, \qquad r \leqq n \leqq 2r. \tag{3}$$

The above is correct for the limited value of n, but in any case we shall have such a term. In the case of n greater than $2r$ we shall have to subtract a function of n and r. By setting in (2) $n = 2r, 2r+1, \cdots, 2r+j$, it will easily be seen that this function is of the second degree in j, if $j \leqq r$. Also it must vanish for $j = 0$ in order to reduce to (3). Hence we may write the subtracted function as $j(Aj+B)$. Setting this expression in (2) and equating coefficients we find

$$p(2r+j) = \frac{r+j+2}{2^{r+1}} - \frac{1}{2^{2r+2}}\frac{j(j+3)}{2}, \qquad j \leqq r+1. \tag{4}$$

In order to find $p(3r+j)$, it should be observed that the last term will be a polynomial of the third degree in j, which may be seen as before, and it must vanish for $j = 0, 1$, in order to reduce to (4). Writing it as $j(j-1)(Aj+B)$ and inserting it in (2), we find on comparing coefficients that

$$p(3r+j) = \frac{2r+j+2}{2^{r+1}} - \left(\frac{1}{2^{r+1}}\right)^2 \frac{(r+j)(r+j+3)}{2} + \left(\frac{1}{2^{r+1}}\right)^3 \frac{j(j-1)(j+4)}{3!}. \tag{5}$$

It will be seen that this is true for $j \leqq r+2$.

This suggests the form

$$p(ir+j) = \sum_{k=1}^{k=i} (-1)^{k+1} \left(\frac{1}{2^{r+1}}\right)^k {}_{(i-k)r+j}C_{k-1} \frac{(i-k)r+j+k+1}{k}, \qquad j \leqq r+i-1, \tag{6}$$

where ${}_mC_k$ is the coefficient of x^k in the expansion of $(1+x)^m$. It will be found by substitution in (2) that this satisfies the equation for $j \leqq r+i-1$, and hence gives the solution.

For $n = 50$, $r = 5$, we find $p(50) = .55188$.

3047 [1923, 449]. Proposed by ARNOLD DRESDEN, University of Wisconsin.

Prove that for any positive integer n,

$$\sum \frac{1}{\Pi_{i=1}^{t} p_i!\, k_i^{p_i}} = 1,$$

where k_i and p_i are positive integers, k_i being the distinct elements of any t-partite partition of $n (t = 1 \cdots n)$ and p_i the number of parts of the partition which are equal to k_i.

SOLUTION BY E. T. BELL, University of Washington.

The theorem is the result of equating coefficients of x^n ($|x| < 1$) in

$$1 + x + x^2 + \cdots = 1 + \frac{1}{1!}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) + \frac{1}{2!}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)^2 + \cdots,$$

which is the expanded form of the identity

$$\frac{1}{1-x} = e^{\log(1/1-x)}.$$

3049 [1923, 450]. Proposed by H. GROSSMAN (Student), College of the City of New York.

Prove that every factor of $2^{2^n} + 1$ is congruent to 1 mod 2^{n+1}, and that no two different numbers of the form $2^{2^n} + 1$ have a common factor.

SOLUTION BY A. S. WIENER, Cornell University.

Since $2^{2^n} + 1$ is odd, its factors must be odd. Let p be any prime factor of $2^{2^n} + 1$. Then $2^{2^n} + 1 \equiv 0$, mod p, and $2^{2^n} \equiv -1$, mod p. Hence $2^{2^{n+1}} \equiv 1$, mod p.

Suppose that the exponent to which 2 belongs, mod p, is d. Then $2^{n+1} = Md$. Hence, $M = 2^\nu$, where $\nu \geqq 0$. Then $d = 2^{n+1-\nu}$. Suppose $\nu > 0$. Then $2^n = 2^{\nu-1}2^{n+1-\nu}$. Hence, $2^{2^n} \equiv 1$, mod p, which contradicts $2^{2^n} \equiv -1$, mod p. Our assumption was false and $\nu = 0$. Hence, $d = 2^{n+1}$, and the exponent to which 2 belongs mod p is 2^{n+1}.

$2^{p-1} \equiv 1$, mod p. Hence, $p - 1 = M2^{n+1}$ and $p \equiv 1$, mod 2^{n+1}.

Let N be any factor of $2^{2^n} + 1$ and assume $N = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$. $p_r \equiv 1$, mod 2^{n+1}, by the last congruences. Hence, $p_r^{\alpha_r} \equiv 1$, mod 2^{n+1}, and, therefore, $\prod_1^n p_r^{\alpha_r} \equiv 1$, mod 2^{n+1}. Hence, $N \equiv 1$, mod 2^{n+1}, *i.e.*, every factor of $2^{2^n} + 1$ is congruent to 1, mod 2^{n+1}. Suppose $2^{2^n} + 1$ and $2^{2^m} + 1$ have a common factor N, and let p be a prime factor of N. Then $2^{2^n} + 1$ and $2^{2^m} + 1$ have a common factor p. Now the exponent of 2 belonging to mod p is 2^{n+1} and 2^{m+1}. This is impossible unless $n = m$. Hence, no two different numbers of the form $2^{2^n} + 1$ have a common factor; and the theorem is proved.

A more general problem would be: Prove that every odd factor of $a^{2^n} + 1$ is congruent to 1 mod 2^{n+1}; two numbers $a^{2^n} + 1$ and $a^{2^m} + 1$ have no common odd factor unless $n = m$.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

Professor E. C. COKER, for 18 years professor of mathematics at Winthrop College, Rock Hill, S. C., has been appointed professor of astronomy and mathematics at the University of South Carolina. His successor at Winthrop College is Dr. G. T. PUGH.

At Lehigh University, Professor P. A. LAMBERT has been promoted to be head of the department of mathematics and astronomy.

The Franklin Institute of Philadelphia has awarded a Franklin medal and an honorary certificate of membership to Sir ERNEST RUTHERFORD, of Cambridge University.

The National Academy of Sciences has awarded its Watson medal to Professor C. V. L. CHARLIER, of the University of Lund, and its Henry Draper medal to Professor A. S. EDDINGTON, of Cambridge University. Professor Eddington will lecture on general relativity at the University of California during the first semester of 1924–25, and will conduct a seminar on sidereal astronomy.

New York University has conferred the honorary degree of doctor of laws on Professor M. I. PUPIN, of Columbia University.

Professor A. B. COBLE has been elected a member of the National Academy of Sciences.

3049 [1923, 450]. Proposed by H. GROSSMAN (Student), College of the City of New York.

Prove that every factor of $2^{2^n} + 1$ is congruent to 1 mod 2^{n+1}, and that no two different numbers of the form $2^{2^n} + 1$ have a common factor.

SOLUTION BY A. S. WIENER, Cornell University.

Since $2^{2^n} + 1$ is odd, its factors must be odd. Let p be any prime factor of $2^{2^n} + 1$. Then $2^{2^n} + 1 \equiv 0$, mod p, and $2^{2^n} \equiv -1$, mod p. Hence $2^{2^{n+1}} \equiv 1$, mod p.

Suppose that the exponent to which 2 belongs, mod p, is d. Then $2^{n+1} = Md$. Hence, $M = 2^\nu$, where $\nu \geqq 0$. Then $d = 2^{n+1-\nu}$. Suppose $\nu > 0$. Then $2^n = 2^{\nu-1}2^{n+1-\nu}$. Hence, $2^{2^n} \equiv 1$, mod p, which contradicts $2^{2^n} \equiv -1$, mod p. Our assumption was false and $\nu = 0$. Hence, $d = 2^{n+1}$, and the exponent to which 2 belongs mod p is 2^{n+1}.

$2^{p-1} \equiv 1$, mod p. Hence, $p - 1 = M2^{n+1}$ and $p \equiv 1$, mod 2^{n+1}.

Let N be any factor of $2^{2^n} + 1$ and assume $N = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$. $p_r \equiv 1$, mod 2^{n+1}, by the last congruences. Hence, $p_r^{\alpha_r} \equiv 1$, mod 2^{n+1}, and, therefore, $\prod_1^n p_r^{\alpha_r} \equiv 1$, mod 2^{n+1}. Hence, $N \equiv 1$, mod 2^{n+1}, *i.e.*, every factor of $2^{2^n} + 1$ is congruent to 1, mod 2^{n+1}. Suppose $2^{2^n} + 1$ and $2^{2^m} + 1$ have a common factor N, and let p be a prime factor of N. Then $2^{2^n} + 1$ and $2^{2^m} + 1$ have a common factor p. Now the exponent of 2 belonging to mod p is 2^{n+1} and 2^{m+1}. This is impossible unless $n = m$. Hence, no two different numbers of the form $2^{2^n} + 1$ have a common factor; and the theorem is proved.

A more general problem would be: Prove that every odd factor of $a^{2^n} + 1$ is congruent to 1 mod 2^{n+1}; two numbers $a^{2^n} + 1$ and $a^{2^m} + 1$ have no common odd factor unless $n = m$.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

Professor E. C. COKER, for 18 years professor of mathematics at Winthrop College, Rock Hill, S. C., has been appointed professor of astronomy and mathematics at the University of South Carolina. His successor at Winthrop College is Dr. G. T. PUGH.

At Lehigh University, Professor P. A. LAMBERT has been promoted to be head of the department of mathematics and astronomy.

The Franklin Institute of Philadelphia has awarded a Franklin medal and an honorary certificate of membership to Sir ERNEST RUTHERFORD, of Cambridge University.

The National Academy of Sciences has awarded its Watson medal to Professor C. V. L. CHARLIER, of the University of Lund, and its Henry Draper medal to Professor A. S. EDDINGTON, of Cambridge University. Professor Eddington will lecture on general relativity at the University of California during the first semester of 1924–25, and will conduct a seminar on sidereal astronomy.

New York University has conferred the honorary degree of doctor of laws on Professor M. I. PUPIN, of Columbia University.

Professor A. B. COBLE has been elected a member of the National Academy of Sciences.

Dr. B. H. Brown has been promoted to an assistant professorship of mathematics at Dartmouth College.

Assistant Professor Harris Rice, of Worcester Polytechnic Institute, has been promoted to a full professorship.

Dr. Norbert Wiener, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

Assistant Professor Ernest Flammer, of Queen's University, Kingston, Ontario, has been promoted to an associate professorship of mathematics.

Assistant Professor E. J. Oglesby, of New York University, has been promoted to an associate professorship of mathematics.

Assistant Professor H. S. Everett, of Bucknell University, has been promoted to an associate professorship of mathematics.

Dr. J. A. Nyswander has been appointed assistant professor of mathematics at Swarthmore College.

Associate Professor J. J. Luck, of the University of Virginia, has been promoted to a full professorship of mathematics.

Dr. L. T. E. Thompson has been appointed physicist at the Naval Proving Ground, Dahlgren, Va.

Assistant Professor R. A. Sheets, of Denison University, has been appointed assistant professor of mathematics at Miami University.

Assistant Professor W. G. Simon, of Western Reserve University, has been promoted to an associate professorship of mathematics.

Assistant Professor C. O. Williamson, of the College of Wooster, has been promoted to a full professorship of mathematics.

Assistant Professor F. W. Reed, of Ohio University, has been promoted to an associate professorship.

Assistant Professor J. J. Nassau, of the Case School of Applied Science, has been promoted to an associate professorship of mathematics and astronomy.

Associate Professor C. L. Arnold, of Ohio State University, has been promoted to a full professorship of mathematics.

At the University of Cincinnati, Associate Professor C. N. Moore has been promoted to a full professorship and Dr. I. A. Barnett to an assistant professorship of mathematics.

Professor W. A. Hamilton, formerly of Beloit College, who has been a special lecturer at the University of Wisconsin during the past year, has been appointed head of the department of mathematics at Antioch College, Yellow Springs, Ohio.

Associate Professor K. P. Williams, of Indiana University, has been promoted to a full professorship of mathematics.

Assistant Professor L. C. Emmons, of Michigan Agricultural College, has been promoted to a full professorship of mathematics.

At the University of Michigan, Associate Professors J. W. BRADSHAW and T. H. HILDEBRANDT have been promoted to full professorships. Assistant Professor C. J. COE and Instructors W. M. COATES and D. K. KAZARINOFF are on leave of absence studying in Europe.

Dr. C. A. GARABEDIAN, of Harvard University, has been appointed assistant professor of mathematics at Northwestern University.

President W. A. GRANVILLE, of Gettysburg College, has resigned to become educational director of the United States National Life and Casualty Company, Chicago.

Associate Professor W. D. MACMILLAN, of the University of Chicago, has been promoted to a full professorship of mathematical astronomy.

Dr. FREDRICK WOOD, of the University of Wisconsin, has been appointed professor of mathematics at Lake Forest College.

At the University of Illinois, Dr. C. C. CAMP has been promoted to be an associate in mathematics. Professor H. BLUMBERG has been granted leave of absence for the academic year 1924–1925.

At the University of Missouri, Professor G. E. WAHLIN, of the University of Illinois, has been appointed professor; Associate Professor LOUIS INGOLD has been promoted to a full professorship, to become effective at the expiration of his leave of absence for the year 1924–1925; Dr. E. F. ALLEN has been promoted to an assistant professorship; and Dr. HERMAN BETZ, of Yale University, has been appointed assistant professor.

Mr. L. V. ROBINSON, of the University of Virginia, has been appointed assistant professor of mathematics and astronomy at Oklahoma City College.

Mr. W. M. WHYBURN, of the South Park Junior College, Beaumont, Texas, has been appointed assistant professor of mathematics at Texas Agricultural and Mechanical College.

Miss MARY CAMPBELL, of the University of Texas, has been appointed professor and head of the department of mathematics at South Park Junior College.

Associate Professor G. H. CRESSE, of the University of Arizona, has been promoted to a full professorship of mathematics.

Associate Professor I. L. MILLER, of South Dakota State College, has been promoted to a full professorship of mathematics.

Dr. F. M. WEIDA, of Iowa State University, has been appointed assistant professor of mathematics at Montana State College.

Assistant Professor L. S. DEDERICK, of the United States Naval Academy, has been appointed professor of mathematics at the University of British Columbia.

At the University of California, Assistant Professors B. A. BERNSTEIN and FRANK IRWIN have been promoted to associate professorships and Dr. SOPHIA H. LEVY has been promoted to an assistant professorship of mathematics. At the Southern Branch, at Los Angeles, Associate Professor G. E. F. SHERWOOD

has been promoted to a full professorship, and Dr. P. H. Daus to an assistant professorship.

Dr. Harold Hotelling, of Princeton University, has been appointed junior assistant at the Food Research Institute, Stanford University.

Dr. H. M. Jeffers, of Iowa State University, has been appointed assistant astronomer at the Lick Observatory.

Dr. James Ouspensky of Leningrad, a delegate from the Russian Academy of Sciences to the meeting of the International Mathematical Congress at Toronto, was a recent visitor at the University of Michigan. While at Michigan, he presented to the mathematics club a summary of some remarkable results obtained in 1917 by a student of his, Vinogradoff by name, on the distribution of residues and non-residues of powers.

Associate Professor R. W. Brink is in Europe on sabbatical leave from the University of Minnesota.

The following appointments to instructorships are announced:

Wellesley College, Miss Ethel L. Anderton;
Smith College, Miss Bess M. Eversull;
New York University, Dr. Constance R. Ballantine;
Lehigh University, Mr. C. A. Balof;
University of Florida, Mr. C. G. Phipps;
Purdue University, Mr. J. C. Bennett;
University of Iowa (astronomy), Dr. D. H. Menzel;
University of Missouri, Dr. L. H. MacFarlan;
University of Wisconsin, Mr. H. S. Pollard;
University of Michigan, Dr. J. A. Shohat, Messrs. A. P. Maslow, G. S. Van Fleet, W. C. Green.

Professor J. E. Hodgson, of West Virginia University, died April 11, 1924.

Dr. R. S. Woodward, retired president of the Carnegie Institution, died June 30, 1924, at the age of seventy-four years.

Immediately after the close of the International Mathematical Congress at Toronto, Professor Charles de la Vallée Poussin of the University of Louvain started upon a lecture tour in the United States under the auspices of the Educational Foundation of the Commission for Belgian Relief. His first visit was at the University of Chicago where he stayed for a week, during which time he gave three lectures in French on the general topic "L'approximation des fonctions de variables réelles et les fonctions quasi-analytiques." These lectures were commented upon in English, topic by topic, by Professor Maurice Fréchet of the University of Strasbourg, who was a member of the Chicago staff for the summer quarter. These lectures were greatly appreciated by the members of the Chicago faculty and the two hundred or more graduate students in the department. Similar lectures were subsequently given by Professor de la Vallée Poussin at the Universities of California, Michigan, Minnesota, Wisconsin and other universities.

Work has been slowly going on for some years past at the Astrophysical Observatory of Arcetri, near Florence, Italy, on the hill sacred to the memory of Galileo (for it was here that he made many of his observations and discoveries) for the building of a Sun-Tower similar to that already existing near the Mount Wilson Observatory in California. The Tower is very near Galileo's last residence, which was also the scene of his death, and is intended to serve as a lasting memorial to this great master of science of the sixteenth century. Had it not been for the unexpected increase in the price of materials due to the war, the Tower would have been finished some years ago. Owing however to this cause, it is estimated that, notwithstanding a recent generous contribution from the Ministry of Public Instruction of Italy, a further sum of 30,000 Lire will be needed for its completion. Contributions are therefore requested from scientists and others in all parts of the world. They should be addressed to Cav. Isacco Ciabattari, *Società Leonardo da Vinci, Via dei Corsi, 5, Firenze, Italy.*

The following research courses in mathematics are to be offered in 1924–25 at the "Institute De Mathématiques" de l'Universite de Strasbourg, France, in addition to the usual fundamental courses on analysis, astronomy and mechanics. These research courses are particularly fitted to prepare candidates for the "Diplôme d'études superieures de mathématiques" and for the Doctorates.

First Semester (Nov. 1924 to Febr. 1925). By Professor BAUER, Theory of quanta (2 hours).—By Professor CERF, Singular solutions of differential equations (1 hour).—By Professor FRÉCHET, Theory of abstract sets (3 hours).

Second Semester (March to June, 1925). By Professor BAUER, Constitution of atoms (2 hours).—By Professor FRÉCHET, Smoothing of empirical functions (3 hours).—By Professor THIRY, Hydrodynamics (Selected topics) (2 hours).—By Professor VALIRON, A new theory of integral and meromorphic functions (2 hours).—By Professor VILLAT, On some generalizations of Lamé's differential equations and on minimal surfaces (2 hours).

Isis, an International Review devoted to the History of Science and Civilization, is the official organ of the History of Science Society. It is edited by GEORGE SARTON with associate editors: C. H. HASKINS of Harvard University, R. C. ARCHIBALD of Brown University, J. K. WRIGHT of the American Geographical Society. F. BARRY of Columbia University edits its "Queries and Answers," H. E. BARNES of Smith College its "Teaching and Personalia," and F. E. BRASCH of the Dept. of Terrestrial Magnetism, Washington, D. C., its "News of the History of Science Society" and "Obituaries."

A new edition of the final report of the National Committee on Mathematical Requirements, entitled *The Reorganization of Mathematics in Secondary Education,* has been printed and is ready for distribution. A nominal charge of twenty cents per copy has been found necessary in order to defray packing and shipping costs. Orders for this report with the necessary remittance should be sent to the Dartmouth Press, Hanover, New Hampshire.

THE CARUS MONOGRAPHS.

The first Carus Monograph is ready for the printer. It is entitled: "*The Calculus of Variations*" by G. A. BLISS, of the University of Chicago, and is intended for readers who have not specialized in mathematics beyond the calculus. It will make a book of about 190 pages, size of type page 3½ by 6 inches, and will contain 45 wax engravings. It will be printed in clear readable type, with ample margins, on high quality paper, and will be bound in stiff cloth covers.

It will be published for the Mathematical Association of America by the Open Court Publishing Company, of Chicago, Illinois, and will be sold by the Association directly to its individual and institutional members at a pre-publication price of \$1.25 per copy, covering the actual cost of production and mailing in separate cartons. It will be sold after publication to the general public at \$2.00 per copy by the Open Court Publishing Company with its widely distributed clientele both in America and in foreign countries.

The Committee of Publication feel justified in assuming that every member of the Association will desire to subscribe for this and the subsequent numbers of the Carus Monograph series, publication of which is made possible by the generous gift of MRS. MARY HEGELER CARUS, sole trustee of the Edward C. Hegeler Trust Fund. For a full account of the terms and spirit of this gift see this MONTHLY, October, 1921, pages 352–354, and June, 1923, pages 151–155. In order to simplify the procedure as much as possible for both officers and members of the Association, the Secretary will merely add this item of \$1.25 to the statement of annual dues for 1925 to be sent out in December, 1924, and a copy of Monograph number one will be mailed to each member as soon as his return shall have been received,—unless this item is crossed off.

The manuscript of the second Monograph is also nearly ready on "*Functions of a Complex Variable*" by D. R. CURTISS of Northwestern University. Its publication will follow as soon as practicable.

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the Editor-in-Chief W. B. Ford, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. Carver, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, Oberlin, Ohio.

The following are dates of Section meetings of the Association in 1924 (unless otherwise specified):

Illinois, Elgin, May 2–3
Iowa, Iowa State College, Ames, May 2–3
Kansas, Topeka, February 2
Kentucky, Center College, April
Maryland–Virginia–District of Columbia Annapolis, December 8, 1923
Michigan, Ann Arbor, April 3
Minnesota, Hamline University, St. Paul, May 24
Missouri, Kansas City, November or December
Ohio, Ohio State University, Columbus, April 4–5
Rocky Mountain, Laramie, April, 1925
Southeastern, University of Georgia, Athens March 7–8
Texas, San Antonio, November 29–30

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924

NUMBER 9, NOVEMBER

PUBLISHED BY THE ASSOCIATION

LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others

THE INTERNATIONAL MATHEMATICAL CONGRESS AT TORONTO.

By invitation of the University of Toronto and the Royal Canadian Institute, the International Mathematical Congress was held at the University of Toronto, August 11–16, 1924, in conjunction with the meeting of the British Association for the Advancement of Science which was held August 6–13. Previous congresses had been held as follows: Zurich 1896, Paris 1900, Heidelberg 1904, Rome 1908, Cambridge 1912, Strasbourg 1920. Because many members of the Mathematical Association have made inquiries as to the names of those in attendance and because the proceedings will probably not appear for a year or more, it has seemed desirable to give a fairly full account of the Congress. The list of members of the Congress actually in attendance as here given was furnished by the secretary, Professor J. L. Synge:

ARGENTINA.

C. D. PERRINE, Observ. Nacional, Cordoba.
R. A. VAGO, Arg. Embassy, Washington.

BELGIUM.

A. DEMOULIN, Gand.
L. GODEAUX, Univ. of Brussels.
G. LEMAITRE, Brussels.
E. MERLIN, Gand.
C. SERVAIS, Brussels.
C. DE LA VALLÉE POUSSIN, Univ. of Louvain.

CANADA.

F. D. ADAMS, McGill Univ.
J. R. AMBROISE, Toronto.
D. BUCHANAN, Univ. of Br. Columbia.
C. A. CHANT, Univ. of Toronto.
R. H. COATS, Bur. of Statistics, Ottawa.
H. J. DAWSON, Royal Milit. Coll., Kingston.
A. T. DE LURY, Univ. of Toronto.
L. L. DINES, Univ. of Saskatchewan.
W. P. DOBSON, Hydro-Elec. Power Comm., Toronto.
J. M. DUNCAN, Toronto.
H. B. DWIGHT, Hamilton.
J. D. FERNANDEZ, Toronto.
ALAN FERRIER, Ottawa.
J. C. FIELDS, Univ. of Toronto.
W. FINDLAY, McMaster Univ.
B. A. GILL, Queen's Univ.
A. H. S. GILLSON, McGill Univ.
J. C. GLASHAN, Ottawa.
C. F. GUMMER, Queen's Univ.
C. B. HAMILTON, Toronto.
J. W. HAYWARD, Quebec.
H. P. L. HILLMAN, Hydro-Elec. System, Toronto.
T. H. HOGG, Toronto.
L. HUME, Dept. Natl. Defence, Ottawa.
A. F. HUNTER, Toronto.
T. T. IRVING, Toronto.
L. V. KING, McGill Univ.
H. R. KINGSTON, Univ. of Western Ont.
S. LANZON, Toronto.
A. LEVÉILLÉ, Univ. of Montreal.
G. H. LING, Univ. of Saskatchewan.
C. H. LIGHTHOUSE, Aeronaut. Dept., Ottawa.
N. B. MACLEAN, Univ. of Manitoba.
J. MATHESON, Queen's University.
D. MCARTHUR, Montreal.
W. MCKNIGHT, Tech. Coll., Halifax.
J. MCLEISH, Dept. of Mines, Ottawa.
J. C. MCLENNAN, Univ. of Toronto.
A. R. MCMICHAEL, Toronto.
H. MCTAGGART, Toronto.
N. MILLER, Queen's Univ.
A. MORPHY, Toronto.
D. H. NELLES, Geod. Surv., Ottawa.
A. NEVEILLE, Univ. of Montreal.
N. OGILVIE, Geod. Surv., Ottawa.
J. G. PARKER, Imper. Life Assur. Co., Toronto.
J. PATTERSON, Toronto.
W. J. PATTERSON, Univ. of Western Ont.
J. S. PLASKETT, Dom. Astrophys. Observ., Victoria.
A. POULIOT, Quebec.
L. N. RICHARDSON, Royal Milit. Coll., Kingston.
C. A. ROSE, Am. Nickel Corp., Toronto.
T. R. ROSEBRUGH, Univ. of Toronto.
C. H. RUST, Elec. Light Co., Toronto.
F. SANDERSON, Toronto.
N. A. SHAW, McGill Univ.
R. G. STAGG, Toronto.
A. F. STEVENSON, Toronto.
C. T. SULLIVAN, McGill Univ.
J. L. SYNGE, Univ. of Toronto.
J. J. TRAILL, Hydro-Elec. Co., Toronto.

Miss E. G. Waddell, Univ. of Toronto.
L. A. H. Warren, Univ. of Manitoba.
T. R. Wilkins, Brandon Coll.
H. H. Wolfenden, Grimsby.
W. A. P. Wood, Can. Life Assur. Co., Toronto.
R. B. Young, Toronto.

China.

S. D. Carothers, Hong Kong.

Cuba.

Y. I. Del Corral, Havana.

Czecho-Slovakia.

B. Bydžovský, Univ. of Prag.

Denmark.

T. Bonnesen, Copenhagen.
E. Schou, Copenhagen.
J. F. Steffensen, Hallerup.

France.

Jules Andrade, Univ. of Besancon.
Commandant Barre, Paris.
J. Bosler, Observ. de Marseille.
L. Brillouin, Collège de France.
E. Cartan, Sorbonne.
Gen. V. Charbonnier, Ministr. de la Marine.
J. Chazy, Paris.
V. R. d'Adhemar, Univ. of Lille.
Jules Drach, Paris.
L. Dunoyer, Neuilly.
F. Faure, Dir. de la Rev. polit. et parl.
M. Fréchet, Univ. of Strasbourg.
P. Haag, Clermont-Ferrand.
M. Huber, Paris.
M. Janet, Univ. of Rennes.
G. Koenigs, Sorbonne.
J. Lemoine, Paris.
J. LeRoux, Univ. of Rennes.
A. Levy, Lycée Saint-Louis.
Jean Mascart, Observ., Lyon.
M. Pigeaud, Paris.
Dr. Planiol, École d'Aeronautique, Paris.
J. B. Pomey, Paris.
R. Wurmser, Collège de France.

Georgia.

A. Razmadze, Tiflis.

Great Britain.

Miss E. E. Austin, Bromley.
Arch. Barr, Westerton, Scotland.
T. H. Beare, Univ. of Edinburgh.
A. L. Bowley, Univ. Coll., London.
S. Chapman, Univ. of Manchester.
A. W. Conway, Univ. of Ireland.
P. Cormack, Univ. of Dublin.
A. L. Cortie, Stonyhurst Coll.
J. G. Crowther, London.
A. C. Dixon, Univ. of Belfast.
A. T. Doodson, Tidal Inst., Liverpool.
A. S. Eddington, Cambridge Univ.
R. A. Fisher, Harpenden.
A. P. M. Fleming, Met.-Vickers Elec. Co., London.
A. Fowler, Royal Coll. of Sc.
Henry Fowler, Derby.
I. G. Gibbon, Min. of Health, London.
G. R. Goldsbrough, Sunderland.
A. E. Grasett, Staff Coll., Camberley.
J. G. Gray, Univ. of Glasgow.
G. Greenhill, London.
H. R. Hasse, Univ. of Bristol.
J. B. Henderson, Royal Naval Coll.
G. W. O. Howe, Univ. of Glasgow.
F. R. W. Hunt, Kent.
H. Jeffreys, St. Johns Coll., Cambridge.
C. F. Jenkin, Oxford Univ.
P. A. MacMahon, Cambridge.
E. H. Parker, Glasgow.
W. P. Phelps, London.
H. C. Plummer, Artill. Coll.
A. Robertson, Univ. of Bristol.
L. F. Richardson, Westminster Training Coll.
Ida B. Saxby, Cardiff, S. Wales.
F. J. Selby, Teddington.
Napier Shaw, London.
W. F. Sheppard, Cardona.
D'Arcy Thompson, Univ. of St. Andrews.
F. P. White, St. Johns Coll., Cambridge.
L. Woollard, The Admiralty, London.
W. H. Young, Univ. of Wales.
G. U. Yule, St. Johns Coll., Cambridge.

Holland.

J. A. Barrau, Univ. of Groningen.
W. Kapteyn, Univ. of Utrecht.
W. Van der Woude, Univ. of Leiden.
J. Wolff, Univ. of Utrecht.

India.

G. Prasad, Hindu Univ., Benares.
C. V. Raman, Calcutta.

Italy.

E. Bortolotti, Univ. of Bologna.
M. de Franchis, Univ. of Palermo.
G. Fubini, Univ. of Turin.
F. G. Gianfranceschi, Univ. Gregoriana, Rome.
C. Gini, Univ. of Padua.
G. Giorgi, Inst. of Elec. Engineers, Rome.
G. Muzi, Univ. of Pisa.
G. Peano, Univ. of Turin.

S. PINCHERLE, Univ. of Bologna.
U. PUPPINI, Scuola Ingegneri, Bologna.
F. SEVERI, Univ. degli Studi, Rome.
L. TONELLI, Univ. of Bologna.

NORWAY.

R. BIRKELAND, Univ. of Christiania.
J. BJERKNES, Bergen.
W. BJERKNES, Geophys. Inst., Bergen.
O. ORE, Univ. of Christiania.
C. STØRMER, Univ. of Christiania.

POLAND.

W. SIERPINSKI, Univ. of Warsaw.
S. ZAREMBA, Univ. of Cracovie.

PORTUGAL.

DA COSTA LOBO, Lisbon.
F. DE VASCONCELLOS, Univ. of Lisbon.

ROUMANIA.

G. TZITZEICA, Univ. of Bucarest.

RUSSIA.

M. GUNTHER, St. Petersbourg.
V. A. KOSTITZIN, Moscow.
N. KRYLOFF, Univ. of Kieff.
J. OUSPENSKY, Leningrad.
STEKLOFF, Univ. of Leningrad.
A. V. VASILIEVE, Moscow.

SERVIA.

M. PETROVITCH, Univ. of Belgrade.

SPAIN.

D. J. ALVAREZ-UDE, Madrid.
M. CASTRO BONEL, Univ. of Madrid.
D. TORROJA Y MIRET, Univ. of Barcelona.

SWEDEN.

E. HOLMGREN, Upsala Observ.
J. MALMQUIST, Univ. of Stockholm.
L. E. PHRAGMEN, Kungl. Vetenskapsakad., Stockholm.

SWITZERLAND.

L. J. CRELIER, Univ. of Berne.
H. FEHR, Univ. of Geneva.
R. FUETER, Univ. of Zurich.
M. PLANCHEREL, Univ. of Zurich.

UNITED STATES.

C. R. ADAMS, Brown Univ.
H. L. ALDEN, Univ. of Virginia.
P. L. ALGER, Schenectady, N. Y.
MISS M. ALLEN, Mt. Holyoke Coll.
N. H. ANNING, Univ. of Michigan.
R. C. ARCHIBALD, Brown Univ.
H. T. R. AUDE, Colgate Univ.
M. J. BABB, Univ. of Pennsylvania.
CLARA L. BACON, Goucher Coll.
GRACE M. BAREIS, Ohio State Univ.
L. A. BAUER, Dept. Terr. Magn., Washington.
E. T. BELL, Univ. of Washington.
B. A. BERNSTEIN, Univ. of California.
W. J. BERRY, Brooklyn Poly. Inst.
G. A. BLISS, Univ. of Chicago.
R. L. BORGER, Ohio Univ.
A. BOYAJIAN, Gen. Elec. Co., Pittsfield, Mass.
W. E. BROOKE, Univ. of Minnesota.
F. P. BRACKETT, Pomona Coll.
H. E. BRAY, Rice Inst.
G. BREIT, Dept. Terr. Magn., Washington.
W. C. BRENKE, Univ. of Nebraska.
L. J. BRIGGS, Bur. of Standards, Washington.
R. W. BRINK, Univ. of Minnesota.
E. W. BROWN, Yale Univ.
H. S. BROWN, Hamilton Coll.
E. BUCKINGHAM, Bur. of Standards, Washington.
J. A. BULLARD, U. S. Naval Acad.
W. G. BULLARD, Syracuse Univ.
W. D. CAIRNS, Oberlin Coll.
F. CAJORI, Univ. of California.
G. A. CAMPBELL, Am. T. & T. Co., New York.
I. S. CARROLL, Syracuse.
J. R. CARSON, New York.
C. C. CARTER, Bluffs, Ill.
C. W. CARTER, Am. T. & T. Co., New York.
E. H. CLARKE, Hiram Coll.
G. R. CLEMENTS, U. S. Naval Acad.
A. B. COBLE, Univ. of Illinois.
J. B. COLEMAN, Univ. of South Carolina.
JULIA T. COLPITTS, Iowa State Coll.
LENNIE P. COPELAND, Wellesley Coll.
G. H. CRESSE, Univ. of Arizona.
H. CREW, Northwestern Univ.
LOUISE D. CUMMINGS, Vassar Coll.
H. L. CURTIS, Bur. of Standards, Washington.
D. R. CURTISS, Northwestern Univ.
MARIAN E. DANIELLS, Iowa State Coll.
S. C. DAVISSON, Indiana Univ.
H. M. DADOURIAN, Trinity Coll., Conn.
L. E. DICKSON, Univ. of Chicago.
T. C. DICKSON, Watertown Arsenal.
ELEANOR C. DOAK, Mt. Holyoke Coll.
B. F. DOSTAL, Univ. of Michigan.
A. DRESDEN, Univ. of Wisconsin.
J. A. EIESLAND, West Virginia Univ.
J. S. ELSTON, Travelers Ins. Co., Hartford, Conn.
G. C. EVANS, Rice Inst.
H. S. EVERETT, Bucknell Univ.
H. B. FERNALD, New York.
P. FIELD, Univ. of Michigan.
A. FISHER, Western Union Tel. Co., New York.
J. A. FOBERG, Dept. of Pub. Instr., Harrisburg, Pa.

W. B. Ford, Univ. of Michigan.
T. C. Fry, Am. T. & T. Co., New York.
W. F. Ferhardt, Dayton, Ohio.
J. L. Gibson, Univ. of Utah.
R. E. Gilman, Brown Univ.
O. E. Glenn, Univ. of Pennsylvania.
J. W. Glover, Univ. of Michigan.
W. S. Hamilton, Univ. of Wisconsin.
May N. Harwood, Syracuse Univ.
M. W. Haskell, Univ. of California.
L. A. Hazeltine, Stevens Tech. Coll.
Olive C. Hazlett, Mt. Holyoke Coll.
C. M. Hebert, New York.
N. H. Heck, Coast & Geod. Surv., Washington.
E. R. Hedrick, So. Branch, Univ. of California.
Robert Henderson, Equit. Life Assur. Co., New York.
E. Hille, Princeton Univ.
T. R. Hollcroft, Wells Coll.
Helma L. Holmes, Univ. of Texas.
Anna M. Howe, Newcomb Coll.
Goldie P. Horton, Univ. of Texas.
R. S. Hoyt, River Edge, N. J.
D. Hull, Univ. of Notre Dame.
W. J. Humphreys, Weather Bur., Washington.
J. I. Hutchinson, Cornell Univ.
Emma Hyde, Kansas State Agric. Coll.
S. A. Joffe, Mut. Life Ins. Co., New York.
L. C. Karpinski, Univ. of Michigan.
E. H. Kennard, Cornell Univ.
A. Korzybski, New York.
H. W. Kuhn, Ohio State Univ.
Elizabeth R. Laird, Mt. Holyoke Coll.
P. A. Lambert, Lehigh Univ.
A. E. Landry, Catholic Univ. of Amer.
D. D. Laun, Univ. of Chicago.
Elizabeth LeStourgeon, Univ. of Kentucky.
H. C. Levinson, Chicago.
Mayme I. Logsdon, Univ. of Chicago.
L. A. McColl, Western Elec. Co., New York.
C. C. MacDuffee, Ohio State Univ.
W. Marshall, Purdue Univ.
H. H. Marvin, Univ. of Nebraska.
T. E. Mason, Purdue Univ.
S. J. Mauchly, Dept. Terr. Magn., Washington.
G. A. Miller, Univ. of Illinois.
G. R. Mirick, Lincoln School, New York.
Eugenie M. Morenus, Sweet Briar Coll.
F. Morley, Johns Hopkins Univ.
E. C. Molina, Am. T. & T. Co., New York.
C. N. Moore, Univ. of Cincinnati.
M. Morris, Case School of Appl. Sc.
H. M. Morse, Cornell Univ.
E. J. Moulton, Northwestern Univ.
F. R. Moulton, Univ. of Chicago.
J. R. Musselman, Johns Hopkins Univ.
A. McAdie, Readville, Mass.
G. F. McEwen, Univ. of California.
J. McGiffert, Rensselaer Poly. Inst.
J. J. Nassau, Case School of Appl. Sc.
Edna P. Pepper, Chicago.
T. A. Pierce, Univ. of Nebraska.
J. Pierpont, Yale Univ.
E. A. Porter, Indianapolis.
V. E. Pound, Univ. of Buffalo.
L. J. Reed, Johns Hopkins Univ.
J. N. Rice, Catholic Univ. of Amer.
R. G. D. Richardson, Brown Univ.
G. M. Robison, Trinity Coll., N. C.
C. K. Robbins, Purdue Univ.
E. D. Roe, Jr., Syracuse Univ.
Mrs. E. D. Roe, Syracuse, N. Y.
W. H. Roever, Washington Univ.
J. H. Rogers, Univ. of Missouri.
J. Rosenbaum, Milford, Conn.
D. A. Rothrock, Indiana Univ.
F. H. Safford, Univ. of Pennsylvania.
Ida M. Schottenfels, Chicago.
W. H. Sherk, Univ. of Buffalo.
W. A. Shewhart, Western Elec. Co., New York.
J. A. G. Shirk, Kansas State Tea. Coll.
J. A. Shohat, Univ. of Chicago.
L. Silberstein, Eastman Kodak Co.
W. G. Simon, Western Reserve Univ.
T. McN. Simpson, Randolph-Macon Coll.
T. M. Simpson, Univ. of Florida.
Mary Emily Sinclair, Oberlin Coll.
C. H. Sisam, Colorado Coll.
E. B. Skinner, Univ. of Wisconsin.
C. S. Slichter, Univ. of Wisconsin.
H. L. Slobin, Univ. of New Hampshire.
A. W. Smith, Colgate Univ.
Clara E. Smith, Wellesley Coll.
Helen Smith, Iowa State Coll.
Sarah E. Smith, Mt. Holyoke Coll.
Virgil Snyder, Cornell Univ.
May J. Sperry, Syracuse Univ.
C. E. St. John, Mt. Wilson Observ.
W. F. G. Swann, Univ. of Chicago.
K. D. Swartzel, Univ. of Pittsburgh.
J. S. Taylor, Mass. Inst. of Tech.
E. M. Thomas, Boston.
W. M. Thornton, Univ. of Virginia.
M. O. Tripp, Wittenberg Coll.
W. H. Tschappat, Ord. Dept., Washington.
Bird M. Turner, West Virginia Univ.
A. L. Underhill, Univ. of Minnesota.
G. Van Biesbroeck, Yerkes Observ.
H. S. Vandiver, Cornell Univ.
J. A. L. Waddell, New York.
J. H. Weaver, Ohio State Univ.
A. H. Wheeler, Worcester, Mass.
H. S. White, Vassar Coll.
W. Whited, Pa. State Highway Dept.
A. W. Whitney, Natl. Bur. of Cas. & Surety Underwr., New York.
R. L. Wilder, Univ. of Texas.

W. L. G. Williams, Cornell Univ.
W. F. Willcox, Cornell Univ.
E. B. Wilson, Harvard Univ.
Elizabeth W. Wilson, Washington.
Ruth G. Wood, Smith Coll.
E. W. Woolard, Weather Bur., Washington.
B. F. Yanney, Coll. of Wooster.
C. H. Yeaton, Oberlin Coll.
O. J. Zobel, New York.

At the opening session in Convocation Hall on Monday morning addresses of welcome were given by Minister H. S. Beland, M.D., in English and French on behalf of the Dominion government, and by President Robert Falconer on behalf of the University. Addresses followed by Dr. J. C. Fields, Chairman of the Organizing Committee, who was elected president of the Congress, and by M. de la Vallée Poussin, president of the International Mathematical Union. The list of delegates from various sections, societies, colleges and universities was read by Professor Koenigs, the general secretary of the Union. A group photograph of the Congress was taken at noon and was placed on sale later in the week by the University of Toronto Press.

Monday afternoon and Tuesday, Wednesday, Friday and Saturday mornings were devoted to sessions for reading contributed papers in the following sections: I, Algebra, theory of numbers, analysis; II, Geometry; III (*a*), Mechanics, mathematical physics; III (*b*), Astronomy, geophysics; IV (*a*), Electrical, mechanical, civil and mining engineering; IV (*b*), Aeronautics, naval architecture, ballistics, radio-telegraphy; V, Statistics, actuarial science, economics; VI, History, philosophy, didactics. The number of papers read in these various sections were respectively 51, 38, 24, 11, 28, 14, 23, 13, 15 other papers being given before III (*a*) and III (*b*) jointly, and 4 others before IV (*a*) and IV (*b*) jointly. Excellent abstracts of the majority of papers were distributed at the time of registration; these are still available to a limited extent through the University of Toronto Press. Besides these, eight general lectures were given at other times during the week:

C. Størmer, "Modern Norwegian researches on the aurora borealis;"

F. Severi, "Géométrie algébrique;"

E. Cartan, "La théorie des groupes et les recherches récentes de géométrie différentielle;"

W. H. Young, "Some characteristic features of twentieth century pure mathematical research;"

L. E. Dickson, "Outline of the theory to date of the arithmetics of algebras;"

S. Pincherle, "Opérations fonctionelles;"

J. LeRoux, "Sur l'intégration des équations aux derivées partielles par des intégrales définies;"

J. Pierpont, "Non-Euclidean geometry from a non-projective standpoint."

It will be of interest to all mathematicians of America to note that Professor Fueter of the Editorial Committee for the works of Euler announced that the cost of printing these is so enhanced that only by obtaining an increased number of subscriptions will it be possible to complete the enterprise with any degree of promptness.

The business meeting of the International Mathematical Union held on

finish to a week of festivities highly appreciated by all in attendance. Sums contributed by the members of the Congress provided for a wreath deposited Saturday afternoon at the memorial to the University's soldier dead and for a permanent tablet which is to be placed thereon.

W. D. CAIRNS.

THE MATHEMATICAL ASSOCIATION.

The Trustees of the Association have approved of the organization of a Nebraska Section. Account of the organization meeting appeared in the October number of the MONTHLY.

The following forty-eight individuals and eight institutions, on applications duly certified, have been elected to membership in the Association:

To Individual Membership.

HARRIET ANDERSON, B.S. (Iowa State Coll.). Prof., Grand Island Coll., Grand Island, Nebr.
ETHEL L. ANDERTON, A.M. (Yale). Instr., Wellesley Coll., Wellesley, Mass.
MAURICE BACKER, A.B. (Westminster). Henderson, Ky.
L. F. BENSON, B.S. (East Texas State Teachers Coll.). Forney, Texas.
W. W. BIGELOW, A.B. (Beloit). Computer, Coast & Geodetic Surv., Washington, D. C.
GREGORY BREIT, Ph.D. (Johns Hopkins). Dept. of Terrestrial Magnetism, Washington, D.C. *36th St. and Broad Branch Rd.*
MARY A. CAMPBELL, A.M. (Texas). Head of dept. of math., South Park Jr. Coll., Beaumont, Texas.
JULIA R. S. CHELLBORG, B.S. (Hunter). Asst. prof., Hunter Coll., New York, N. Y.
RUFUS CRANE, A.B. (Middlebury), B.S. (Mass. Inst. of Tech.). Asst. prof. of eng., Ohio Wesleyan Univ., Delaware, Ohio.
C. W. R. CRUM, M.D. Brunswick, Md.
L. A. DEESZ, B.S. (Carnegie Inst.). Asst. supt. of elec. dept., Colo. Fuel and Iron Co., Pueblo, Colo.
BERTHA K. DUNCAN, A.M. (Texas). Head of dept. of math., Grenada Coll., Grenada, Miss.
W. M. EWING. Junior student, Rice Inst., Houston, Texas.
J. M. HARTSFIELD. Junior student, Rice Inst., Houston, Texas.
(Recommended by Professors Rice and Ford.)
ANNIE W. FLEMING, A.M. (California). Asst. prof., Iowa State Coll., Ames, Ia.
J. F. FOX. Mathematician, Coast & Geodetic Surv., Washington, D. C.
ALICE A. GRANT, A.B. (Toronto; McMaster). 238 Gano St., Providence, R. I.
F. C. HENROTEAU, Dr. of physics and math. (Brussels). Head of astrophysical dept., Dominion Observ., Ottawa, Can.
A. L. HILL, A.B. (Doane). Head of dept. of math., Peru, Nebr.
ELINOR V. HOLLIS, M.S. (Chicago). 11 Boynton St., Worcester, Mass.
H. K. HUGHES, A.B. (Iowa). Grad. asst., Univ. of Iowa, Iowa City, Ia.
J. F. HULSE, A.B. (Coll. of Ozarks). 705 W. 22½ St., Austin, Texas.
MABEL HUTCHINS, A.B. (Blue Mountain). Head of dept. of math., Blue Mountain Coll., Blue Mountain, Miss.
MARY F. JACKSON, B.S. (Nebraska). Teacher, High School, Lincoln, Nebr.
STELLA B. KIRKER, A.M. (Nebraska). Head of dept. of math., Senior High School, Lincoln, Nebr.
G. A. KREINES, B.S. (New York Univ.). 1001 Lafayette Ave., Brooklyn, N. Y.
OLGA LARSON, A.M. (Missouri). Asst. prof., Florida State Coll. for Women, Apopka, Fla.
E. P. MARTINSON, A.M. (Nebraska). Instr., Colo. School of Mines, Golden, Colo.
L. J. MCBANE, B.S. (Case School). Case School of Appl. Sc., Cleveland, Ohio.
E. L. MICKELSON, A.B. (Minnesota). Instr., Hamline Univ., St. Paul, Minn.
H. G. MILLINGTON, C.E. (Rensselaer). Asst. prof., Coll. of Eng., Univ. of Vt., Burlington, Vt.

J. J. Nassau, Ph.D. (Syracuse). Asso. prof. of math. and astr., Case School of Appl. Sc., Cleveland, Ohio.
A. W. Osterhout, A.B. (Cotner). Dept. of math., Cotner Coll., Bethany, Nebr.
R. D. Perry, B.S. (Southwest Texas Tea. Coll.). Instr., Teachers Coll., Greeley, Colo.
J. D. Phenix, A.B. (Texas). Teacher, High School, Austin, Texas.
H. M. Phillips, B.S. (Syracuse; St. Lawrence). De Kalb Junction, N. Y.
H. S. Pollard, A.B. (Olivet). Asst., Univ. of Iowa, Iowa City, Ia.
W. R. Porter, B.S. (Carnegie Inst.). Instr., Mass. Agric. Coll., Amherst, Mass.
G. C. Priester, M.S. (Minnesota). Asst. prof. of math. and mech., Univ. of Minnesota, Minneapolis, Minn.
P. K. Rees, A.B. (Southwestern). Teacher, Jr. High School, Austin, Texas.
W. A. Rees, A.B. (Southwestern). Teacher, High School, Austin, Texas.
G. A. Ross, A.M. (George Washington). Teacher, Central High School, Washington, D. C.
L. L. Smail, Ph.D. (Columbia). Asst. prof., Univ. of Oregon, Eugene, Ore.
Mabel M. Stewart, A.B. (Greenville, Ill.), B.S. (Central State Tea. Coll., Okla.). Edmond, Okla.
G. W. Sublette, A.B. (North Missouri State Normal). 1929 Third Ave. S., Minneapolis, Minn.
S. B. Townes, A.B. (Oklahoma). 125 W. 2d St., Oklahoma City, Okla.
Esther M. Weaver, M.S. (Chicago). Instr., Northwestern Univ., Evanston, Ill.
Mrs. Grace R. West, B.S. (North Texas State Tea. Coll.). Teacher, High School, Austin, Texas.

To Institutional Membership.

University of California, Southern Branch, Los Angeles, Calif. E. R. Hedrick, Official representative.
Georgia School of Technology, Atlanta, Ga. Pres. M. L. Brittain, Official representative.
Illinois Wesleyan University, Bloomington, Ill. H. P. Pettit, Official representative.
Blue Mountain College, Blue Mountain, Miss. Pres. W. T. Lowrey, Official representative.
Northwest Missouri State Teachers College, Maryville, Mo.
Otterbein College, Westerville, Ohio. B. C. Glover, Official representative.
Allegheny College, Meadville, Pa. O. P. Akers, Official representative.
The Drexel Institute, Philadelphia, Pa. H. C. Wolff, Official representative.

W. D. Cairns, *Secretary-Treasurer.*

ANNUAL MEETING OF THE TEXAS SECTION.

The third annual meeting of the Texas section of the Mathematical Association of America was held in conjunction with the mathematics section of the Texas State Teachers Association at Ft. Worth, Texas, on Friday morning and afternoon, December 1, 1923. Dr. C. N. Wunder, dean of Southwestern University, Georgetown, Texas, and professor of mathematics, presided over the morning session as chairman of the mathematics section of the Texas State Teachers Association. The afternoon session was presided over by Professor L. R. Ford, of Rice Institute, Houston, Texas, chairman of the Texas section of the Association.

There were fifty-five (55) present, including the following members of the Association:

J. M. Bledsoe, Myrtle C. Brown, J. E. Burnam, J. W. Calhoun, J. P. Downer, H. J. Ettlinger, G. C. Evans, L. R. Ford, A. J. Hargett, W. A. Nelson, Hugh Porter, E. R. Tucker, C. N. Wunder.

The following papers were presented:

(1) "Some interesting theorems of elementary geometry" by Professor J. M. BLEDSOE, East Texas State Teachers College;

(2) "The esthetic element in mathematics" by Professor J. W. CALHOUN, University of Texas;

(3) "Plane geometry constructions by means of the compasses alone" by Professor H. J. ETTLINGER, University of Texas;

(4) "Numerical integration of differential equations" by Professor L. R. FORD, Rice Institute;

(5) "Some further applications of Duhamel's theorem including integration of a non-uniformly convergent series termwise" by Professor H. J. ETTLINGER, University of Texas;

(6) "Mathematics of economics" by Professor G. C. EVANS, Rice Institute.

The present officers were reëlected to serve for the next year. It was voted to instruct the secretary to ascertain the sentiment of the members by a mail questionnaire on the proposition of changing the time of meeting, so as to have a separate meeting at Christmas time.

H. J. ETTLINGER, *Secretary-Treasurer.*

ON THE INTEGRABILITY OF A CONTINUOUS FUNCTION.

By H. J. ETTLINGER, University of Texas.

In a recent paper [1] the writer gave a proof of the integrability of a continuous function, based on Moore's form of Duhamel's theorem.[2] It is the object of this paper to give a proof of what is essentially Moore's form of Duhamel's theorem from which the integrability of a continuous function is obtained as a direct corollary. As in the former paper, use is made of a limit theorem on areas.[3] The present proof gives a principle for obtaining an outer sum which is a monotonically decreasing function of n, which in simplicity leaves nothing to be desired.

We will now state the principal theorem.

FUNDAMENTAL THEOREM ON SUMMATION: *Let the interval $I : a \leqq x \leqq b$ be divided into n subdivisions in any manner whatsoever, each of length $\Delta x_i = \Delta x_i(n)$, such that if P is any fixed point of I whose abscissa is x_P, and $\Delta x_P = \Delta x_P(n)$ is a subdivision which contains P for every value of n, then* $\lim n \rightarrow \infty\ \Delta x_P = 0$. *Let $H_i = H_i(n)$ be any set of numbers corresponding to Δx_i, such that* (1) $|H_i| \leqq K$

[1] H. J. Ettlinger, "A Simple Form of Duhamel's Theorem and Some New Applications," this MONTHLY (*1922*, 239–250).

[2] R. L. Moore, "On Duhamel's Theorem," *Annals of Mathematics*, second series, vol. 13, 1912, pp. 161–166. For other forms of this theorem and substitutes thereof, see the preceding reference, this MONTHLY (*1922*, 240–243).

[3] *L. c.*, this MONTHLY (*1922*, 243). See also H. J. Ettlinger, "An Elementary Proof of a Fundamental Lemma Concerning the Limit of a Sum," *Bulletin of the American Mathematical Society*, 1923, vol. 29, pp. 219–223.

The following papers were presented:

(1) "Some interesting theorems of elementary geometry" by Professor J. M. BLEDSOE, East Texas State Teachers College;

(2) "The esthetic element in mathematics" by Professor J. W. CALHOUN, University of Texas;

(3) "Plane geometry constructions by means of the compasses alone" by Professor H. J. ETTLINGER, University of Texas;

(4) "Numerical integration of differential equations" by Professor L. R. FORD, Rice Institute;

(5) "Some further applications of Duhamel's theorem including integration of a non-uniformly convergent series termwise" by Professor H. J. ETTLINGER, University of Texas;

(6) "Mathematics of economics" by Professor G. C. EVANS, Rice Institute.

The present officers were reëlected to serve for the next year. It was voted to instruct the secretary to ascertain the sentiment of the members by a mail questionnaire on the proposition of changing the time of meeting, so as to have a separate meeting at Christmas time.

H. J. ETTLINGER, *Secretary-Treasurer.*

ON THE INTEGRABILITY OF A CONTINUOUS FUNCTION.

By H. J. ETTLINGER, University of Texas.

In a recent paper [1] the writer gave a proof of the integrability of a continuous function, based on Moore's form of Duhamel's theorem.[2] It is the object of this paper to give a proof of what is essentially Moore's form of Duhamel's theorem from which the integrability of a continuous function is obtained as a direct corollary. As in the former paper, use is made of a limit theorem on areas.[3] The present proof gives a principle for obtaining an outer sum which is a monotonically decreasing function of n, which in simplicity leaves nothing to be desired.

We will now state the principal theorem.

FUNDAMENTAL THEOREM ON SUMMATION: *Let the interval* $I : a \leqq x \leqq b$ *be divided into n subdivisions in any manner whatsoever, each of length* $\Delta x_i = \Delta x_i(n)$, *such that if P is any fixed point of I whose abscissa is* x_P, *and* $\Delta x_P = \Delta x_P(n)$ *is a subdivision which contains P for every value of n, then* $\lim n \rightarrow \infty\ \Delta x_P = 0$. *Let* $H_i = H_i(n)$ *be any set of numbers corresponding to* Δx_i, *such that* (1) $|H_i| \leqq K$

[1] H. J. Ettlinger, "A Simple Form of Duhamel's Theorem and Some New Applications," this MONTHLY (*1922*, 239–250).

[2] R. L. Moore, "On Duhamel's Theorem," *Annals of Mathematics*, second series, vol. 13, 1912, pp. 161–166. For other forms of this theorem and substitutes thereof, see the preceding reference, this MONTHLY (*1922*, 240–243).

[3] *L. c.*, this MONTHLY (*1922*, 243). See also H. J. Ettlinger, "An Elementary Proof of a Fundamental Lemma Concerning the Limit of a Sum," *Bulletin of the American Mathematical Society*, 1923, vol. 29, pp. 219–223.

for every value of i ($\leqq n$) *and* n, *where* K *is a positive constant, and* (2) *if* $H_P = H_P(n)$ *is that number which corresponds to* Δx_P, *let* $\lim n \to \infty\ H_P = f(x_P)$, *where* $f(x)$ *is a single-valued continuous function over* I. *Then*

$$\lim_{n\to\infty} \sum_1^n H_i \Delta x_i \text{ exists.}$$

Proof: Designate the points of division of I by $x_i = x_i(n)$, $(i = 1, 2, \cdots, n-1)$, the internal points of division, together with the end points $x_0 = a$ and $x_n = b$. Now for any particular value of n consider all the interior points of division $x_1(2)$; $x_1(3)$, $x_2(3)$; $x_1(4)$, $x_2(4)$, $x_3(4)$; $x_1(5)$, $x_2(5)$, $x_3(5)$, $x_4(5)$; $\cdots$; $x_1(n)$, $x_2(n)$, $\cdots$, $x_{n-1}(n)$. For any particular value of n let the total number of these points be $p = p(n)$. It is clear that p will be at least as great as $n - 1$, since although any point of a given set $x_i(s)$ may coincide with a point of a set previously chosen $x_j(r)$, where $r < s$, none of the points $x_i(n)$, that is, the last $n - 1$ points chosen, may coincide with each other. The totality of the pairs of consecutive internal points of division, together with a and b, divide I into $p + 1$ subdivisions which we shall call $\delta_i = \delta_i(n)$.

Let $Y_i = Y_i(n)$ be the largest value of $f(x)$ in δ_i, and M and m the largest value and smallest value respectively of $f(x)$ in I. Form the outer sums

$$\begin{aligned} \bar{A}(1) &= Y_1\delta_1 = M(b-a), \\ \bar{A}(2) &= Y_1\delta_1 + Y_2\delta_2, \\ &\cdots\cdots\cdots\cdots \\ \bar{A}(n) &= Y_1\delta_1 + Y_2\delta_2 + \cdots + Y_{p+1}\delta_{p+1} = \sum_1^{p+1} Y_i\delta_i, \end{aligned} \tag{1}$$

where on the right side of the equation, the dependence of the numbers on n is not explicitly stated for brevity's sake. Now the outer sum $\bar{A}(n)$ is a monotonic decreasing function of n, for if the number of subdivisions is increased from n to $n + 1$, any subdivision $\delta_i(n)$ will either remain unchanged, *i.e.*, unsubdivided, or it will be divided into several parts; but in any case the maximum value of $f(x)$ in any one of these new subdivisions of δ_i cannot be greater than $Y_i(n)$. Now for all values of n

$$\bar{A}(n) \geqq m(b-a).$$

Hence by a well-known limit theorem $\bar{A}(n)$ will approach a limit, A, as $n \to \infty$. It is to be noticed that since $p \geqq n - 1$, p will increase without limit as $n \to \infty$. The limit A is the *upper Darboux integral*, whether $f(x)$ be continuous or not, provided $f(x)$ is bounded.

Now let $\Delta x_i = \Delta x_i(n) = x_i(n) - x_{i-1}(n)$ and form the sum

$$\sum_1^n H_i \Delta x_i. \tag{2}$$

We may compare this sum with (1) if we properly correlate the $\delta_i(n)$ of (1) with the corresponding $\Delta x_i(n)$ of which it is a part and choose the same $H_i(n)$

for all those $\delta_i(n)$'s which are part of the same $\Delta x_i(n)$. If we take the difference between (2) and (1) written in terms of δ_i's, we have

$$\sum_1^{p+1} [Y_i - H_i]\delta_i. \tag{3}$$

If $\delta_P = \delta_P(n)$ is that subdivision which for each value of n contains the point P where P is any fixed point of I whose abscissa is x_P, and $Y_P = Y_P(n)$ and $H_P = H_P(n)$ the corresponding values of $Y_i(n)$ and $H_i(n)$ respectively, then at the same time that $n \to \infty$, $\delta_P \to 0$, $Y_P \to f(x_P)$ and $H_P \to f(x_P)$ since $\Delta x_P \to 0$ and $f(x)$ is continuous. We note also that the numerical value of the difference between Y_i and H_i is never greater than the larger of the two numbers $|K - |M||$, $|K - |m||$. Hence we may apply the lemma on the limit of such a sum [1] and obtain

$$\lim_{p\to\infty} \sum_1^{p+1} H_i\delta_i = \lim_{n\to\infty} \sum_1^{n} H_i\Delta x_i = A.$$

Hence the Fundamental Theorem on Summation is proved.

We may choose as a particular set of numbers H_i, the values of $f(x)$, $f(t_i)$, where $t_i = t_i(n)$ is any abscissa in $\Delta x_i(n)$, for if t_P is the abscissa corresponding to Δx_P, then since $f(x)$ is continuous we have $\lim n \to \infty\ f(t_P) = f(x_P)$ and $f(t_i)$ is never greater than the larger of the two numbers $|M|$, $|m|$. Hence

Corollary I. *$f(x)$ is integrable over I.* We will remark that it can be shown that the limit A is the same for two different modes of subdivision, by considering the totality of intervals defined by each pair of consecutive points of the total number of points of both subdivisions and choosing the values of $f(x)$ on these intervals in accordance with the principle of choice of H_i in the proof of the theorem above. Hence $A = \int_a^b f(x)dx$, and we have

Corollary II.

$$\lim_{n\to\infty} \sum_1^{n} H_i\Delta x_i = \int_a^b f(x)dx.$$

Finally we will note that if in the original theorem $y_i = y_i(n)$, where $y_i(n)$ is the smallest value of $f(x)$ in $\delta_i(n)$, be chosen for each $\delta_i(n)$, we obtain a monotonically increasing inner sum by means of which it is readily proved that the *lower Darboux integral* exists, provided $f(x)$ is bounded.[2]

---

[1] H. J. Ettlinger, *Bulletin of the American Mathematical Society, l. c.*, p. 219; *cf.* remark on unequal subdivisions on p. 221, preceding case 2.

[2] It may be readily seen from Moore's form of Duhamel's theorem (*l. c.*) that the upper and lower Darboux integrals are equal if $f(x)$ is continuous "almost everywhere" over I, that is, except for a null set. The above argument also lends itself readily to a very simple proof that if $f(x)$ is *measurable* over I, it is integrable over I in the sense of Lebesgue. The details of such a proof have been carried through by W. L. Ayres, a graduate student at the University of Texas.

THE HISTORY OF MODERN CALCULATING MACHINES, AN AMERICAN CONTRIBUTION.

By L. LELAND LOCKE.

During the past two decades the calculating machine has been developed and commercialized to such an extent that it may be said to rival logarithms in importance as a labor-saving device. Coming as the climax of three centuries of somewhat unproductive experimentation, its perfecting lacks the picturesqueness which belongs to the invention of logarithms. Just how soon the latter device will be remembered only as a curiosity in the development of mathematics it is difficult to predict. Certain it is that the calculating machine has not attracted the attention of the mathematician to the extent it deserves, witness the complete absence of literature on the subject in American technical journals and an almost equal void in foreign publications.

The available information concerning such machines is chiefly to be found in patents, descriptive articles on the mechanical features of particular machines, catalogs of collections and exhibitions, advertising material (sometimes with historical notes of more or less value), and a few general treatises. Moreover, these various elements, with diverse and often competing aims, have produced an ambiguous terminology which is a source of dissatisfaction to the careful reader. Inaccuracies in statement, once made, have been repeated, and incorrect dating, particularly where priority is involved, has become a serious fault. The first of the following notes is merely suggestive in the matter of nomenclature in laying a foundation for the second note, which is a compilation of facts bearing on one of the more important questions of priority.

1. Terminology. Mathematics is a science which has slowly evolved with the culture of the race and from the needs of everyday life, the concepts and terms being the result of a long-continued process of refinement. The designing of calculating machines is an art which has been derived from the science itself and it has the disadvantage of possessing a terminology appropriated from and based on that of mathematics, this terminology occasionally being none too well defined. It is but natural that the inventor should choose the most high-sounding phrases to describe his work, a practice scarcely so serious in its effects as the very prevalent tendency of some manufacturers to endeavor to make a machine which is designed for a particular purpose function in every conceivable situation. The term "calculating machine" is often applied to a mechanism which has no mechanical capacity for carrying out the four processes of addition, subtraction, multiplication, and division. Such machines were primarily designed for one or more of these four fundamental operations, the remaining processes being worked out by means of applied formulas. For example, any adding machine may be used for calculations involving subtraction, multiplication, and division, as it is possible to work out all of these through the medium

of addition, with pencil and paper as well as with a machine. In order to be mathematically correct, however, the name *Calculating Machine* should be reserved for such machines as have the mechanical capacity for working out each of the four operations by a direct mathematical method.

No serious attempt has yet been made to standardize this terminology although Lenz [1] has given a brief statement of the practice used in the German Patent Office. For the purpose of the next note the following convenient terminology is suggested.

Machine. Consider a decimal scale, where nine units have been registered in any order. If, when the tenth unit is registered in this order, provision is made automatically to carry one to the next higher order, this mechanical feature may be said to transform the device into a *machine.* The Japanese soroban is probably the most versatile calculating device ever created but lacks the one feature essential to classify it as a machine.

Counting Machine. A machine designed to receive entries in the units' or ones' order only, the register in the higher orders being the accumulations from the carry, may be called a *counting machine.* Such a machine is exemplified in any one of the various types of meters.

Adding Machine. A machine designed to receive entries in all orders, successively or simultaneously, may be called an *adding machine.*

Calculating Machine. Two additional features will be deemed necessary to produce a calculating machine: (1) a carriage by which the numeral wheels or registering dials and the selector mechanism may have their relative positions shifted; (2) provision for the performance of subtraction directly. By "directly" is meant that the number to be subtracted is entered on the set-up and combined with the number recorded in the machine, either by a reversal of the numeral wheels or by mechanically produced over-addition. In subtraction by over-addition the difference between the subtrahend and the next higher power of ten is added to the minuend. The 1 from the carry is taken care of by inserting a succession of 9's or by providing a cut-off for the carry at the proper point. In an article on "Calculating Machine Mathematics" in *The Mathematics Teacher,* XV, 7, the use of the word *supplement* was suggested for the difference between a number and infinity, infinity being here defined as a power of ten which will carry the 1 off the machine. An earlier use of the word in a similar sense by Barr in certain British patents has since been noted.

For descriptive purposes the parts of the common calculating machine, with their functions, may be named as follows:

Set-up Mechanism. The device by which a given number is entered on the machine will be designated the *set-up mechanism.* The older form of set-up was usually a slide, either a straight line or an arc of a circle. This form is now being rapidly replaced by the keyboard. The set-up mechanism is sometimes called the *installing mechanism,* a somewhat more desirable term.

Selector Mechanism. The mechanism which selects the proper mechanical

[1] K. Lenz: *Die Rechenmaschinen und das Maschinenrechnen,* Leipzig, 1915, p. 22.

movements to correspond to the number set up is designated as the *selector* or *differential mechanism*. It may well be called the brains of the machine, the design of which determines one of the commoner classifications of machines. The Leibniz-Thomas stepped cylinder is shown in fig. 2 and the Baldwin cam-operated radial pins in fig. 3. Among other types may be mentioned the rack-and-pinion of the Millionaire and Mercedes machines, the mechanical Pythagorean table of the Bollée and the Millionaire, and the squirrel cage or lantern wheel of the Monroe.

Registering Mechanism. The set of circular or cylindrical dials or numeral wheels upon which results are usually recorded is called the *registering mechanism.*

Carry Mechanism. The function of the carry mechanism is fully implied in the term.

Control Mechanism. All devices for the prevention of inaccurate operation, mechanical or human, such as checks for overrotation, locking devices and stops, will be included under the term *control mechanism.*

Erasing Mechanism. The various types of devices used to return the numeral dials to the zero position are known as *erasing* or *zero-setting mechanisms.*

2. A new classification of calculating machines. The classifications heretofore used have been either on a basis of function, as adding, listing, multiplying, dividing, etc., or of the employment of essential mechanical elements, as the stepped cylinder, cam-operated radial pins, mechanical Pythagorean table, etc. The new classification is based on the sequence of operations as follows:

Type I.	Monophase machines,
Type II.	Non-reversible cycle machines,
Type III.	Reversible cycle machines.

In type I the carry action occurs simultaneously with the digital selection or registration. Such machines may be unidirectional or reversible, the latter type not having been developed commercially. In type II the digital registration and the carrying of tens occur in cyclic order. The cycle is divided into two parts, the first part serving for digital registration, and the second part for the carrying of the tens from the lower to the higher numeral wheels. Obviously this cyclic action is not reversible and in order to reverse from positive to negative numeral

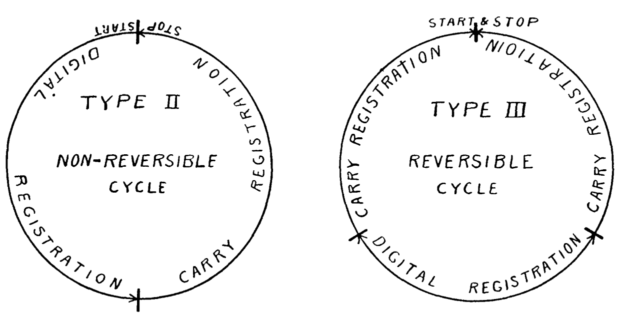


FIG. 1.

wheel registrations, special reversing mechanism must be installed which will not reverse the cycle. Such mechanism may be manually or automatically controlled. In type III the digital registration and the carrying of the tens occur in reversible cyclic order. The zone of digital registration is located midway in the cycle, with a carrying zone provided on each side. The carry action will therefore function after the digital registration in either direction of operation. In this type no special reversing mechanism is required, merely a reversal of direction of the driving shaft which operates the machine. Figure 1 shows the two latter types in diagrammatic form.

3. The Baldwin-Odhner priority claims. To Pascal belongs the honor of having conceived and constructed the first calculating machine of which certain models and a description have been preserved. Of more than fifty models constructed by Pascal, several are extant, the oldest of date 1642. Following the classification above, this would be called an adding machine. However it is interesting to note Pascal's genius in discovering the process of subtraction by overaddition, very similar to the method commonly employed with keydrive machines. Of the many attempts following the work of Pascal, only those in direct sequence in the development of the commercial calculating machine of today will be mentioned. The first of these was the multiplying machine of Leibniz, who designed the stepped cylinder as a selector mechanism, which became the distinctive feature of one of the two great classes of commercial calculating machines.

The next real advance was due to Thomas de Colmar, who utilized the stepped cylinder of Leibniz in a machine built in 1820, although it is said that he was not familiar with the work of the latter. The Thomas invention is the prototype of practically all commercial machines built before 1875, and of a goodly share of those developed since that time. In fig. 2, A is the stepped cylinder, on the

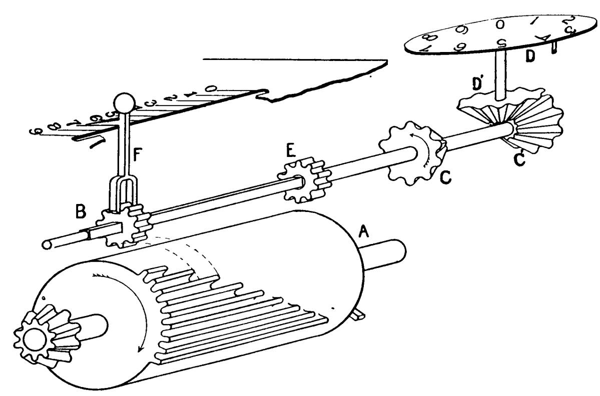


FIG. 2.

surface of which are nine cogs of length varying from one to nine units. This cylinder always rotates in one direction as indicated by the arrow. On the

shaft above and parallel to the cylinder a pinion B slides to a position indicated by a scale on the edge of the groove in the upper plate. When the pointer on handle F indicates 7, the pinion will engage with the seven longest cogs on the cylinder and miss the eighth.

The motion is transmitted by the shaft to the pair of bevel gears C and C'. The dial D with a bevel gear D' on the lower end of its shaft is mounted on a carriage. By means of a slide lever this bevel gear may be placed in mesh with either C or C' at will, one of which rotates it forwardly for addition, the other rotating it reversely for subtraction. Thus the Thomas machine is transformed by a lever shift from an adding and multiplying machine into one adapted to subtraction and division.

Frank Stephen Baldwin, on September 8, 1873, applied for and on February 2, 1875, was granted a patent on the first[1] practical calculating machine which at all times had the capacity to add, subtract, multiply, and divide with no resetting of the mechanism and with no form of conversion for any of the processes. It is the purpose of this note to describe the machine briefly and to submit evidence as to the validity of the two qualifying adjectives, *first* and *practical*, the former of which has frequently by implication been denied.

Description. In the Baldwin machine the stepped cylinders of Leibniz and Thomas are replaced by a single cylinder. The selector mechanism is a series of sets of nine radially placed pins, one set for each decimal order and located in planes at right angles to the axis of the cylinder. By means of a rotatable cam these pins may be made at will to project beyond the surface of the cylinder as shown in fig. 3. In this figure the handle which operates the cam is located

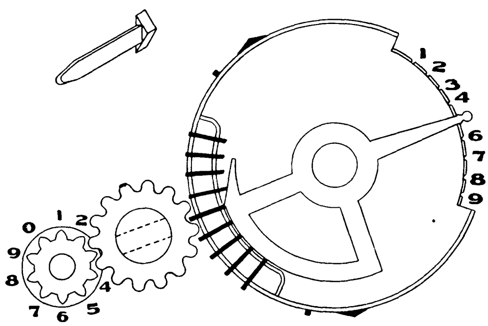


FIG. 3.

at 5, corresponding to which five of the pins project beyond the surface and thus engage cogs of the intermediate pinion, which in turn is in mesh with the recording or numeral dials. The five pins are said to be active and the four which do not

[1] Possibly the nearest approach to such a machine was that of Stanhope (1777), which did not pass the experimental stage. This invention was based on certain mechanical principles inherently weak which would seem to preclude its development into a practical machine without radical changes in design.

project are inactive. This set of active and inactive pins[1] controlled by a rotatable cam in the plane of the pins is the "active and inactive pin principle," which is the distinctive feature of the second great type of calculating machine.

A second feature of the Baldwin machine is the "wedge" shown in figure 3, which passes through a slot (indicated by dotted lines) in the shaft of the intermediate gear. When the numeral wheel passes from 9 to 0 or from 0 to 9 a stud or post on the dial pushes the slide through the shaft toward the large cylinder as shown in figure 4. The bevel edge on the larger end of the wedge throws a

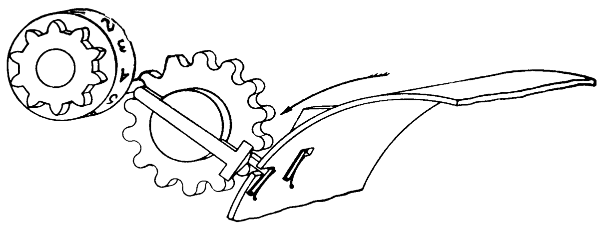


FIG. 4.

finger on the surface of the cylinder into the path of the pinion which operates the next numeral wheel and 1 is carried to that wheel. The large cylinder or set-up barrel may be moved to the right or left to bring the pins opposite any order of the dials or numeral wheels, which are mounted in the frame of the machine.

Chronology. On October 5, 1872, Mr. Baldwin filed with the United States Patent Office a caveat, attested on September 28, 1872, containing complete drawings and description of this machine. On September 8, 1873, application was filed for a patent which was granted on February 2, 1875, as U. S. No. 159244. A complete model of the machine was submitted with the application, and is now in the possession of the Smithsonian Institution. The design of this machine was therefore, by documentary evidence, completed before September 28, 1872.

Practicability. Of the first ten machines built, eight were sold as follows: Pennsylvania R.R., Philadelphia, (1); Fairman Rogers, Philadelphia, (1); State Insurance Department, Albany, (1); Insurance Companies, New York, (3); War Department, Washington, (1); Interior Department, Washington, (1).

All of the above were sold before 1876. The pertinent question here is, were these machines in the experimental stage, or were they calculating machines. This question is fully answered in the one letter which space allows.

[1] The cam operated radial pin was utilized in a machine designed by Roth in 1841. The disposition of the parts and general design in no way resemble the work of Baldwin. This machine was never developed commercially and was unknown to Baldwin and probably to Odhner, so that it may safely be said to have had no influence in the development of this type of machine.

The works of inventors whose machines were not developed to a commercial stage are milestones in the pathway of progress and should not be subject to disparagement. However a distinction must be made between those inventions which terminated in a blind alley and those which led to the development of commercial machines.

that Mr. Baldwin shall receive the credit which has been so long and so persistently withheld and which is his just due.

Thomas de Colmar was the inventor of the first practical non-reversible cycle calculator. Frank Stephen Baldwin was the inventor of the first practical reversible cycle calculator. Probably 90 per cent. of all calculators which have achieved commercial success since 1875 must be classed as direct successors of the Baldwin patent.

THE THEOREM ON THE MOMENT OF MOMENTUM.

By O. D. KELLOGG, Harvard University.

1. Introduction. In volume XXI of the MONTHLY, Professor Huntington [1] pointed out the inadequacy, and in some cases also the erroneousness, of the treatments of the theorem on the moment of momentum in the current elementary texts in mechanics. He then gave a correct and admirably simple treatment of the theorem in the case of uniplanar motion of a rigid body, on the assumption that a rigid body may be regarded as a system of a finite number of particles rigidly connected.

In conducting courses during the last four years, I found that not only were the elementary texts unsatisfactory with regard to this theorem, but that many of the classical books on mechanics were also. For this reason, and also because, in the interests of simplicity, Professor Huntington restricted himself in point of generality, I venture to offer the presentation which I have developed in my lectures. I do this under no misapprehensions as to the likelihood of novelty in method or results, but because the theorem is so important that it is high time that correct presentations of it became current.

2. Statement of the Theorem. We shall first consider a system of n particles, which may be free, or subject to constraints. The hypotheses of the theorem are Newton's second law, and his third law extended as indicated below. These laws, of course, postulate the existence of absolute motion. They may be stated as follows:

II. *The product of the vector acceleration of a particle by its mass is proportional to the resultant of the forces acting on the particle.*

III. *The forces exerted by two particles on each other are equal in magnitude, and opposite in sense.* We assume further that they are *directed along the line joining the particles.*

Forces satisfying this extended third law we call *internal forces.* All others we call *external.*

By the *momentum* of a particle, we mean the product of its vector velocity by its mass. The velocity may be either the absolute velocity, or the velocity relative to some specified moving point. By the *moment* of a vector $\boldsymbol{v}$, passing

[1] E. V. Huntington, "The Theorem of Rotation in Elementary Mechanics," this MONTHLY (*1914*, 315–320).

that Mr. Baldwin shall receive the credit which has been so long and so persistently withheld and which is his just due.

Thomas de Colmar was the inventor of the first practical non-reversible cycle calculator. Frank Stephen Baldwin was the inventor of the first practical reversible cycle calculator. Probably 90 per cent. of all calculators which have achieved commercial success since 1875 must be classed as direct successors of the Baldwin patent.

THE THEOREM ON THE MOMENT OF MOMENTUM.

By O. D. KELLOGG, Harvard University.

1. Introduction. In volume XXI of the MONTHLY, Professor Huntington [1] pointed out the inadequacy, and in some cases also the erroneousness, of the treatments of the theorem on the moment of momentum in the current elementary texts in mechanics. He then gave a correct and admirably simple treatment of the theorem in the case of uniplanar motion of a rigid body, on the assumption that a rigid body may be regarded as a system of a finite number of particles rigidly connected.

In conducting courses during the last four years, I found that not only were the elementary texts unsatisfactory with regard to this theorem, but that many of the classical books on mechanics were also. For this reason, and also because, in the interests of simplicity, Professor Huntington restricted himself in point of generality, I venture to offer the presentation which I have developed in my lectures. I do this under no misapprehensions as to the likelihood of novelty in method or results, but because the theorem is so important that it is high time that correct presentations of it became current.

2. Statement of the Theorem. We shall first consider a system of n particles, which may be free, or subject to constraints. The hypotheses of the theorem are Newton's second law, and his third law extended as indicated below. These laws, of course, postulate the existence of absolute motion. They may be stated as follows:

II. *The product of the vector acceleration of a particle by its mass is proportional to the resultant of the forces acting on the particle.*

III. *The forces exerted by two particles on each other are equal in magnitude, and opposite in sense.* We assume further that they are *directed along the line joining the particles.*

Forces satisfying this extended third law we call *internal forces.* All others we call *external.*

By the *momentum* of a particle, we mean the product of its vector velocity by its mass. The velocity may be either the absolute velocity, or the velocity relative to some specified moving point. By the *moment* of a vector $\boldsymbol{v}$, passing

[1] E. V. Huntington, "The Theorem of Rotation in Elementary Mechanics," this MONTHLY (*1914*, 315–320).

through a point P, with respect to a point Q, we mean the vector (or outer) product of $\boldsymbol{v}$ by the vector $\overrightarrow{QP}$, or $\overrightarrow{QP} \times \boldsymbol{v}$. By the *moment of momentum* of a particle relative to a point Q, we mean the moment with respect to Q of the momentum of the particle relative to Q. The moment of a *system* of vectors relative to Q will be understood to mean the vector sum of the moments with respect to Q of the individual vectors.

With these, the usual definitions, and on the basis of the hypotheses II and III, we may now state the theorem:

THEOREM. *The time rate of change of the moment of momentum relative to a point Q of a system of n particles is equal to the moment relative to Q of the external forces, if* and *only if* the point Q

(*a*) *has zero acceleration,*

(*b*) *is the center of mass of the system,* or

(*c*) *is such that its acceleration vector always passes through the center of mass.*[1]

3. Proof. The proof of the theorem is simple. Newton's second law may be written, with proper choice of units, in the form:

$$m_i \ddot{\boldsymbol{r}}_i = \sum_j \boldsymbol{F}_{ij} + \boldsymbol{F}_i, \qquad (i = 1, 2, \cdots n), \tag{1}$$

where $\boldsymbol{r}_i$ is the position vector of the ith particle drawn from a fixed origin, O, where $\boldsymbol{F}_{ij}$ is the internal force on the ith particle due to the jth, and where $\boldsymbol{F}_i$ is the resultant of the external forces on the ith particle. The extended third law states that $\boldsymbol{F}_{ji} = - \boldsymbol{F}_{ij}$, and that this vector is parallel to the vector joining the corresponding particles, or $\boldsymbol{r}_j - \boldsymbol{r}_i$. Hence $(\boldsymbol{r}_j - \boldsymbol{r}_i) \times \boldsymbol{F}_{ji} = 0$, or $\boldsymbol{r}_j \times \boldsymbol{F}_{ji} - \boldsymbol{r}_i \times \boldsymbol{F}_{ji} = \boldsymbol{r}_j \times \boldsymbol{F}_{ji} + \boldsymbol{r}_i \times \boldsymbol{F}_{ij} = 0$. If we now sum with respect to i and j, we see that $\sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} = 0$, and this holds whether the vectors $\boldsymbol{r}_i$ are measured from a fixed point or not, provided they are measured from the *same* point.

We now introduce the general point of reference, Q. Let $\boldsymbol{q}$ denote the position vector of Q measured from the fixed point O, and let $\boldsymbol{s}_i$ denote the position vector of the ith particle measured from Q, so that

$$\boldsymbol{r}_i = \boldsymbol{q} + \boldsymbol{s}_i. \tag{2}$$

We are now ready to form the equation between moments, by taking the vector products of the two members of each of the equations (1) by the corresponding $\boldsymbol{s}_i$ and adding. The result is

$$\sum_i \boldsymbol{s}_i \times m_i \ddot{\boldsymbol{r}}_i = \sum_i \boldsymbol{s}_i \times \boldsymbol{F}_i, \tag{3}$$

[1] The usual practice with respect to this theorem has been to prove it on the (usually tacit) assumption that Q is fixed. It has then been applied in cases where Q is not fixed. The more careful writers, of course, avoid this error, but I have not seen any statement equivalent to the above. Stäckel, in his article in the *Encyk. d. Math. Wiss.*, IV, 6 (1908), p. 475, makes no statement about the point Q. Routh states the theorem for the cases (*a*) and (*b*), and considers cases in which Q may be a point of the instantaneous axis when the system is replaced by a rigid body; *Elem. Dyn. of a Rigid Body*, 5th ed. (1891), p. 179. Appell confines himself to cases (*a*) and (*b*); *Traité de Méc. Rat.*, 2d ed. (1904), vol. II, p. 23, p. 34. Webster states that the theorem holds for any point Q; *Dynamics* (1904), p. 96. Similarly, Jeans; *Theoret. Mech.* (1907), p. 296.

Under any circumstances, however, it will be noted that the theorems derived will hold with at least as great a degree of generality as any of the principles of mechanics which are based on Newton's laws and the assumption that the pairs of forces of reaction are directed along the lines joining their points of application.[1]

ON A TRANSFORMATION BY PAPER FOLDING.

By C. A. RUPP, University of Texas.

1. Introduction. He who delves in obscure corners of the mathematical world may well be pleased to find only two predecessors listed in his field in the standard histories and indexes. Cajori,[2] under the heading Paper Folding, confines himself to a quotation from Klein,[3] who devotes a paragraph to the work of Wiener and Row.

Klein states that in 1893 a German and a Hindu hit on the device of using paper folding in geometric constructions. Wiener [4] apparently found out how to construct the nets of the regular polyhedra by paper folding. His memoir has been inaccessible to the author, but its title would seem to indicate that there is no overlapping.

Beman and Smith were sufficiently interested in Klein's note concerning T. Sundara Rowe (or Row) to go to the trouble of hunting up the original book, which was published in India in 1893, and was therefore rather inaccessible to Occidental scholars. The book proved so interesting that they revised it and published it in this country, where it has had several editions.[5]

Several elementary analytic geometries amuse the freshmen with the following problem: Take a piece of paper having a straight edge, mark a point not on the edge, fold the paper so that a point of the edge coincides with the marked point, and prove that the creases so formed are tangent to a parabola whose directrix is the edge of the paper and whose focus is the fixed point.

The main object of this paper is to study and identify the mechanical folding transformation which turns a straight line into a parabola. Incidentally we shall show how to construct by paper folding the pedal and podoid of any given curve, as well as the negative pedal and negative podoid. We shall see that it is just as easy for the freshman to construct an ellipse or hyperbola by folding as it is to make the parabola by such a method. The central object of the

---

[1] Since the setting in type of this paper, Professor Murnaghan has kindly called to my attention the fact that Painlevé published a completely satisfactory statement and proof of the theorem with which we are concerned nearly thirty years ago, in his *Leçons sur l'intégration des équations différentielles de la mécanique* (Paris, A. Hermann, 1895, pp. 9–13). Painlevé states the theorem as in § 2 above; it has not the same generality as in (A), § 3, nor is his proof quite so elementary.

[2] Cajori, *History of Elementary Mathematics*, MacMillan, 1921, page 265.

[3] Klein, *Famous Problems in Elem. Geom.*, Beman and Smith, Ginn, 1897, p. 42.

[4] Dyck, *Katalog Munchener Math. Ausstellung von 1893*, Nachtrag, Seite 52.

[5] Row, *Geometric Exercises in Paper Folding*, third ed., rev. by Beman and Smith, Open Court Publishing Co., 1917.

Under any circumstances, however, it will be noted that the theorems derived will hold with at least as great a degree of generality as any of the principles of mechanics which are based on Newton's laws and the assumption that the pairs of forces of reaction are directed along the lines joining their points of application.[1]

ON A TRANSFORMATION BY PAPER FOLDING.

By C. A. RUPP, University of Texas.

1. Introduction. He who delves in obscure corners of the mathematical world may well be pleased to find only two predecessors listed in his field in the standard histories and indexes. Cajori,[2] under the heading Paper Folding, confines himself to a quotation from Klein,[3] who devotes a paragraph to the work of Wiener and Row.

Klein states that in 1893 a German and a Hindu hit on the device of using paper folding in geometric constructions. Wiener[4] apparently found out how to construct the nets of the regular polyhedra by paper folding. His memoir has been inaccessible to the author, but its title would seem to indicate that there is no overlapping.

Beman and Smith were sufficiently interested in Klein's note concerning T. Sundara Rowe (or Row) to go to the trouble of hunting up the original book, which was published in India in 1893, and was therefore rather inaccessible to Occidental scholars. The book proved so interesting that they revised it and published it in this country, where it has had several editions.[5]

Several elementary analytic geometries amuse the freshmen with the following problem: Take a piece of paper having a straight edge, mark a point not on the edge, fold the paper so that a point of the edge coincides with the marked point, and prove that the creases so formed are tangent to a parabola whose directrix is the edge of the paper and whose focus is the fixed point.

The main object of this paper is to study and identify the mechanical folding transformation which turns a straight line into a parabola. Incidentally we shall show how to construct by paper folding the pedal and podoid of any given curve, as well as the negative pedal and negative podoid. We shall see that it is just as easy for the freshman to construct an ellipse or hyperbola by folding as it is to make the parabola by such a method. The central object of the

[1] Since the setting in type of this paper, Professor Murnaghan has kindly called to my attention the fact that Painlevé published a completely satisfactory statement and proof of the theorem with which we are concerned nearly thirty years ago, in his *Leçons sur l'intégration des équations différentielles de la mécanique* (Paris, A. Hermann, 1895, pp. 9–13). Painlevé states the theorem as in § 2 above; it has not the same generality as in (A), § 3, nor is his proof quite so elementary.

[2] Cajori, *History of Elementary Mathematics*, MacMillan, 1921, page 265.

[3] Klein, *Famous Problems in Elem. Geom.*, Beman and Smith, Ginn, 1897, p. 42.

[4] Dyck, *Katalog Munchener Math. Ausstellung von 1893*, Nachtrag, Seite 52.

[5] Row, *Geometric Exercises in Paper Folding*, third ed., rev. by Beman and Smith, Open Court Publishing Co., 1917.

paper is the consideration of folding as a mechanical means of effecting a transformation. Row seems not to have considered the possibility of transforming one curve into another by paper folding.

2. Definitions. Our outfit consists of a flat piece of paper on which are marked a fixed point O and an arbitrary curve C. We will choose O as the origin of our coördinate system, and we will call it the *pole* of the transformation.

Operation 1 is defined as the process of forming the crease c by folding the paper so that an arbitrary point P falls upon the pole O. Note that to make this construction feasible we must have paper of reasonable transparency, as ordinary typewriter paper. This utilization of the transparency of the paper to guide us in making our folds accurately is a departure from the methods of Sundara Row. We also see that Operation 1 associates a line with every point in the finite plane with the exception of the pole, this line being the perpendicular bisector of the line joining the point and the pole.

Operation 2 is the process of finding M, the mid-point of OP, as the intersection of the creases c and OP.

Operation 3 is the process of folding a crease γ which will be tangent to a curve C. By a trial and error adjustment of the paper before we make the crease we can get the tangency about as closely as we wish. The accuracy is certainly comparable with that of drawing the tangent to a given curve with the aid of a ruler and a good eye.

Operation 4 is defined as making a crease γ' through the origin and perpendicular to γ. This is readily done by finding that crease through O which makes the two halves of γ coincide. Of course the problem of making a crease through a given point and at right angles to a given line is one of the simplest solved by Row.

If we call the intersection of γ and γ' the point K, we may define *Operation 5* as the process of determining L as the image of the pole O in K. Row would effect this quite simply by folding along γ and, while the paper was still folded, pricking the paper through at O with a sharp point. It seems a bit more sporting not to use a needle, but to proceed as follows: Make γ and γ'. Refold the paper along the crease γ, and, without unfolding, crease the paper through O perpendicular to γ'. The unfolded paper shows four creases, γ, γ', and two creases parallel to γ, one passing through O and the other, γ'', through the required point L, which is thereby determined as the intersections of γ' and γ''.

These five constructions will be enough for this paper. The reader is urged to assure himself of the practicability of each of these folding operations.

3. Theorems. If the points of Operation 1 be taken at short intervals along the curve C, it will be seen that the corresponding creases c envelop a curve C', which we call the transform of C. In just a moment we shall prove that the creases c do have an envelope, and we shall identify it, but for the present it seems convenient to consider the locus of M as P travels along the curve C.

THEOREM 1. *The transformation of a curve by Operation 2 repeated is a homothetic transformation, with the ratio of similitude one half.*

PROOF: By the definition of Operation 2, the point M is half-way between the point P and the origin. If P moves along an arbitrary curve C, M will generate a similar curve of half the size.

Let us now turn to the question of the envelope of the creases c. If we take the equation of the arbitrary curve C in the form $y = f(x)$ referred to O as the origin, and take the coördinates of the point P on C to be (x, y), then, since the crease c is the perpendicular bisector of OP, the equation of c, in the running coördinates X, Y, is

$$2(xX + yY) = x^2 + y^2.$$

The envelope of the creases c is found by the ordinary methods of the calculus to be, in parametric form,

$$X = \frac{2xy + (y^2 - x^2)f'}{2(y - xf')}, \qquad Y = \frac{y^2 - x^2 - 2xyf'}{2(y - xf')},$$

where f' denotes differentiation of $y = f(x)$.

THEOREM 2. *The transformation of a curve by Operation 1 repeated is equivalent to a homothetic transformation followed by a negative pedalization, the pole of both transformations being the origin.*

PROOF: By Operation 1 the crease c is perpendicular to the segment OP at its mid-point M. By definition, the first negative pedal of a curve is the envelope of the perpendiculars drawn to the radii vectores at their extremities. It follows that C', the envelope of the creases c, is the first negative pedal of the M-curve, which, by Theorem 1, is homothetic to C.

We see that the folding transformation effected by the repetition of Operation 1 gives the same transform that we get by first shrinking C about the origin to half its size, and then taking the negative pedal of the result. Let the student prove that first taking the negative pedal of C, and shrinking the result to half its size about the origin will again give C'.

THEOREM 3. *The inverse folding transformation is identical with the PODOIDAL transformation.*

PROOF: Reversing our machinery to see with what original curve C we must begin if we wish our folding to yield us a given curve C', we see that we are to pedalize C' in the origin, and dilate to twice the size, *i.e.*, to take the reflection of the pedal of a curve in the tangents of that curve, a transformation which has been studied by Brocard,[1] Loria,[2] and others, under the name of the PODOID.

This identification of the folding transformation with a reasonably well-known birational point-line transformation shows us how to produce all kinds of curves. We needed only to have known that the podoid of a parabola in its

[1] Brocard, *Notes de bibliographie des courbes geometriques*, Bar le Duc, 1897, page 221. Brocard et LeMoyne, *Courbes Geometriques*, Vuibert, 1919 (Tome I), p. 27.

[2] Loria, *Ebene Kurven* (zweite auf. Teubner, 1910), II, Seite 324.

focus is its directrix to have easily foreseen the result of the problem in paper folding stated in the Introduction. Now that we have the clue to the transformation we can grind out theorems by the score. For example, under the folding transformation a unicursal quartic whose node is the pole will give a conic in general position.

Row,[1] in his chapter of conics, confines himself to the construction of points on the curves, with the exception of noting on page 116 that his crease is really tangent to the parabola. We wish to construct an ellipse as the envelope of its tangents. Remembering that the reflection of the focus in the tangents is the director circle, whose center is the other focus, and whose radius is the major axis, we see that C is a circle surrounding the origin. More generally, we can state the

THEOREM 5. *The folding transform of a circle is a conic whose foci are the center of the circle and the pole of the transformation. If the pole is at the center of the circle, the transform is itself a circle. If the pole is within the circle, the transform is an ellipse. If the pole is outside the circle, the transform is a hyperbola whose asymptotes are perpendicular to the tangents to the circle from the pole of the transformation.*

The last part of this theorem is an excellent problem for a good class in analytic geometry.

Paper folding is a more powerful method of construction than one might think. Consider the problem of finding the pedal of an arbitrary curve C.

THEOREM 6. *The succession of Operations 3 and 4 upon the points of an arbitrary curve yields the points of the pedal in the origin.*

PROOF: The point K on the pedal which corresponds to the point P on the curve is found by dropping a perpendicular from O on the tangent at P of the curve C. But this point K is the point of intersection of the creases γ and γ' of the Operations 3 and 4.

THEOREM 7. *The succession of Operations 3, 4, and 5 on the points of a curve C yields the points of the podoid with respect to the origin.*

PROOF: By the definition of the podoid, a point P' on it is symmetric to the origin in the tangent to C at P. The crease γ' of Operation 4 is by hypothesis perpendicular to γ, which is tangent to C at P, and hence the point L of Operation 5, originally defined as the image of O in the point K, is also the image of O in the line γ. L is accordingly the point P' of the podoid.

4. Note. The process of paper folding that we have considered is essentially one involving the line of intersection of two planes. The author is now preparing a paper extending the idea of folding to space of three dimensions, and also to surfaces any part of which is applicable to any other part thereof.[2]

---

[1] Row, *loc. cit.*, Chap XIII, passim.

[2] For an analytical discussion of the parabola, ellipse, and hyperbola as obtained by paper folding see "Construction of the Conic Sections" by A. J. Lotka, *Scientific American Supplement* for Feb. 17, 1912, page 112. EDITORS.

THE USE OF PROGNOSTIC TESTS IN SECTIONIZING COLLEGE FRESHMEN IN MATHEMATICS.[1]

By M. A. NORDGAARD, Antioch College.

With the advent of the large enrollments in our colleges and universities the question of effective freshman instruction has become paramount. Notably is this true in the departments of English and of mathematics. Latin and Greek are not continued by the student poorly prepared in these subjects; the natural sciences and modern languages have beginning courses in the college curriculum; the political sciences need not demand any specific preparation; but high school English and mathematics are the two branches that must be *continued* in college, either at the instance of the faculty or of the students themselves, and these departments are obliged to build on a substructure that is often both porous and miscellaneous.

As regards mathematics the most fruitful procedure has seemed to be to sectionize the freshmen according to ability, and let the subject matter as well as the mode of presentation be adapted to the status of the section. It is generally admitted that both native ability and specific preparation ought to enter into this classification. Ten years ago, at a certain eastern university, we sectionized the freshmen at the end of the first semester on the basis of their current work as appraised by the individual teacher, and formed as many sections for each hour as there were classes reciting at that hour. This seems still to be the current practice, though occasionally the classification is effected at the end of the first six or nine weeks. The plan has three inherent disadvantages: (1) there can be no standard as between the different sections of different hours, and in a small school the gradations are too few; (2) the classification comes too late; and (3) the personal equation generally prevents the appraisal of the individual instructors from being very satisfactory for this purpose.

When two years ago I took charge of the mathematics department at Antioch College, the dean and I decided to sectionize along rather finer lines and at the very beginning of the course. So on registration day every freshman took a mathematical placement examination consisting of two parts, each requiring one hour. For Part Two, I gave such questions from arithmetic, algebra, and geometry as are usually given in "finals" in high school. It failed utterly to differentiate the students into groups, and its results had to be disregarded: it was clearly *not the kind of test* to give students who had been on a vacation anywhere from three months to three years.

Part One proved to be a good "sieve," and we have used no other type of placement test since then. It also required one hour,—twenty minutes each on arithmetic, algebra, and plane geometry. It is a prognostic test, very much on the order of the prognostic tests used so successfully in grade and high schools the last ten years. In building it up I used as a pattern the recent psychological and vocational tests. It is a "hurdle test," and consists of thirty questions,

[1] Summary of a paper read at a meeting of the Ohio Section at Columbus, April 4, 1924.

whose answers can be quickly and easily graded. On the results we classified our two hundred and fifty freshmen (mathematics is required of all freshmen at Antioch) into five groups, *A, B, C, D, E,*—two divisions in each group. With a similar procedure our two hundred and eighty odd freshmen of the next year were divided into six groups.

It has proved a workable method of sectionizing. For, though the college time table permitted any freshman mathematics student to be shifted into any division without a conflict, we felt justified in shifting only about five per cent of the more than five hundred students sectionized in this manner during the semester, and an equal number at the end of the semester. For the others the placement test has sifted them into groups sufficiently homogeneous to work up an *esprit de corps,* to determine separate modes of presentation, and to advance at varying speeds.

But in order to investigate scientifically the relations of the scores on the prognostic test to records in current work, I made a few studies; I submit the results of two of these. As a gauge I used the scale propounded by A. R. Crathorne in his article "The Theory of Correlation Applied to School Grades" (see *The Reorganization of Mathematics in Secondary Education,* Chapter X). In his investigation Dr. Crathorne computed the correlation coefficients of subjects whose relationships were known to be close or remote. The following are indicative:

College algebra—trigonometry, same time, same instructor	0.737,
College algebra—trigonometry, same time, different instructor . . .	0.620,
Latin (boys), two consecutive courses	0.680,
English (boys), two consecutive courses	0.579,
Latin, two courses, two years intervening	0.430,
English, two courses, two years intervening	0.410,
Domestic science—civics	0.260,
Manual training—Latin	0.070.

From the study of seventy-four such correlation coefficients, Crathorne sets up the following scale:

Extremely high	High	Medium	Low	Extremely low	
1.00	.70	.55	.40	.25	0

For my first study I selected from the 1922 freshmen a group of eighteen students in the top section, whose "external environment" had been identical, —tests at the same time and place, the same instructor, etc. The following correlations (Pearson formula) obtained:

Prognostic test—1st semester "finals"	0.531,
Five week class test—semester "finals"	0.422,
Prognostic test—1st semester marks	0.294,
Five week class test—semester marks	0.107,
Prognostic test—five week test	0.211,
Prognostic test—Thurstone psychological test	0.318.

Between college records and the scores of a prognostic test involving tools acquired in high school we should expect a correlation equal to that between two courses in English or Latin, with two years intervening,—that is, between .40 and .50.

The homogeneous character of the group may be a contributing reason for the low correlations. Even so we note that the placement scores bear a closer relation to the semester marks and "finals" than do the scores of the class test taken at the end of five weeks. As a gauge of a close relation I found the correlation between the marks of the first and second semesters to be 0.686.

I also made a study of two groups of this year's freshmen. The first group included fifteen students from sections A and B, taught by one instructor; the second group of twenty-six people were from sections E and F and taught by another instructor. The low correlation of the second group was traceable to four quite indifferent students whose vocational experience and inclinations gave them an unduly high grade in the arithmetic and geometry part of the placement examination. Their misplacement was soon detected and they were shifted. The algebra placement and the semester ranks had a high correlation. Following are the correlation coefficients:

	Group *I*	Group *II*
Prognostic test—Semester ranks	0.676	0.404
Prognostic test—Average semester tests	0.664	0.447
Algebra placement—Average semester tests	0.523	0.760
Thurstone psychological test—Average semester tests	0.365	0.239
Thurstone psychological test—Semester ranks	0.220	0.316.

In secondary education, as observed in the University High Schools of Iowa and of Minnesota and the experimental schools of Teachers College, New York City, one of the chief values of prognostic tests has been to gauge what results may be expected by the teacher in charge of a group. And we have found the prognostic tests at Antioch of no less value in college work. Knowing the calibre of inferior and medium students we need not expect the impossible either from them or from the instructor. The test also helps to single out students from whom much may be expected, at the very beginning of their course, before they get into slovenly habits of study due to lack of competition.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. A NUMBER THEORETIC FUNCTION SATISFYING THE FUNCTIONAL EQUATION $\psi(m\cdot n) = \psi(m) + \psi(n)$.

BY A. J. KEMPNER, University of Illinois.

1. *Definition of a number theoretic function satisfying the functional equation:*

$$\psi(m\cdot n) = \psi(m) + \psi(n).$$

We assume $p_1, p_2, \cdots$ to be distinct primes and define $\psi(m)$ in the following manner:

1. For $m = p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_n^{\alpha_n}$, $\alpha_1, \alpha_2, \cdots, \alpha_n$ integers $\geqq 1$, let $\psi(m) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$;

2. $\psi(1) = 0$.

Then we shall have for positive integral values of m, n, k, $\psi(mn) = \psi(m) + \psi(n)$, $\psi(m^k) = k\cdot\psi(m)$, so that $\psi(m)$ satisfies the functional equation of the logarithmic function.

2. *Extension of $\psi(m)$ to include rational values of m.*

Let $p_1, \cdots, p_n, q_1, \cdots, q_\nu$ be distinct primes, $m_1 = p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_n^{\alpha_n}$, $m_2 = q_1^{\beta_1}\cdot q_2^{\beta_2}\cdots q_\nu^{\beta_\nu}$, the α's and β's positive integers, $m = m_1/m_2$. We set $\psi(m) = \psi(m_1/m_2) = \Sigma\alpha - \Sigma\beta$.

3. *Further extension of $\psi(m)$ to include argument values $m = (m_1/m_2)^{\alpha/\beta}$.*

We first observe that every number of the form $(m_1/m_2)^{\alpha/\beta}$ can be written in the form [1]

$$\frac{p_1^{\alpha_1/\alpha_1'}\cdots p_n^{\alpha_n/\alpha_n'}}{q_1^{\beta_1/\beta_1'}\cdots q_\nu^{\beta_\nu/\beta_\nu'}}, \tag{I}$$

where the p's and q's are defined as above, all α's and β's positive integers and each pair α_i, α_i' and each pair β_j, β_j' relatively prime. We admit also that the numerator or denominator, or both, may be unity.

We shall need the fact that the representation of a number in the form I is unique. For simplicity of notation we indicate the simple proof of a special case. Assume

$$\frac{p_1^{\alpha_1/\alpha_1'}\cdot p_2^{\alpha_2/\alpha_2'}}{q_1^{\beta_1/\beta_1'}\cdot q_2^{\beta_2/\beta_2'}} = \frac{r_1^{\gamma_1/\gamma_1'}\cdot r_2^{\gamma_2/\gamma_2'}}{s_1^{\delta_1/\delta_1'}\cdot s_2^{\delta_2/\delta_2'}}$$

[1] It is understood that whenever a rational exponent occurs, the real positive value of the radical is to be chosen.

to be two distinct representations I of the same number. Then

$$\frac{(p_1^{\alpha_1\alpha_2'}\cdot p_2^{\alpha_2\alpha_1'})^{1/\alpha_1'\alpha_2'}}{(q_1^{\beta_1\beta_2'}\cdot q_2^{\beta_2\beta_1'})^{1/\beta_1'\beta_2'}} = \frac{(r_1^{\gamma_1\gamma_2'}\cdot r_2^{\gamma_2\gamma_1'})^{1/\gamma_1'\gamma_2'}}{(s_1^{\delta_1\delta_2'}\cdot s_2^{\delta_2\delta_1'})^{1/\delta_1'\delta_2'}}.$$

Multiply across and raise both sides to the power $(\alpha_1'\cdot\alpha_2'\cdot\beta_1'\cdot\beta_2'\cdot\gamma_1'\cdot\gamma_2'\cdot\delta_1'\cdot\delta_2')$; comparing both sides and remembering the uniqueness of factorization of integers into prime factors, the uniqueness of the representation follows immediately. We admit now as argument values all numbers $(m_1/m_2)^{\alpha/\beta}$ and define, thus including our previous definitions:

For a given $m = (m_1/m_2)^{\alpha/\beta}$, *write* m *in the form* I *and let*

$$\psi(m) = \Sigma\alpha_i/\alpha_i' - \Sigma\beta_j/\beta_j'.$$

If $m_1 = 1$, *we take of course* $\Sigma(\alpha_i/\alpha_i') = 0$, *if* $m_2 = 1$, $\Sigma(\beta_j/\beta_j') = 0$. The functional equation $\psi(mn) = \psi(m) + \psi(n)$ still holds on account of the uniqueness of (1).

4. *Proof that* $\psi(m)$ *covers the half plane everywhere dense.*

We consider the graph of $y = \psi(x)$ in a rectangular system of coördinates. $\psi(x)$ is defined for an everywhere dense set of real argument values. We show that the function as defined in § 3 is represented by a set of points which cover everywhere dense the whole half plane $x > 0$. Everywhere dense means that if we select any point in this half plane and draw around it a circle of arbitrarily small radius, there will always be at least one point of the graph inside the circle,—and therefore an infinite number of points.

In a paper which I have not seen since twelve years and which is not available to me at present, "Axiomatische Untersuchungen über die Vektoraddition," 1909, R. Schimmack proved, I believe, that every solution of the functional equation $\psi(m\cdot n) = \psi(m) + \psi(n)$ which is discontinuous must be very strongly discontinuous, namely, of the type described above.

We make use of the following classical theorem on prime numbers:[1] *For a given arbitrarily small positive* η *there exists an integer* $N = N(\eta)$ *such that for every* $n > N$ *there lies at least one prime between* n *and* $n(1 + \eta)$.

All the more, there must be a prime between $n(1 - \eta)$ and $n(1 + \eta)$. To prove the theorem of this paragraph it is sufficient to show the following: *Given any* $\gamma > 0$ *and any rational* $\alpha \neq 0$, *then we can find three primes* p_1, p_2, p_3 *such that* $|(p_1\cdot p_2/p_3)^\alpha - \gamma| < \epsilon$, ϵ *arbitrarily small* > 0.

Proof: Let $\gamma^{1/\alpha} = c$, $(\gamma + \epsilon)^{1/\alpha} = c + \epsilon_1$, $(\gamma - \epsilon)^{1/\alpha} = c - \epsilon_2$, and ϵ_3 the smaller of ϵ_1, ϵ_2, so that $\epsilon_3 \to 0$ together with ϵ. We have to show

$$-\epsilon < \left(\frac{p_1p_2}{p_3}\right)^\alpha - \gamma < \epsilon,$$

$$(\gamma - \epsilon)^{1/\alpha} < \frac{p_1p_2}{p_3} < (\gamma + \epsilon)^{1/\alpha},$$

$$c - \epsilon_3 < \frac{p_1p_2}{p_3} < c + \epsilon_3.$$

[1] See, for example, Landau, *Handbuch der Primzahlen.* The theorem is fundamental.

Choose $\epsilon_4 = \epsilon_3/c$, then also $\epsilon_4 \to 0$ with ϵ, and

$$\frac{p_1p_2}{c} \cdot \frac{1}{1-\epsilon_4} > p_3 > \frac{p_1p_2}{c} \cdot \frac{1}{1+\epsilon_4} \cdot$$

Finally introduce $\epsilon_5 > 0$ so that simultaneously

$$\frac{1}{1-\epsilon_4} > 1 + \epsilon_5, \qquad \frac{1}{1+\epsilon_4} < 1 - \epsilon_5;$$

obviously $\epsilon_5 \to 0$ with ϵ. Then our preceding inequality is satisfied if we find three primes satisfying

$$\frac{p_1p_2}{C}(1-\epsilon_5) < p_3 < \frac{p_1p_2}{C}(1+\epsilon_5).$$

Let N be a number so large that for $n > N$ there lies a prime between $n(1 - \epsilon_5)$ and $n(1 + \epsilon_5)$. Then select for p_1 and p_2 two primes so large that $p_1p_2/c > N$.

5. *Extension of $\psi(m)$ to include all positive argument values.* Our last step shall be to extend our function again, this time in such a manner that it shall be defined for all positive values of the argument, and so that the functional equation shall still hold. The method for doing this is well established, provided we accept the theorem of Zermelo that every set may be well ordered. Indeed, all that we need to do is to paraphrase a few paragraphs from a paper by Hamel[1] in the sense that we everywhere apply multiplication where he employs addition.

The argument values for which $\psi(x)$ is defined up to the present consist exactly of all numbers of the form a^α, a any positive rational number, α any rational number. This set we exclude from the continuum of positive numbers. Let the remaining set be C', which is, since the deleted set is denumerable, still of the density of the continuum. Assume the set well ordered and a_1 its first element (a_1 some number not of the form a^α). We then consider the denumerable set $a_1^{\alpha_1}$, α_1 ranging over all rational numbers $\neq 0$, and define $\psi(a_1^{\alpha_1}) = \alpha_1$. For an x of the form $a^\alpha \cdot a_1^{\alpha_1}$ we define $\psi(x) = \psi(a^\alpha) + \psi(a_1^{\alpha_1})$. Since no a^α equals an $a_1^{\alpha_1}$ and no product of an a^α and an $a_1^{\alpha_1}$ equals an a^α, there can be no interference between the old functional values and the newly introduced ones. The new function is in a sense superimposed upon the old one, its argument values fit into lacunæ left by the arguments of the old function.

We continue in the same manner. We now imagine from the positive set C' the (denumerable) subset $a^\alpha \cdot a_1^{\alpha_1}$ deleted. The remaining set, still of the density of the continuum, we assume to have a first element, a_2, by virtue of Zermelo's theorem. Consider the set of numbers $a_2^{\alpha_2}$, α_2 rational, $\neq 0$. Then no $a_2^{\alpha_2}$ = any $a^\alpha \cdot a_1^{\alpha_1}$, or, which is the same, no $a^\alpha \cdot a_1^{\alpha_1} \cdot a_2^{\alpha_2} = 1$. Defining $\psi(a_2) = \alpha_2$ and agreeing that $\psi(a^\alpha \cdot a_1^{\alpha_1} \cdot a_2^{\alpha_2}) = \psi(a^\alpha \cdot a_1^{\alpha_1}) + \psi(a_2^{\alpha_2})$, the functional equation will still hold.

Proceeding thus a finite number of times we at each step enlarge the domain of our argument values, so that the argument may assume any number of the

[1] *Mathematische Annalen*, vol. 60 (1905), pp. 459–462.

II. On the Number of Divisions in Finding a G.C.D.

By H. Grossman, Student, College of the City of New York.

The number of divisions to be effected in finding the G.C.D. of two numbers by the Euclidian algorithm does not exceed five times the number of digits in the smaller number (when this number is written in the scale of ten).[1]

Proof: Let a_n and a_{n+1} be the two numbers $(a_n < a_{n+1})$.

Finding their G.C.D. by the Euclidian algorithm:

$$\begin{aligned} a_{n+1} &= a_n m_n + a_{n-1} \qquad (0 < a_{n-1} < a_n), \\ a_n &= a_{n-1} m_{n-1} + a_{n-2} \qquad (0 < a_{n-2} < a_{n-1}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_3 &= a_2 m_2 + a_1 \qquad (0 < a_1 < a_2), \\ a_2 &= a_1 m_1, \end{aligned}$$

where a_1 is the G.C.D. of a_n and a_{n+1}, and it requires n divisions to find it.

We wish to prove that n does not exceed five times the number of digits in a_n. Obviously a_n is a minimum for a given n when $m_{n-1} = m_{n-2} = \cdots = m_2 = a_1 = 1$ and $m_1 = 2$. (If $m_1 = 1$, $a_2 = a_1$.) In this case, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 8$, $a_6 = 13$, etc. That is, the smallest pair of numbers to find whose G.C.D. requires n divisions are the nth and $(n + 1)$th numbers in the sequence 1, 2, 3, 5, 8, 13, $\cdots$, in which $a_n = a_{n-1} + a_{n-3}$.

First we will prove that $a_{i+5} > 10a_i$, $(i = 1, 2, \cdots)$. This is seen to be true when $i = 1, 2, 3$. When $i > 3$, we note that $a_i = a_{i-1} + a_{i-2} = 2a_{i-2} + a_{i-3}$; whereas $a_{i+5} = 2a_{i+3} + a_{i+2} = 3a_{i+2} + 2a_{i+1} = \cdots = 21a_{i-2} + 13a_{i-3}$. Hence $a_{i+5} > 10a_i$.

From this it follows that, if $n > 5k$, a_n has at least $k + 1$ digits; so that n does not exceed five times the number of digits in a_n.

III. On the Exponential Notation for Circular Functions.

By H. L. Slobin, University of New Hampshire.

From De Moivre's Theorem,

$$(\cos \theta + i \sin \theta)^{p/q} = \cos (p\theta/q) + i \sin (p\theta/q);$$

and, when $\theta = \pi$,

$$2 \cos (p\pi/q) = (-1)^{p/q} + (-1)^{-p/q}, \qquad 2i \sin (p\pi/q) = (-1)^{p/q} - (-1)^{-p/q},$$

where $(-1)^{p/q}$ means the pth power of the complex qth root of -1 having the least positive amplitude.

If we let $(-1)^{1/q} = x$, we have

$$(1) \quad 2 \cos (p\pi/q) = x^p + x^{-p}, \qquad (2) \quad 2i \sin (p\pi/q) = x^p - x^{-p}.$$

[1] This theorem was proved by G. Lamé, *Comptes Rendus*, Paris (1844), vol. 19, pp. 867–9. For a number of references to this and similar results, see L. E. Dickson, *History of the Theory of Numbers*, vol. 1, p. 394. Editor.

From these two forms we have of course similar forms for all the circular functions.

The purpose of this note is to call special attention to the usefulness of these forms as a direct and elementary instrument for the solution of problems which otherwise necessitate a knowledge of advanced analysis.

The following problem from the *Educational Times* will illustrate a particular use of these forms.

Problem: If $\Sigma \cos^4 (\pi/7) = \cos^4 (\pi/7) + \cos^4 (2\pi/7) + \cos^4 (3\pi/7)$, find $\Sigma \cos^4 (\pi/7)$, $\Sigma \sin^4 (\pi/7)$, $\Sigma \tan^4 (\pi/7)$, $\Sigma \sec^4 (\pi/7)$.

Solution: By (1), $16 \cos^4 (\pi/7) = (x + x^{-1})^4$, $16 \cos^4 (2\pi/7) = (x^2 + x^{-2})^4$, $16 \cos^4 (3\pi/7) = (x^3 + x^{-3})^4$. Hence, since $x^7 = -1$, if we let $x^6 - x^5 + x^4 - x^3 + x^2 - x = k$, we have $\Sigma \cos^4 (\pi/7) = (18 + 5k)/16$.

But from $x^7 + 1 = (x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1) = 0$, we have $k = -1$, so that $\Sigma \cos^4 (\pi/7) = 13/16$.

Similarly,

$$\Sigma \sin^4 (\pi/7) = (18 - 3k)/16 = 21/16,$$
$$\Sigma \tan^4 (\pi/7) = 976k + 1347 = 371,$$
$$\Sigma \sec^4 (\pi/7) = 16(31k + 57) = 416.$$

Obviously these formulæ may be used to verify the known trigonometric identities; and they also often lead to interesting forms not so readily obtained otherwise.

Thus let $p/q = r$. Then $2 \cos r\pi = (-1)^r + (-1)^{-r}$, and we have immediately $4 \cos^2 r\pi = (-1)^{2r} + (-1)^{-2r} + 2 = 2 \cos 2r\pi + 2$, or on proceeding to the limit, since the cosine is a continuous function, and $r\pi$ may be made to approach any arbitrary angle α, $2 \cos^2 \alpha = 1 + \cos 2\alpha$.

Or again, $2 \cos nr\pi = (-1)^{nr} + (-1)^{-nr}$, which, if n is odd, is equal to

$$[(-1)^r + (-1)^{-r}][(-1)^{(n-1)r} - (-1)^{(n-3)r} + \cdots + (-1)^{-(n-1)r}],$$

so that $(\cos n\alpha)/\cos \alpha = 2 \cos (n - 1)\alpha - 2 \cos (n - 3)\alpha + \cdots \pm 1$. Thus

$$\cos 9\alpha/\cos \alpha = 2 \cos 8\alpha - 2 \cos 6\alpha + 2 \cos 4\alpha - 2 \cos 2\alpha + 1.$$

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

The Mathematical Theory of Relativity. By A. S. EDDINGTON. Cambridge University Press, 1923. ix + 247 pages.

This book, as stated in the preface, is the outgrowth of a mathematical supplement to the French edition of the author's "*Space, Time and Gravitation.*" While, in its present form, its direct dependence on the earlier work has been minimized, it does presuppose such of the less technical knowledge of the theory

From these two forms we have of course similar forms for all the circular functions.

The purpose of this note is to call special attention to the usefulness of these forms as a direct and elementary instrument for the solution of problems which otherwise necessitate a knowledge of advanced analysis.

The following problem from the *Educational Times* will illustrate a particular use of these forms.

Problem: If $\Sigma \cos^4 (\pi/7) = \cos^4 (\pi/7) + \cos^4 (2\pi/7) + \cos^4 (3\pi/7)$, find $\Sigma \cos^4 (\pi/7)$, $\Sigma \sin^4 (\pi/7)$, $\Sigma \tan^4 (\pi/7)$, $\Sigma \sec^4 (\pi/7)$.

Solution: By (1), $16 \cos^4 (\pi/7) = (x + x^{-1})^4$, $16 \cos^4 (2\pi/7) = (x^2 + x^{-2})^4$, $16 \cos^4 (3\pi/7) = (x^3 + x^{-3})^4$. Hence, since $x^7 = -1$, if we let $x^6 - x^5 + x^4 - x^3 + x^2 - x = k$, we have $\Sigma \cos^4 (\pi/7) = (18 + 5k)/16$.

But from $x^7 + 1 = (x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1) = 0$, we have $k = -1$, so that $\Sigma \cos^4 (\pi/7) = 13/16$.

Similarly,

$$\Sigma \sin^4 (\pi/7) = (18 - 3k)/16 = 21/16,$$
$$\Sigma \tan^4 (\pi/7) = 976k + 1347 = 371,$$
$$\Sigma \sec^4 (\pi/7) = 16(31k + 57) = 416.$$

Obviously these formulæ may be used to verify the known trigonometric identities; and they also often lead to interesting forms not so readily obtained otherwise.

Thus let $p/q = r$. Then $2 \cos r\pi = (-1)^r + (-1)^{-r}$, and we have immediately $4 \cos^2 r\pi = (-1)^{2r} + (-1)^{-2r} + 2 = 2 \cos 2r\pi + 2$, or on proceeding to the limit, since the cosine is a continuous function, and $r\pi$ may be made to approach any arbitrary angle α, $2 \cos^2 \alpha = 1 + \cos 2\alpha$.

Or again, $2 \cos nr\pi = (-1)^{nr} + (-1)^{-nr}$, which, if n is odd, is equal to

$$[(-1)^r + (-1)^{-r}][(-1)^{(n-1)r} - (-1)^{(n-3)r} + \cdots + (-1)^{-(n-1)r}],$$

so that $(\cos n\alpha)/\cos \alpha = 2 \cos (n - 1)\alpha - 2 \cos (n - 3)\alpha + \cdots \pm 1$. Thus

$$\cos 9\alpha/\cos \alpha = 2 \cos 8\alpha - 2 \cos 6\alpha + 2 \cos 4\alpha - 2 \cos 2\alpha + 1.$$

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

The Mathematical Theory of Relativity. By A. S. EDDINGTON. Cambridge University Press, 1923. ix + 247 pages.

This book, as stated in the preface, is the outgrowth of a mathematical supplement to the French edition of the author's "*Space, Time and Gravitation.*" While, in its present form, its direct dependence on the earlier work has been minimized, it does presuppose such of the less technical knowledge of the theory

of relativity as is there given. Assuming these general ideas, the author attempts to provide a complete and consistent mathematical treatment of the subject. Mathematical in structure only, for the point of view is physical throughout. As the discussion by no means follows the beaten track, we proceed to give a somewhat detailed account of the contents.

The starting point is the four-dimensional space-time manifold. Its geometry is taken as Riemannian, *i.e.*, governed by a geometrical invariant ds^2, expressed as a quadratic differential form in the differentials of the coördinates. Physically, this is the interval, on which all observation is based. By noting that in a "small" region the coefficients may be taken as constant, and that in any case this is the simplest possibility, we are led to the flat space-time of special relativity. On physical grounds it is found that

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2,$$

essentially, where c is the velocity of light. The derivation of the Lorentz transformation, and of the usual formulæ and consequences of the special theory then follow.

The chief method used in passing from the results of the special to those of the general theory is the Principle of Equivalence, which asserts that, for limited regions, the laws governing phenomena in the general theory are identical with those of the restricted theory. While this principle is frequently used, we are warned that it is rather a suggestion than an infallible dogma, and must be constantly checked by experiment. This warning is at variance with the reverence for the principle of equivalence expressed by most early writers on relativity including Eddington himself.

To treat the more general space-time, the tensor calculus is introduced. The physical meaning of tensors, as well as the necessity of their use in studying physical questions, is made clear. The difficulties apt to be experienced with the manipulations of tensors will be lightened owing to the full explanations given.

Gravitation is handled by assuming that in empty space the coefficients of the quadratic form

$$ds^2 = \Sigma g_{ij}dx_idx_j$$

satisfy the equations:

$$G_{ij} = \lambda g_{ij},$$

where G_{ij} is the contracted Riemann-Christoffel curvature tensor ($B_{ijk}{}^k$), and λ is either zero or a small constant. The Schwarzschild solution of these equations for the "symmetrical, static" field is derived by direct computation, given in quite clear and elegant form. Then, assuming further that the paths of particles are geodesics, and those of light rays geodesics of zero length, the three well-known experimental discrepancies with the Newtonian theory are discussed. Effects on the moon's motion are calculated by approximate methods, and found to be of negligible amount.

In the relativistic description of mechanics and hydrodynamics, the material energy tensor T_{ij} makes its appearance. We note that the author defines the proper-density of a perfect fluid as the scalar obtained by contracting this tensor, which is not the usual convention. The analogy of the vanishing of the divergence of the energy tensor with the classical equations of hydrodynamics is pointed out. Further, by identifying this tensor with one involving the g_{ij} whose divergence vanishes identically, we are led to the equations of gravitation in the presence of matter, the generalized Poisson equation:

$$G_j^i - \tfrac{1}{2}g_j^i(G - 2\lambda) = -\,8T_j^i.$$

The relation of this law to the Principle of Stationary Action is considered, though the author does not consider this Principle so fundamental as some writers do.

The intuitive conception of curvature is explained at some length, and a geometrical statement of Einstein's law of gravitation is attempted. The spherical world of de Sitter, and the cylindrical world of Einstein are described, and their relative merits compared.

Electricity, and its relation to the new theory, is then taken up. The general tensor form of Maxwell's equations is obtained by the principle of equivalence. Various consequences of them are drawn, including a justification of the assumption that light rays pursue nul-geodesics.

The concluding chapter of the book is devoted to the theories which attempt to correlate gravitational and electromagnetic phenomena by using geometries more general than that of Riemann. The author first gives Weyl's theory, described as "the greatest advance in the relativity theory after Einstein's work," and then a more general geometry of his own. In this geometry, the starting point is the Γ_{jk}^i which occur in the equations of parallel displacement. As these also occur in the equations of the geodesics, this geometry is identical with the "geometry of paths" of Veblen and Eisenhart.[1] A tensor analogous to the ordinary curvature tensor is set up, and from it and its derived tensors, expressions are constructed which may be identified with physical quantities on the basis of similarity of properties.

A noteworthy point here is Eddington's distinction between true or natural geometry, like the Riemannian geometry which "actually" holds for space-time; and "world geometry" which is merely a geometry providing graphical representation for physical relations like the pv space of an indicator diagram. He considers Weyl's geometry and his own generalization as mere world geometries which, however, are useful for the additional light they throw on the actual physical relations. This view is quite consoling to the pragmatist who considers the substitution of the ten g_{ij} of Einstein for the single potential of Newton, forced on him by experimental evidence, quite sufficient complication without replacing these by forty Γ_{jk}^i to achieve a formal symmetry of gravitation and electricity.

[1] *Cf.* L. P. Eisenhart, *Annals of Mathematics*, vol. 24 (1923), p. 367.

Throughout, the book is quite clearly written (a quality too rare in texts on relativity) and is to be heartily recommended to any one seeking a comprehensive grasp of the theory.

PHILIP FRANKLIN.

Introduction to the Mathematical Theory of the Conduction of Heat in Solids. By H. S. CARSLAW. Second edition, completely revised. London, Macmillan and Co., Ltd., 1921. xii + 268 pages. Price 30 shillings.

With the possible exception of the theory of vibrating strings, there is scarcely any chapter in physics which has given a stronger impetus to the development of mathematical analysis than the theory of the conduction of heat. Both theories were originally expanded in a purely formal manner; rigorous justifications came later.

The typical non-stationary problem in the conduction of heat leads to the following mathematical considerations. There is given a set of functions depending upon one or several space variables x, y, $\cdots$ and a time variable t, which functions satisfy a partial differential equation, the equation of conduction. It is required to pick out a sub-set of solutions of the differential equation which satisfy the boundary conditions for all values of t. The resulting functions ordinarily satisfy a condition of orthogonality with respect to the space variables on a certain range. The next step is to express the solution of the given problem in terms of the functions in the sub-set in such a manner that the solution approaches the prescribed initial state as a limit when t approaches zero.

There are quite a few delicate points involved in this process. For instance, are all the roots of the characteristic equation which determines the particular solutions real? Or, a much deeper question: Can the initial distribution of the heat in the solid be expressed in terms of the reduced set obtained by letting t approach zero in the expression for the particular solutions determined by the boundary conditions? Finally, in what sense does the solution approach the initial state when t approaches zero?

These questions are fundamental for the mathematical theory of the conduction of the heat, and it seems to the reviewer that a book on this subject has to be judged largely upon the kind of treatment it accords to these considerations. It is natural that older treatises on the theory in question should be lacking in this respect, but it is surprising how much inexactness is still to be found in much-used texts of fairly recent date.

Professor Carslaw's *Introduction to the theory of Fourier's series and integrals and the mathematical theory of the conduction of heat* appeared in a second edition some years ago. The first edition was perhaps the best comprehensive treatise on the subject, though, maybe, it did not fulfill all requirements as to rigor and disposition of the subject matter even at the time of its publication. In the revision the few sins against rigor seem to have been pretty well amended and the book stands the test proposed above with ease. But the general outlook of the book is not modern. It is true that there are a good many details that are

ment. The author in his discussion of them suffers from a self-imposed handicap because the mathematical processes used will be considered difficult by many of those for whom the book is intended.

A great deal is being written at the present time on the subject of economic research, with special emphasis on the problem of forecasting. Efforts are being made to devise methods by means of which variations in one series may be used to forecast variations in another. To the solution of this important problem, the author [2] is making important contributions. It is the opinion of one who has worked into the field of economic research from the mathematical side, and who has confessedly brought with him his love for quantitative statement, that there is more hope of success along lines of analysis than along those whose main appeal is to the eye. A comparison between two graphs may suggest a relation between the two, yet the question as to whether this relation actually exists must be determined by mathematical analysis often of great complexity. That future progress in economic research will be made along mathematical, rather than graphical, lines is a thesis which probably would not be supported by the author or by Mr. Carl Snyder, who has written a timely introduction to the volume under consideration.

C. C. MORRIS.

The Quantum Theory. By FRITZ REICHE, translated by H. S. HATFIELD and H. L. BROSE. New York, E. P. Dutton and Company, 1924. 183 pages. Price $2.50.

This little book contains a systematic and compact review of the quantum theory and of its applications to thermal radiation, specific heats, optical spectra and X-rays. The simpler parts of the mathematical theory are given in detail, but the more complicated parts are simply sketched. This plan is a good one, but unfortunately the author has not always succeeded in drawing a clear sketch: for instance, the discussion of the lattice-theory of atomic heats can hardly mean much to a reader unfamiliar with the complete theory. Impartiality is carried to such a point that some readers may not realize how largely the earlier formulations of the theory have been superseded. A very valuable feature is the abundance of references, which appear to cover the literature down to 1920.

Errors of fact or of translation are scarce. We will note only that the translator seems unwilling to believe that "Impulsmoment" really means merely "moment of momentum," and falls back later upon the erroneous translation "impulse."

In the absence of a preface one cannot be sure for what class of readers the book was intended by the author. It is quite unsuited for use by a class and would hardly do even as a first introduction for a more experienced reader. It will, however, serve admirably as a good index to the quantum theory as it existed four or five years ago.

E. H. KENNARD.

[2] "The Theory of Quadrature in Economics," by K. G. Karsten, *Journal American Statistical Society*, March, 1924.

ment. The author in his discussion of them suffers from a self-imposed handicap because the mathematical processes used will be considered difficult by many of those for whom the book is intended.

A great deal is being written at the present time on the subject of economic research, with special emphasis on the problem of forecasting. Efforts are being made to devise methods by means of which variations in one series may be used to forecast variations in another. To the solution of this important problem, the author [2] is making important contributions. It is the opinion of one who has worked into the field of economic research from the mathematical side, and who has confessedly brought with him his love for quantitative statement, that there is more hope of success along lines of analysis than along those whose main appeal is to the eye. A comparison between two graphs may suggest a relation between the two, yet the question as to whether this relation actually exists must be determined by mathematical analysis often of great complexity. That future progress in economic research will be made along mathematical, rather than graphical, lines is a thesis which probably would not be supported by the author or by Mr. Carl Snyder, who has written a timely introduction to the volume under consideration.

C. C. MORRIS.

The Quantum Theory. By FRITZ REICHE, translated by H. S. HATFIELD and H. L. BROSE. New York, E. P. Dutton and Company, 1924. 183 pages. Price $2.50.

This little book contains a systematic and compact review of the quantum theory and of its applications to thermal radiation, specific heats, optical spectra and X-rays. The simpler parts of the mathematical theory are given in detail, but the more complicated parts are simply sketched. This plan is a good one, but unfortunately the author has not always succeeded in drawing a clear sketch: for instance, the discussion of the lattice-theory of atomic heats can hardly mean much to a reader unfamiliar with the complete theory. Impartiality is carried to such a point that some readers may not realize how largely the earlier formulations of the theory have been superseded. A very valuable feature is the abundance of references, which appear to cover the literature down to 1920.

Errors of fact or of translation are scarce. We will note only that the translator seems unwilling to believe that "Impulsmoment" really means merely "moment of momentum," and falls back later upon the erroneous translation "impulse."

In the absence of a preface one cannot be sure for what class of readers the book was intended by the author. It is quite unsuited for use by a class and would hardly do even as a first introduction for a more experienced reader. It will, however, serve admirably as a good index to the quantum theory as it existed four or five years ago.

E. H. KENNARD.

[2] "The Theory of Quadrature in Economics," by K. G. Karsten, *Journal American Statistical Society*, March, 1924.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 30, nos. 5–6, May–June, 1924: "A characterization of surfaces of translation" by E. P. Lane, 231–232; "Concerning a suggested and discarded generalization of the Weierstrass factorization theorem" by L. L. Dines, 233–236; "The class number relations implicit in the Disquisitiones Arithmeticae" by E. T. Bell, 236–238; "Number of cycles of the same order in any given substitution group" by G. A. Miller, 239–246; "Algebras and their arithmetics" by L. E. Dickson, 247–257.

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES, series 9, volume 3, no. 3, 1924: "Sur les intégrales multiples des variétés algébriques" by S. Lefschetz, 319–343.

THE MESSENGER OF MATHEMATICS, volume 53, no. 11, March, 1924: "Theta expansions useful in arithmetic" by E. T. Bell, 166–176.

PHILOSOPHICAL MAGAZINE AND JOURNAL OF SCIENCE, ser. 6, volume 48, no. 283, July, 1924: "On the chance of an electron being ejected photoelectrically from an atom by X-rays" by G. E. M. Jauncey, 81–88.

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, ser. 2, volume 21: "An analytical treatment of the 3-bar curve" by F. V. Morley, 140–160; "On lines of electric induction and the conformal transformations of a space of four dimensions" by H. Bateman, 256–270.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 10, no. 6, June, 1924: "A statistical discussion of sets of precise astronomical measurements, II: proper motions" by E. B. Wilson and W. J. Luyten, 228–231; "An explanation of the gaps in the distribution of the asteroids according to their periods of revolution" by E. W. Brown, 248–253; "Note on some statistical consequences of the luminosity law" by W. J. Luyten, 260–264; "On parametric representations of continuous surfaces" by B. de Kerékjártó, 267–271.—No. 7, July, 1924: "Second note: Electrodynamics in the general relativity theory" by G. Y. Rainich, 294–298.

RENDICONTI DEL CIRCOLO MATHEMATICO DI PALERMO, volume 48, part 1, 1924: "A generalization of evolutes" by J. L. Walsh, 23–27.

SCIENTIFIC MONTHLY, volume 19, no. 1, July, 1924: "The origin, nature, and influence of relativity" by G. D. Birkhoff, 18–29.—No. 2, August, 1924: "The origin, nature and influence of relativity" (concluded) by G. D. Birkhoff, 180–187.

TOHÔKU MATHEMATICAL JOURNAL, volume 21, 1922: "On plane algebraic curves which are invariant under a quadric Cremona transformation" by A. Emch, 310–326.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 26, no. 1, January, 1924: "An existence theorem for the characteristic number of a certain boundary value problem" by H. T. Davis, 1–16; "A theorem on the factorization of polynomials of a certain type" by L. L. Dines, 17–24; "A fundamental class of geodesics on any closed surface of genus greater than one" by H. M. Morse, 25–60; "The Hilbert integral and Mayer fields for the problem of Mayer in the calculus of variations" by G. A. Larew, 61–67; "Normal congruences and quadruply infinite systems of curves in space" by J. Douglas, 68–100; "The equivalence of certain regular transformations" by L. L. Silverman, 101–112; "MacLaurin expansion of the interpolation polynomial determined by $2n + 1$ evenly spaced points" by G. Rutledge, 113–123; "On covariants of linear algebras" by C. C. MacDuffee, 124–132; "A generalized problem in weighted approximations" by D. Jackson, 133–154.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICAL CLUB OF ADELPHI COLLEGE, Brooklyn, N. Y. [*1922*, 24.]

The purpose of the club has been to promote interest in mathematics outside the classroom. During the course of its existence there have been many interesting discussions at the meetings. Such topics as the history of mathematics from early times down to modern times, women mathematicians, mathematical puzzles and tricks have been greatly favored by the members. Even the Einstein theory was discussed with interest where understanding was possible.

Meetings are held regularly at the College. At the beginning of each semester, a party is given to the new freshman members, and at the end of the year, a closing party is held. An occasional supper meeting is also held.

Dr. Joseph Bowden, professor of mathematics at Adelphi College, was the first president of the club and has held the position of honorary president ever since. The club owes a great part of its success to Dr. Bowden who has given it much of his time.

This year has been a memorable one in the history of the Adelphi College Mathematical Club for it celebrated its twenty-fifth anniversary this year. This was the most important meeting ever held by the club. Many alumni attended and the event proved successful also as a reunion. A play was given in which the activities of the club were featured. In the play, all terms used were mathematical and when these terms were applied to everyday life, the result was most unusual and interesting. The author and coach was Miss Mildred Goss '24 who was also toastmistress. Among those who made addresses were Dr. Joseph Bowden, Mr. L. L. Locke, first treasurer of the club, Miss Ruth Van Gaasbeek, president, and Miss Goss.

At the close of the entertainment, Miss Goss, in behalf of the senior members, presented to the president a gavel. Two members, who were waitresses in the play, brought in a birthday cake with twenty-five candles. Dr. Bowden, as honorary president, cut the first slice and presented it to the president, Miss Gaasbeek. Refreshments were served. Many of the older members expressed a desire to attend future meetings of the club. This proves that the anniversary meeting accomplished a double purpose, entertainment and revival of interest.

The officers of the club are: honorary president, Dr. J. Bowden; president, Miss R. Van Gaasbeek; vice-president, Miss Mildred Newman; secretary, Miss Violet Miller; treasurer, Miss Rose Brody.

(Report by Miss Violet Miller, secretary.)

THE WHITE MATHEMATICS CLUB, University of Kentucky, Lexington, Ky. [*1923*, 336.]

The officers of the White Mathematics Club for the year 1923–1924 were: president, Professor P. P. Boyd; commissary, Professor Flora Elizabeth LeStourgeon; secretary, Professor E. L. Rees.

The following papers were presented at the meetings in 1923–1924:

October 25, 1923. "Old methods of computing" by Professor P. P. Boyd.

November 8, 1923. "Some fundamentals in the theory of integral equations" by Professor F. E. LeStourgeon.

November 22, 1923. "Involutions" by Mr. W. R. Hutcherson, Gr.

December 6, 1923. "The error function" by Professor H. H. Downing.

January 17, 1924. "History of algebraic symbolism" by Professor J. M. Davis.

February 21, 1924. "Interpretations of inverse hyperbolic and circular functions as areas" by Mr. J. C. Nixon, instructor.

March 6, 1924. "Some fallacies in the theory of probability" by Mr. M. C. Brown, Gr.

March 20, 1924. "Graphical solution of equations" by Mr. T. Andrew, instructor.

April 10, 1924. "Calculation of the date of Easter" by Professor H. H. Downing.

May 11, 1924. "The nine point circle" by Miss Helen McGurk '24.

May 23, 1924. "Intrinsic equations" by Miss L. Kuykendall '24. "Constructions of a tangent to an ellipse" by Mr. H. W. Mobley '24.

(Report by Professor E. L. Rees, secretary.)

Denison Mathematics Club, Denison University, Granville, O.
[*1922*, 25.]

September 20, 1921. "Mathematical fallacies" by Professor R. Sheets.
September 27, 1921. "Graphical methods" by Professor F. B. Wiley.
October 25, 1921. Debate: "Resolved that freshman mathematics should be required for graduation from Denison University," affirmative, Miss Finley, Mr. Chandler; negative, Mr. Seasholes, Mr. Burke.
November 8, 1921. History of mathematics (first of a series), by Charlotte Larsen. "Desargues Theorem" by Mr. Lemon, faculty.
November 22, 1921. "Fourth dimension" by Messrs. Linebaugh, Quinn, Powell, Holt.
December 6, 1921. "History of Egyptian mathematics" by Alma Chambers. "Bible numerics" by Mary Packer.
January 17, 1922. Social meeting.
February 28, 1922. "Greek mathematics" by Mr. Robt. Case. "Infinitesimals" by Mr. Davis.
March 14, 1922. Solution of prize problems by Messrs. Bannister, Seasholes, Jones.
April 25, 1922. "The coördinate systems of the relativity theory" by Professor Befeld.
May 5, 1922. Annual banquet.
September 26, 1922. Informal talk by Professor Sheets.
October 10, 1922. "Magic Squares" by Mr. Chandler.
November 7, 1922. "Introduction to the Einstein theory" by Professor F. B. Wiley.
November 21, 1922. "Einstein theory" by Professor F. B. Wiley.
December 5, 1922. "Einstein theory" by Professor F. B. Wiley.
January 16, 1923. Social meeting.
February 13, 1923. "Radio" by Mr. Howe, faculty.
March 13, 1923. "Parallel axes" by Mr. H. B. Lemon, faculty.
April 10, 1923. "Inversions" by Professor A. B. Peckham.
April 24, 1923. "Inversions" by Anne Marshall.
May 11, 1923. Annual banquet with Professor C. T. Bumer of Ohio State University as speaker.
May 22, 1923. "Arithmetical progressions of high order" by Irene Kissling. Prize problems by George Stibitz and Mr. Bannister.
September 25, 1923. "The classification of geometries" by Professor Sheets.
October 9, 1923. "Magic Squares" by Mr. Ellis Powell, president of the club.
October 23, 1923. "Unlimited numbers" by Professor F. B. Wiley.
November 6, 1923. "Trilinear coördinates" by Mr. Donald Fitch.
November 20, 1923. "The number 10" by Miss N. Alspach. "The number" by Mr. Stibitz.
December 4, 1923. The abacus (a demonstration) by Mr. Kato, Mr. Matsuhashi.
December 18, 1923. "Mathematical fallacies" by Messrs. Bannister, Bash, Gay.
January 15, 1924. "A perpetual calendar" by Messrs. Lester Hunt, Leland Powell, Samuel Treharne.
February 5, 1924. Social hour.
February 19, 1924. "Logarithms of negative numbers" by Mr. L. Bone, Miss H. Dunlap, Mr. K. Holt.
March 18, 1924. "Rubbing elbows with infinity" by Professor F. B. Wiley.
April 29, 1924. Prize problem by Mr. Powell.
May 2, 1924. Annual banquet with Professor C. H. Yeaton of Oberlin College as speaker.
May 13, 1924. "Probability" by Professor F. B. Wiley.
May 29, 1924. Prize problem by Mr. Powell.

(Report by Professor Wiley.)

The Newtonian Society of the State College of Washington, Pullman, Wash.
[*1922*, 27.]

The officers for the year 1923–1924 were: president, Mildred Hunt '24; secretary and treasurer, Katheryn Maloney '25; reporter to the college paper, Mildred Allgood '27. The following programs were given during the year:
October 30, 1923. "Sets of points" by Professor C. A. Isaacs. "Limits" by Mildred Hunt '24.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3098. Proposed by JEAN WINSTON, University of Cincinnati.

Find the involutes of the parabola $y = x^2$.

3099. Proposed by S. A. COREY, Des Moines, Iowa.

Let H, I, J, and K be the complex quaternion units, $\frac{1}{2}[1 - \theta(i + k)]$, $\frac{1}{2}[1 + j + \theta(i - k)]$, $\frac{1}{2}[1 + j - \theta(i - k)]$, $\frac{1}{2}[1 - j + \theta(i - k)]$, respectively, which have the "multiplication table"

$$\begin{array}{c|cccc} & H & I & J & K \\ \hline H & H & I & H & -I \\ I & H & I & -H & I \\ J & J & -K & J & K \\ K & -J & K & J & K. \end{array}$$

Prove that there exists a set of four matrices, involving no imaginaries, which have the same "multiplication table."

3100. Proposed by H. E. TREFETHEN, Colby College.

Show that the segment between the axes tangent to the astroid at any point and the radius of the fixed circle to its point of contact with the generating circle bisect each other.

SOLUTIONS.

3050 [1924, 49]. Proposed by C. N. MILLS, State Normal School, Aberdeen, South Dakota.

Eliminate x, y, z from the equations

$$x^4/a^{5/4}b^{3/4} + y^4/a^{3/4}b^{5/4} = mz^2,$$
$$x^{3/2}/(az)^{1/4} = x + y = y^{3/2}/(bz)^{1/4},$$

and show that, if $ab > 0$, m cannot be less than 2^9.

SOLUTION BY HAZEL E. SCHOONMAKER.

From the second equations we have $x^6 = ay^6/b$, from which we obtain x. This value of x inserted in the first equation gives

$$z = \pm \frac{(a^{1/6} + b^{1/6})^{1/2}y^2}{(a^9b^{17}m^{12})^{1/24}}.$$

Inserting the values of x and z in $x + y = y^{3/2}/(bz)^{1/4}$ we obtain

$$y^4(a^{1/6} + b^{1/6})^{9/2} = a^{3/8}b^{3/8}m^{1/2}y^4,$$

whence

$$m = \left(c + \frac{1}{c}\right)^9, \qquad \left(\frac{a}{b}\right)^{1/12} = c.$$

Assume $c + 1/c < 2$. Then $c^2 + 1 < 2c$ or $(c - 1)^2 < 0$. This is impossible if c is real. Hence $m \geqq 2^9$.

NOTE BY EDITORS: In this proof it is assumed that y is not zero. If y is zero, x and z also vanish, and m may have any value; thus in this case the theorem in the problem is not true.

Also solved by J. A. BULLARD, A. C. CLARK, F. S. GACHET, J. S. GEORGES, A. M. HARDING, G. A. KREINS, A. PELLETIER and E. E. WHITFORD.

3052 [1924, 49]. Proposed by DR. JOSEF LEWAMDONSKI, Pfaffsatten, Austria.

The ellipse whose parametric equations are

$$x = a \cos \varphi, \qquad y = b \sin \varphi,$$

intersects the conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

If φ_1, φ_2, φ_3, φ_4 are the eccentric angles of the four points of intersection, prove that

$$\tan \tfrac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = abB/(a^2A - b^2C).$$

SOLUTION BY O. J. PETERSON, University of Michigan.

Denote by Q_1, Q_2, Q_3, Q_4 the four points on the major auxiliary circle of the ellipse whose polar angles are the eccentric angles of the four points of intersection of the ellipse and conic.

Replace y by $(b/a)\,y$ in the equations of the ellipse and conic. Then the ellipse is replaced by its major auxiliary circle $x^2 + y^2 = a^2$, and the given conic is replaced by the conic

$$S \equiv a^2Ax^2 + abBxy + b^2Cy^2 + a^2Dx + abEy + a^2F = 0.$$

The points of intersection of the circle and $S = 0$ are the points Q_1, Q_2, Q_3, Q_4. If α be the angle which an axis of symmetry of $S = 0$ makes with the x-axis,

$$\tan 2\alpha = \frac{abB}{a^2A - b^2C}.$$

Now rotate the axes through the angle α. Let the coördinates of Q_1 referred to the new axes be x_1', y_1', and its polar angle φ_1'; similarly for Q_2, Q_3, Q_4. The slope of the chord Q_1Q_2 equals

$$\frac{y_1' - y_2'}{x_1' - x_2'} = -\frac{x_1' + x_2'}{y_1' + y_2'} = -\frac{\cos \varphi_1' + \cos \varphi_2'}{\sin \varphi_1' + \sin \varphi_2'} = -\cot \frac{\varphi_1' + \varphi_2'}{2}.$$

and the slope of Q_3Q_4 equals $-\cot \dfrac{\varphi_3' + \varphi_4'}{2}$.

Since Q_1Q_2 and Q_3Q_4 are opposite common chords of a circle and a conic with an axis parallel to the x'-axis, the angles which these chords make with the x'-axis are supplementary, and hence the slope of Q_1Q_2 is the negative of the slope of Q_3Q_4; that is,

$$-\cot \frac{\varphi_1' + \varphi_2'}{2} = +\cot \frac{\varphi_3' + \varphi_4'}{2}.$$

and

$$\frac{\varphi_1' + \varphi_2'}{2} + \frac{\varphi_3' + \varphi_4'}{2} = n\pi.$$

Since $\varphi_1 = \varphi_1' + \alpha$, $\varphi_2 = \varphi_2' + \alpha$, $\varphi_3 = \varphi_3' + \alpha$, $\varphi_4 = \varphi_4' + \alpha$,

$$\frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{2} = 2\alpha + n\pi,$$

and finally,

$$\tan \frac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = \tan 2\alpha = \frac{abB}{a^2A - b^2C}.$$

NOTE BY THE EDITORS: A proof of the theorem regarding the common chords of a circle and a conic is given in the note on the solution of Problem 2990 (*1924*, 51).

The part of the solution following the derivation of $\tan 2\alpha$ may be shortened by a direct computation of the angles from a figure. Thus: the perpendiculars from the center to the chords Q_1Q_2 and Q_3Q_4 of the circle have polar angles which may be written $(\varphi_1 + \varphi_2)/2$ and $(\varphi_3 + \varphi_4)/2$, neglecting multiples of π. By the theorem cited above a line parallel to an axis of the conic bisects the angle between the pair of chords, also the angle between the corresponding

pair of perpendiculars. Hence we may write, neglecting multiples of π,

$$\frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{2} = 2\alpha,$$

and the desired result follows at once.

Also solved by A. BOGARD, J. A. BULLARD, WILLIAM HOOVER, C. K. ROBBINS, HAZEL E. SCHOONMAKER and the PROPOSER.

3055 [1924, 101]. Proposed by J. L. RILEY, Tarleton Station, Texas.

Given the non-intersecting circles

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0,$$
$$x^2 + y^2 + a_2x + b_2y + c_2 = 0;$$

it is required to find the four common tangents.

SOLUTION BY ROSCOE WOODS, University of Iowa.

Let O_1, O_2 be the centers of the circles C_1, C_2 whose radii are r_1, r_2 respectively. The two points S_1, S_2 which divide the line O_1O_2 internally and externally in the ratio $r_1 : r_2$ are called the centers of similitude. Two of the common tangents to the two circles C_1, C_2 pass through each center of similitude.

The coördinates of O_1, O_2, S_1, S_2 are $(-a_1, -b_1)$, $(-a_2, -b_2)$ [1] and $\left(-\dfrac{a_1r_2 \pm a_2r_1}{r_2 \pm r_1}, -\dfrac{b_1r_2 \pm b_2r_1}{r_2 \pm r_1}\right)$ respectively. The values of r_1 and r_2 in terms of a_1, b_1, etc., are $\sqrt{a_1^2 + b_1^2 - c_1}$, $\sqrt{a_2^2 + b_2^2 - c_2}$ respectively. The coefficients a_1, b_1, c_1, a_2, b_2, c_2 are assumed to be real.

The equation of a line through S_1 or S_2 with a variable slope m is

$$(y - mx)(r_2 \pm r_1) + b_1r_2 \pm b_2r_1 - m(a_1r_2 \pm a_2r_1) = 0.\text{[2]} \tag{1}$$

The condition that this line be tangent to one of the circles, C_1 say, is that the distance from this line to the point $O_1(-a_1, -b_1)$ be r_1. This condition when simplified is

$$\pm\sqrt{1 + m^2}(r_2 \pm r_1) = m(a_1 - a_2) + (b_2 - b_1). \tag{2}$$

(The ambiguous signs are independent.)

If (2) is rationalized, a quadratic in m results. The roots of this quadratic equation are the slopes of the pair of common tangents to the circles C_1, C_2 through each center of similitude, that is, a pair of tangents is given by a choice of the ambiguous sign. These roots are

$$m = \frac{(a_1 - a_2)(b_1 - b_2) \mp (r_2 \pm r_1)\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 - (r_2 \pm r_1)^2}}{(a_1 - a_2)^2 - (r_2 \pm r_1)^2}. \tag{3}$$

The four values of m obtained from (3) set in (1) give the equations of the four common tangents to C_1, C_2 regardless of the character of the radical. Since these expressions are cumbersome, they are not inserted at length.

In order to discuss the various cases that are possible, note that the sum of the first two terms under the radical in (3) is the square of the distance O_1O_2. Choose the plus sign. The pair of tangents is real, coincident, or imaginary according as the expression under the radical is positive, zero or negative. This expression is positive when one circle lies wholly without the other; is zero when the circles are tangent externally; and is negative when the circles intersect in two real points or one lies wholly within the other. Similarly, when the minus sign is chosen, the pair of tangents is real, coincident, or imaginary according as the expression under the radical sign is positive, zero or negative. This expression is positive when one circle lies wholly without the other or when the two circles intersect in two real points; is zero when the two circles are tangent internally, and is negative when one circle lies wholly within the other. Hence

[1] Note the slight change in notation, $2a_1$ is written for a_1, etc.

[2] Assume $r_1 \neq r_2$ for the present. The case $r_1 = r_2$ is discussed later.

all the tangents are real when one circle lies wholly without the other, one pair real and the other pair imaginary when the circles intersect in two real points, and both pairs are imaginary when one circle lies wholly within the other.

When $r_1 = r_2$, the center of similitude S_2 (given by the minus sign) is infinitely distant. If we refer to (2) with the minus sign before r_1 just one value of m is given, *i.e.*, $m = (b_2 - b_1)/(a_2 - a_1)$ as it should be. In case the circles coincide, the slope is indeterminate unless the direction along which the center O_1 approaches O_2 is given.

If $r_1 \neq r_2$ and the circles are concentric, (2) shows that there are only two values of m, *i.e.*, $m = \pm\sqrt{-1}$. In this case all four common tangents pass through the common center O_1 and are coincident in pairs. They are the isotropic or minimal lines through O_1, namely

$$(y + b_1) \pm i(x + a_1) = 0 \quad \text{where} \quad i = \sqrt{-1}. \tag{4}$$

Finally, it is to be noted that if the radii of both circles are unreal (that is, pure imaginary numbers as is the case since the coefficients a_1, etc., are assumed to be real) all four tangents can be real. If one radius is real and the other unreal all four tangents are unreal.

Also solved by C. S. Atchison, William Hoover, A. Pelletier, W. B. Pierce, Hazel Schoonmaker and J. K. Whittemore.

3058 [1924, 101]. Proposed by LOUIS WEISNER, University of Rochester.

If n is any integer greater than 1, the number of integers less than n and prime to n of the form $c + xd$ is $\phi(n)/\phi(d)$, where d is a divisor of n and c is an integer less than d and prime to d.

Solution by L. C. Mathewson, Dartmouth College.

Let $b_1 = 1, b_2, b_3, \cdots, b_{\phi(n)}$ be the positive integers less than n and prime to n, and let $c_1 = 1, c_2, \cdots, c_{\phi(d)}$ be the positive integers less than d and prime to d.

Separate the b's into sets placing all those of the form $c_1 + xd$ into the first set; all those (if any) of the form $c_2 + xd$ into the second set; etc. No b will be in two different sets, for dividing any one b by d could not give two different c's for remainders.

We shall next show that there are the same number of b's in each set. Since c_i is prime to d, there exists by number theory an integer k such that $c_ik \equiv c_j$, mod d. Multiplying the integers $\equiv c_i$ by k would give the same number of distinct integers $\equiv c_j$. Hence there are at least as many b's in the jth set as in the ith set. Similarly, there are as many in the ith set as in the jth set. Since there are $\phi(n)$ b's and $\phi(d)$ sets, the number in any one set is $\phi(n)/\phi(d)$.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

At the meeting of the British Association for the Advancement of Science at Toronto in August, 1924, Professor Horace Lamb was elected president, for the meeting to be held in Southampton in 1925. Sir W. H. Bragg, as president of the section of Mathematical and Physical Sciences, delivered an address on *Crystal Structure.* On the occasion of the meeting, the University of Toronto conferred the honorary degree of doctor of science on Sir Ernest Rutherford, retiring president of the Association.

Professor J. C. McLennan, of the University of Toronto, has been elected president of the Royal Society of Canada.

Professor R. A. Millikan, of the California Institute of Technology, has received the honorary degree of doctor of science from Trinity College, Dublin.

all the tangents are real when one circle lies wholly without the other, one pair real and the other pair imaginary when the circles intersect in two real points, and both pairs are imaginary when one circle lies wholly within the other.

When $r_1 = r_2$, the center of similitude S_2 (given by the minus sign) is infinitely distant. If we refer to (2) with the minus sign before r_1 just one value of m is given, *i.e.*, $m = (b_2 - b_1)/(a_2 - a_1)$ as it should be. In case the circles coincide, the slope is indeterminate unless the direction along which the center O_1 approaches O_2 is given.

If $r_1 \neq r_2$ and the circles are concentric, (2) shows that there are only two values of m, *i.e.*, $m = \pm\sqrt{-1}$. In this case all four common tangents pass through the common center O_1 and are coincident in pairs. They are the isotropic or minimal lines through O_1, namely

$$(y + b_1) \pm i(x + a_1) = 0 \qquad \text{where} \qquad i = \sqrt{-1}. \tag{4}$$

Finally, it is to be noted that if the radii of both circles are unreal (that is, pure imaginary numbers as is the case since the coefficients a_1, etc., are assumed to be real) all four tangents can be real. If one radius is real and the other unreal all four tangents are unreal.

Also solved by C. S. Atchison, William Hoover, A. Pelletier, W. B. Pierce, Hazel Schoonmaker and J. K. Whittemore.

3058 [1924, 101]. Proposed by LOUIS WEISNER, University of Rochester.

If n is any integer greater than 1, the number of integers less than n and prime to n of the form $c + xd$ is $\phi(n)/\phi(d)$, where d is a divisor of n and c is an integer less than d and prime to d.

Solution by L. C. Mathewson, Dartmouth College.

Let $b_1 = 1, b_2, b_3, \cdots, b_{\phi(n)}$ be the positive integers less than n and prime to n, and let $c_1 = 1, c_2, \cdots, c_{\phi(d)}$ be the positive integers less than d and prime to d.

Separate the b's into sets placing all those of the form $c_1 + xd$ into the first set; all those (if any) of the form $c_2 + xd$ into the second set; etc. No b will be in two different sets, for dividing any one b by d could not give two different c's for remainders.

We shall next show that there are the same number of b's in each set. Since c_i is prime to d, there exists by number theory an integer k such that $c_ik \equiv c_j$, mod d. Multiplying the integers $\equiv c_i$ by k would give the same number of distinct integers $\equiv c_j$. Hence there are at least as many b's in the jth set as in the ith set. Similarly, there are as many in the ith set as in the jth set. Since there are $\phi(n)$ b's and $\phi(d)$ sets, the number in any one set is $\phi(n)/\phi(d)$.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. Burgess, c/o Western Electric Co., 195 Broadway, New York City.

At the meeting of the British Association for the Advancement of Science at Toronto in August, 1924, Professor Horace Lamb was elected president, for the meeting to be held in Southampton in 1925. Sir W. H. Bragg, as president of the section of Mathematical and Physical Sciences, delivered an address on *Crystal Structure*. On the occasion of the meeting, the University of Toronto conferred the honorary degree of doctor of science on Sir Ernest Rutherford, retiring president of the Association.

Professor J. C. McLennan, of the University of Toronto, has been elected president of the Royal Society of Canada.

Professor R. A. Millikan, of the California Institute of Technology, has received the honorary degree of doctor of science from Trinity College, Dublin.

Professor A. S. EDDINGTON, of Cambridge University, has been elected an honorary member of the American Astronomical Society.

Professor M. I. PUPIN, of Columbia University, has received the honorary degree of doctor of science from Princeton University.

The private scientific library of the late Professor A. G. WEBSTER, of Clark University, has been purchased by the Riverbank Laboratories, Geneva, Illinois, and is now housed there as a separate collection.

At the meeting of the International Mathematical Congress held during August in Toronto, Professor S. PINCHERLE, of the University of Bologna, was elected president of the International Mathematical Union, and Professor J. C. FIELDS, of the University of Toronto, was elected president of the Congress for its next meeting.

The third Pan-American Scientific Congress will be held at Lima, Peru, beginning December 20, 1924. The officers of Section II (Mathematical and Physical Sciences) are Vice-Admiral M. M. CARBAJAL, chairman, and Professor J. R. DE LA PUENTE, secretary. The chairman of Subsection I (Pure Mathematics, Rational Mechanics, Mathematical Physics) is Professor E. VILLARÁN.

At Bryn Mawr College, Professor CHARLOTTE A. SCOTT has retired; she is succeeded as head of the department of Mathematics by Professor ANNA J. PELL. Dr. D. V. WIDDER has been appointed associate in mathematics.

Dean E. G. BILL, of Dartmouth College, is serving as chairman of a commission appointed by the College Entrance Examination Board to consider the desirability of psychological tests being given by the Board.

Professor S. LEFSCHETZ, who is on leave of absence from the University of Kansas, has been appointed visiting professor of mathematics at Princeton University for the academic year 1924–1925.

Dr. MAYME I. LOGSDON, of the University of Chicago, has been awarded a foreign fellowship by the General Education Board for study abroad during the year 1925–26. She will spend the time in study in Italian universities. She will sail for Italy in June, 1925.

The officers of the Association now and then receive requests for a possible donation of mathematical books for libraries in the smaller institutions where funds are not available for mathematical books. It is more than probable that individual or institutional members of the Association may have duplicate copies of mathematical books which they would be glad to donate to such institutions if they only knew where the need exists. Any such information will be acted upon if sent to the Secretary of the Association.

The composition for the first Carus Monograph is under way. It is expected that it will come from the press early in January, 1925.

All members of the Association should receive soon a communication from the Joint Committee on Membership of the Association and the Society, indicating the steps which are being taken to secure the membership in one or the other, but especially in *both*, of these organizations. An attempt is made to reach directly every person who is teaching collegiate mathematics and who is not now

supporting both the Society and the Association. Every member of the Association is requested to assist this Committee by his personal influence among non-members and by giving to the Committee any information which may be of use in forwarding this work. Please address the Secretary, Professor W. D. CAIRNS, Oberlin, Ohio, if you have any suggestions to make.

The Commission for Relief in Belgium Educational Foundation announces that a limited number of American Graduate Fellowships for study in Belgium during the academic year 1925–1926 will be awarded by April 1, 1925. Preference in selection is given to applicants between the ages of twenty-five and thirty-three who are unmarried and who intend to take up teaching or research as a profession. Not more than six fellowships will be awarded for 1925–1926; fellowships may be held in any one of twenty subjects, of which mathematics is one. Applications must reach the Committee (Fellowship Committee, C-R-B Educational Foundation, Inc., 42 Broadway, New York City) by February 15, 1925.

CONTENTS.

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

The following are dates of Section meetings of the Association in 1924 (unless otherwise specified):

ILLINOIS, Elgin, May 2–3
IOWA, Iowa State College, Ames, May 2–3
KANSAS, Topeka, February 2
KENTUCKY, Center College, April
MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA Annapolis, December 8, 1923
MICHIGAN, Ann Arbor, April 3
MINNESOTA, Hamline University, St. Paul, May 24
MISSOURI, Kansas City, November or December
OHIO, Ohio State University, Columbus, April 4–5
ROCKY MOUNTAIN, Laramie, April, 1925
SOUTHEASTERN, University of Georgia, Athens March 7–8
TEXAS, San Antonio, November 29–30

The American Mathematical Monthly

No reprints of articles are furnished to contributors free of charge. Two or three copies of an issue are, however, sent to contributors of longer articles. Orders for reprints, at rates indicated below should be made when returning galley proof to the Editor-in-Chief.

	4pp.	8pp.	12pp	16pp.		4pp.	8pp.	12pp.	16pp.
25 copies...	\$1.45	\$2.50	\$375	\$4.55	100 copies..	\$2.35	\$4.05	\$5.70	\$6.80
50 " ...	1.70	3.00	4.45	5.35	150 " ..	2.90	4.95	7.05	8.25
75 " ...	2.05	3.55	5.15	6.00	200 " ..	3.20	5.40	7.70	9.00

Covers extra, on regular stock: 25 copies \$1.75, and 1½c for each additional copy.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

REGISTER OF OFFICERS AND MEMBERS

VOLUME XXXI, 1924

NUMBER 9, NOVEMBER SUPPLEMENT

PRICE 50 CENTS

PUBLISHED BY THE ASSOCIATION

LANCASTER, PA., AND PROVIDENCE, R. I.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;

$5.00 a Year, Single Copies 60 cents, to Others

THE CONTINUUM

and Other Types of Serial Order

with an Introduction to Cantor's Transfinite Numbers

By

EDWARD V. HUNTINGTON

Published by

HARVARD UNIVERSITY PRESS

Cambridge, Mass.

1917

Price - $1.25

From the Bulletin of the American Mathematical Society:

"A remarkably beautiful and satisfying exposition of one of the highly fascinating subjects of modern mathematics. Anyone, whether or not he has been interested in the theory of the continuum and related matters, will find awaiting him here a purely intellectual delight of unusual order. It would be hard to seek out anywhere a more satisfying account of a topic in mathematics."

From the American Mathematical Monthly:

"In introducing beginners to the simple postulational science of a system with order relation, Huntington at once chooses the best means of exemplifying general ideas, and provides an explanation, intelligible to all, of the delicate and difficult real number concept. Ample material is provided to serve as a foundation for further profitable reading, and excellent indications are given to guide the student in such reading. The book is professedly and undeniably available for use in non-mathematical circles without suffering in scientific tone. Simplicity and attractiveness are somehow made compatible with painstaking attention to details of argumentation. The treatment of continuously ordered series, with the emphasis placed on such types as possess an *n*-dimensional framework, is noteworthy."

From the Journal of Philosophy, Psychology, and Scientific Methods:

"The chief text and reference book of those American students who desire an acquaintance with the important subject of the theory of aggregates. Probably the handiest and most up-to-date brief treatment of the subject in existence."

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

REGISTER OF OFFICERS AND MEMBERS

FOR THE ACADEMIC YEAR 1924-1925

LANCASTER, PA., AND ANN ARBOR, MICH.
PUBLISHED BY THE ASSOCIATION
1924

LANCASTER PRESS, INC.
LANCASTER, PA.

For a list of all officers and Council Members or Trustees for preceding years see page 60.

STANDING COMMITTEES OF THE ASSOCIATION.

National Committee on Mathematical Requirements.

A. R. Crathorne, University of Illinois.
C. N. Moore, University of Cincinnati.
E. H. Moore, University of Chicago.
D. E. Smith, Columbia University.
H. W. Tyler, Massachusetts Institute of Technology.
J. W. Young, Dartmouth College, *Chairman.*

Additional Members Representing other Associations and Secondary Schools.

W. F. Downey, English High School, Boston, Mass., representing The Association of Teachers of Mathematics in New England.
Vevia Blair, Horace Mann School, New York, N. Y., representing The Association of Teachers of Mathematics in the Middle States and Maryland.
J. A. Foberg, State Director of Mathematical Instruction, Camp Hill, Pa., representing The Central Association of Science and Mathematics Teachers, *Secretary.*
A. C. Olney, Commissioner of Secondary Education, Sacramento, Cal.
Raleigh Schorling, University of Michigan, Ann Arbor, Mich.
P. H. Underwood, Ball High School, Galveston, Tex.
Eula A. Weeks, Cleveland High School, St. Louis, Mo.

Committee on Libraries.

Florian Cajori, University of California.
E. S. Crawley, University of Pennsylvania.
Solomon Lefschetz, University of Kansas.
W. R. Longley, Yale University.
R. E. Root, U. S. Naval Academy.
W. B. Ford, University of Michigan, *Chairman.*

* Not members of the Board of Trustees.

STANDING COMMITTEES OF THE ASSOCIATION (*continued.*)

Editorial Committee on Carus Monographs.

G. A. Bliss, University of Chicago.
D. R. Curtiss, Northwestern University.
H. E. Slaught, University of Chicago.

Committee on Relations with the Annals of Mathematics.

R. C. Archibald, Brown University.
Alexander Ziwet, University of Michigan.
E. H. Moore, University of Chicago, *Chairman.*

Committee on Mathematical Dictionary.

R. C. Archibald, Brown University.
H. L. Rietz, University of Iowa.
H. E. Slaught, University of Chicago.
D. E. Smith, Columbia University.
E. R. Hedrick, Southern Branch, University of California, *Chairman.*

Committee on Standard Departments of Mathematics.

R. C. Archibald, Brown University.
G. C. Evans, Rice Institute.
Mary Emily Sinclair, Oberlin College.
E. R. Smith, Iowa State College.
J. W. Young, Dartmouth College.
R. D. Carmichael, University of Illinois *Chairman.*

Representatives on Councils of Scientific Organizations.

National Research Council—H. L. Rietz, University of Iowa, for the term ending June 11, 1926.

American Association for the Advancement of Science—W. D. Cairns, Oberlin College, and T. M. Focke, Case School of Applied Science, for the year 1924.

American Section, International Mathematical Union—R. C. Archibald, Brown University; E. H. Moore, University of Chicago; J. W. Young, Dartmouth College; H. L. Rietz, University of Iowa, *ex-officio* as representative on the National Research Council.

Associate Editors of the Official Journal for 1924.

N. H. Anning, University of Michigan.
R. W. Burgess, Brown University.
W. B. Carver, Cornell University.
Otto Dunkel, Washington University.
H. J. Ettlinger, University of Texas.
B. F. Finkel, Drury College.
Tomlinson Fort, Hunter College.
C. F. Gummer, Queen's University.
C. N. Mills, South Dakota State Normal.
F. D. Murnaghan, Johns Hopkins University.
D. E. Smith, Columbia University.
V. H. Wells, Williams College.

SECTIONS OF THE ASSOCIATION.

Illinois.

. J. Moulton, Northwestern University, *Chairman.*
. B. Lytle, University of Illinois, *Vice-Chairman.*
Bessie I. Miller, Rockford College, *Secretary-Treasurer.*

Indiana.

F. H. Hodge, Purdue University, *Chairman.*
E. N. Johnson, Butler College, *Vice-Chairman.*
H. T. Davis, Indiana University, *Secretary-Treasurer.*

Iowa.

E. R. Smith, Iowa State College, *Chairman.*
I. F. Neff, Drake University, *Vice-Chairman.*
J. F. Reilly, University of Iowa, *Secretary-Treasurer.*

Kansas.

J. J. Wheeler, University of Kansas, *Chairman.*
Lucy T. Dougherty, Kansas City Junior College, *Vice-Chairman.*
U. G. Mitchell, University of Kansas, *Secretary-Treasurer.*

SECTIONS OF THE ASSOCIATION (*continued*).

KENTUCKY.

J. M. DAVIS, University of Kentucky, *Chairman.*
A. R. FEHN, Centre College, *Secretary-Treasurer.*

LOUISIANA-MISSISSIPPI.

B. M. WALKER, Mississippi A. and M. College, *Chairman.*
H. E. BUCHANAN, Tulane University, *Vice-Chairman.*
I. C. NICHOLS, Louisiana State University, *Secretary-Treasurer.*
B. E. MITCHELL, Millsaps College, *Member Executive Committee.*
A. C. MADDOX, Louisiana State Normal College, *Member Executive Committee.*

MARYLAND, DISTRICT OF COLUMBIA, AND VIRGINIA.

F. D. MURNAGHAN, Johns Hopkins University, *Chairman.*
HARRY ENGLISH, Washington High Schools, *Secretary-Treasurer.*
J. A. BULLARD, U. S. Naval Academy, *Member Executive Committee.*

MICHIGAN.

E. R. SLEIGHT, Albion College, *Chairman.*
J. P. EVERETT, Western Normal College, *Secretary-Treasurer.*
N. H. ANNING, University of Michigan, *Member Executive Committee.*

MINNESOTA.

W. H. KIRCHNER, University of Minnesota, *Chairman.*
A. L. UNDERHILL, University of Minnesota, *Secretary-Treasurer.*
CLARA L. HANCOCK, Virginia Junior College, *Member Executive Committee.*
R. A. JOHNSON, Hamline University, *Member Executive Committee.*
G. C. PRIESTER, University of Minnesota, *Member Executive Committee.*

MISSOURI.

R. A. WELLS, Park College, *Chairman.*
A. C. ANDREWS, Kansas City Manual Training High School, *Vice-Chairman.*
P. R. RIDER, Washington University, *Secretary-Treasurer.*

NEBRASKA.

W. C. BRENKE, University of Nebraska, *Chairman.*
EMMA E. HANTHORN, Kearney Teachers College, *Secretary-Treasurer.*
R. M. McDILL, Hastings College, *Member Executive Committee.*

OHIO.

HARRIS HANCOCK, University of Cincinnati, *Chairman.*
G. N. ARMSTRONG, Ohio Wesleyan University, *Secretary-Treasurer.*
R. B. WILDERMUTH, Capital University, *Member Executive Committee.*

ROCKY MOUNTAIN SECTION.

J. C. FITTERER, University of Wyoming, *Chairman.*
S. L. MACDONALD, Colorado Agricultural College, *Vice-Chairman.*
PHILIP FITCH, North Side High School, Denver, *Secretary.*
G. H. LIGHT, University of Colorado, *Treasurer.*

SOUTHEASTERN SECTION.

TOMLINSON FORT,* University of Alabama, *Chairman.*
S. M. BARTON, University of the South, *Vice-Chairman.*
W. W. RANKIN, JR., Agnes Scott College, *Secretary-Treasurer.*

TEXAS.

L. R. FORD, Rice Institute, *Chairman.*
A. A. BENNETT, University of Texas, *Vice-Chairman.*
H. J. ETTLINGER, University of Texas, *Secretary-Treasurer.*

---

* Removed from state.

INDIVIDUAL MEMBERS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

Closed for printing November 15, 1924.

Note. This list gives data for the members of the Association as taken from the register issued during November, 1922 (but dated June–July, 1922), unless changes in position, mailing address, etc., have been reported to the Secretary. Records may be inexact or incomplete through the failure of members to return fresh information to the Secretary, but great efforts have been put forth to be full and correct. When members are known to be occupied in other lines than mathematics, this fact is indicated.

ADAMS, Prin. A. S., A.M. (Maine). High School, Woodland, Me.
ADAMS, B. T., A.M. (Baylor). Head of Dept., Jr. Coll., Wichita Falls, Tex.
ADAMS, C. R., Ph.D. (Harvard). Instr., Brown Univ., Providence, R. I. *578 Hope St.*
ADAMS, Prof. E. P., Ph.D. (Harvard). Princeton Univ., Princeton, N. J.
ADAMS, O. S., Sc.D. (Kenyon). U. S. Coast and Geod. Surv., Washington, D. C.
AGARD, Asst. Prof. H. L., Ph.D. (Yale). Williams Coll., Williamstown, Mass.
AGNEW, R. P., A.B. (Allegheny). Grad. Asst., Iowa State Coll., Ames, Ia.
AKERS, Prof. O. P., Ph.D. (Cornell). Allegheny Coll., Meadville, Pa.
ALBERT, Prof. O. W., A.M. (Columbia). Univ. of Redlands, Redlands, Calif. *629 Buena Vista St.*
ALBRIGHT, Prof. G. H., A.M. (Harvard). Colorado Coll., Colorado Springs, Colo.
ALDERTON, Asst. Prof. NINA M., Ph.D. (California). Mills College, Calif.
ALEXANDER, J. W., Ph.D. (Princeton). *29 Cleveland La.*, Princeton, N. J.
ALEY, Pres. R. J., Ph.D. (Penna.). Butler Coll., Indianapolis, Ind.
ALLEN, Prof. B. R., M.S. (Chicago). Galloway Coll., Searcy, Ark.
ALLEN, Asst. Prof. E. F., Ph.D. (Missouri). Univ. of Missouri, Columbia, Mo.
ALLEN, FLORENCE E., Ph.D. (Wisconsin). Instr., Univ. of Wisconsin, Madison, Wis.
ALLEN, GERTRUDE E., B.S. (California). Sup. Teaching Math., Univ. High School, Oakland, Calif. *58th and Grove Sts.*
ALLEN, Asso. Prof. JOSEPH, A.M. (Harvard). Coll. of the City of New York, New York, N. Y.
ALLEN, Prof. L. G. West Texas State Normal Coll., Canyon, Tex.
ALLEN, Prof. R. B., Ph.D. (Clark). Kenyon Coll., Gambier, Ohio.
ALLEN, ZELDA, A.B. (La. State Normal Coll.). Teacher, High School, Homer, La.
ALLISON, Prof. ELI, M.S. (Wesleyan). Brenau Coll., Gainesville, Ga.
AMICK, T. C., Ph.D. (Central Univ.). Prof. and Treas., Elon College, N. C.
AMMERMAN, CHARLES, A.M. (Illinois). Teacher, McKinley Man. Tr. High School, St. Louis, Mo.
ANDERSON, Prof. HARRIET, B.S. (Iowa State). Grand Island Coll., Grand Island, Nebr.
ANDERSON, H. E., A.M. (Augustana). Teacher, High School, Atascadero, Calif.
ANDERSON, Prof. MARY, A.M. (Illinois). Illinois Woman's Coll., Jacksonville, Ill.
ANDERSON, Prof. W. E., Ph.D. (Penna.). Miami Univ., Oxford, Ohio.
ANDERTON, ETHEL L., A.M. (Yale). Instr., Wellesley Coll., Wellesley, Mass. *Wilder Hall.*
ANDREW, THURMAN, B.S. (W. Va. Wesleyan). Instr., Univ. of Kentucky, Lexington, Ky. *224 E. High St.*
ANDREWS, ANNA H., Ph.B. (Wesleyan). Teacher, Public High School, Hartford, Conn. *167 Beacon St.*
ANDREWS, Prof. W. H., Ph.D. (Chicago). Educ., State Agric. Coll., Manhattan, Kans. *630 Moore St.*
ANGELICA, Sister, Ph.D. (Fordham). St. Joseph Coll. for Women, Brooklyn, N. Y. *245 Clinton Ave.*
ANNING, Asst. Prof. N. H., M.A. (Queen's). Univ. of Michigan, Ann Arbor, Mich. *128 Observatory St.*
ARCHER, Rev. PETER, S.J. Dir. of Georgetown Coll. Observ. and Prof. of Math., Georgetown Univ., Washington, D. C.
ARCHIBALD, Prof. R. C., Ph.D. (Strassburg). Brown Univ., Providence, R. I. *9 Charles Field St.*
ARMS, Prof. R. A., Ph.D. (Penna.). Gettysburg Coll., Gettysburg, Pa. *59 Lincoln Ave.*

ARMSTRONG, BEULAH M., Ph.D. (Illinois). Instr., Univ. of Illinois, Champaign, Ill. *602 E. Chalmers St.*
ARMSTRONG, Prof. G. N., D.Tech.Sc. (Munich). Ohio Wesleyan Univ., Delaware, Ohio.
ARMSTRONG, P. L., A.M. (Southwestern Presb. Univ.). *76 E. 5th St., Apt. 9,* Atlanta, Ga.
ARNAUD, J. J., B.S. (Coll. of City of N. Y.). Asso. Physicist, Bur. of Standards, Washington, D. C. *2520 Hall Pl., N.W.*
ARNDT, Prof. W. F. C., Ph.D. (Göttingen). Grey Univ. Coll., Univ. of S. Africa, Bloemfontein, S. Africa.
ARNOLD, Prof. C. L., M.S. (Ohio State). Ohio State Univ., Columbus, Ohio.
ARNOLD, H. E., A.M. (Wesleyan). Instr., Wesleyan Univ., Middletown, Conn.
ARNOLD, KATHERINE S., A.M. (Columbia). Amer. Assoc. of Univ. Women, Washington, D. C. *1634 Eye St., N.W.*
ARNOLD, W. C., M.S. (Chicago). Instr., De Pauw Univ., Greencastle, Ind. *204 S. Vine St.*
ASHCRAFT, Prof. T. B., Ph.D. (Johns Hopkins). Colby Coll., Waterville, Me. *34 Pleasant St.*
ASHMUN, R. N., A.M. (Washington). Mathematician, Int. Boundary Commission, Washington, D. C. *205 New Jersey Ave.*
ASHTON, Prof. C. H., Ph.D. (Munich). Univ. of Kansas, Lawrence, Kans.
ATCHISON, Prof. C. S., Ph.D. (Johns Hopkins). Washington and Jefferson Coll., Washington, Pa. *403 E. Chestnut St.*
ATWATER, FRANCES A., A.M. (Cornell). Instr., Milwaukee-Downer Coll., Milwaukee, Wis.
ATWOOD, OLIVE, A.B. (Carleton). Instr., N. Dak. Agric. Coll., Fargo, N. D. *1414 12th Ave. N.*
AUERBACH, MATILDA, Grad. Teachers Coll. Ethical Culture High School, New York, N. Y. *33 Central Park W.*
AUSTIN, W. A., A.M. (Indiana). Union High School, Venice, Calif.
AUTENRIETH, Asst. Prof. G. C., A.M. (Columbia). Descr. Geom. and Drawg., Coll. of the City of New York, New York, N. Y. *139th St. and Convent Ave.*
AVERS, H. G., A.B. (George Washington). Mathematician, U. S. Coast and Geod. Surv., Washington, D. C.
AYRES, FRANK, Jr., B.S. (Washington Coll., Md.). Instr., A. and M. Coll. of Texas, College Station, Tex. *Campus Box 270.*
AYRES, W. L., A.B. (Southwestern). Fellow in Pure Math., Univ. of Texas, Austin, Tex. *Univ. Sta.*

BABBITT, A. E., A.M. (Illinois). Actuary, Lamar Life Ins. Co., Jackson, Miss.
BABCOCK, Miss WEALTHY, A.M. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans. *1238 Miss.*
BACKER, MAURICE, A.B. (Westminster, Mo.). Teacher, High School, Paintsville, Ky.
BACON, Prof. CLARA L., Ph.D. (Johns Hopkins). Goucher Coll., Baltimore, Md. *2316 N. Calvert St.*
BAGBY, L. C., A.B. (Washburn). Asst., Univ. of Illinois, Urbana, Ill. *408 W. Nevada St.*
BAIDAFF, B. I., Dr. (Buenos Aires). Analisis, Univ. Bs. As., Buenos Aires, Argentina. *Belgrano 887.*
BAILEY, Prof. E. A., A.B. (Georgia). Math. and Sc., La Grange Coll., La Grange, Ga. *701 Vernon St.*
BAILEY, Prof. F. H., A.M. (Harvard). Mass. Inst. of Tech., Cambridge, Mass.
BAILEY, H. W., A.M. (Illinois). Asst., Univ. of Illinois, Urbana, Ill. *431 Nat. Hist.*
BAIRD, A. C., A.B. (Wooster). Vice-Prin., Fifth Ave. High School, Pittsburgh, Pa.
BAKER, Prof. R. P., Ph.D. (Chicago). Univ. of Iowa, Iowa City, Ia. *929 Kirkwood Ave.*
BALDWIN, J. W., A.M. (Michigan). Instr., Jr. Coll., Detroit, Mich. *6433 Roosevelt Ave.*
BALL, W. W. R., M.A. Trinity Coll., Cambridge, Eng.
BALLANTINE, J. P., Ph.D. (Chicago). Instr., Columbia Univ., New York, N. Y.
BANES, GLADYS L., A.B. (Butler). Instr., Butler Coll., Indianapolis, Ind.
BARBOUR, KATE C., A.B. (Oklahoma). High School, Norman, Okla. *313 W. Apache St.*
BAREIS, Asst. Prof. GRACE M., Ph.D. (Ohio State). Ohio State Univ., Columbus, Ohio. *201 W. 11th Ave.*
BARLOW, W. S., Grad. U. S. Naval Acad. Instr., Jr. Coll., Detroit, Mich. *438 W. Ferry Ave.*
BARNARD, R. W., M.S. (Michigan). Fellow, Univ. of Chicago, Chicago, Ill. *1229 E. 57th St.*
BARNETT, Asst. Prof. I. A., Ph.D. (Chicago). Univ. of Cincinnati, Cincinnati, Ohio. *Dept. of Math.*
BARNETT, O. P., B.S. (Brigham Young Univ.). Instr., High School, Provo, Utah. *371 W. 2d St. S.*

BARNEY, IDA, Ph.D. (Yale). *346 Whitney Ave.*, New Haven, Conn.
BARNHART, Prof. C. A., A.M. (Illinois). Univ. of New Mexico, Albuquerque, N. M. *115 S. Maple St.*
BARNHILL, Prof. J. F., A.B. (Kansas). Michigan State Teachers Coll., Ypsilanti, Mich.
BARR, Mrs. LULU A., A.M. (Oklahoma). Jonestown, Miss.
BARRETT, A. J., A.B. (Arkansas). Instr., Carnegie Inst. of Tech., Pittsburgh, Pa.
BARROW, Prof. DAVID, Ph.D. (Harvard). Univ. of Georgia, Athens, Ga. *436 Dearing St.*
BARTOL, Prof. W. C., Ph.D. (Adrian). Math. and Astr., Bucknell Univ., Lewisburg, Pa. *110 Univ. Ave.*
BARTON, HELEN, A.M. (Johns Hopkins). Math. and Dean of Women, Albion Coll., Albion, Mich.
BARTON, Prof. S. M., Ph.D. (Virginia). Univ. of the South, Sewanee, Tenn.
BASOCO, M. A., A.B. (California). So. Branch, Univ. of California, Los Angeles, Calif.
BATCHELDER, Actg. Asst. Prof. P. M., Ph.D. (Harvard). Brown Univ., Providence, R. I.
BATEMAN, Prof. HARRY, Ph.D. (Johns Hopkins). Calif. Inst. of Tech., Pasadena, Calif. *R. F. D. 4, Box 163.*
BATEN, Prof. W. D., A.M. (Texas). Bessie Tift Coll., Forsyth, Ga.
BAUDIN, Asst. Prof. M. C., A.B. (Washington Univ.). Romance Languages, Miami Univ., Oxford, Ohio.
BAUER, Asso. Prof. G. N., Ph.D. (Columbia). Univ. of New Hampshire, Durham, N. H.
BAUER, Dr. L. A. Dir., Dept. of Terrestrial Magnetism, Washington, D. C. *36th St. and Broad Branch Rd.*
BAUMAN, Prof. J. A., A.M. Emeritus, Muhlenberg Coll., Allentown, Pa. *399 Turner St.*
BAYS, Prof. SÉVÉRIN. Faculté des Sciences, Univ. de Fribourg, Bethlehem, Fribourg, Switzerland.
BEAL, Asst. Astr. W. O., M.S. (Chicago). Univ. of Minnesota, Minneapolis, Minn.
BEALL, SARAH. Computer, U. S. Coast and Geod. Surv., Washington, D. C.
BEATLEY, Asst. Prof. RALPH, A.M. (Columbia). Harvard Grad. School of Educ., Cambridge, Mass.
BEATTY, Asst. Prof. SAMUEL. Univ. of Toronto, Toronto, Can.
BEATY, Asso. Prof. E. B., A.M. (California). Oregon Agric. Coll., Corvallis, Ore.
BEAUPRÉ, Prof. V.-E., B.A. (Manitoba). Calc. and Anal. Geom., École Polytech., and Univ. de Montreal, Montreal, Can. *228 St. Denis St.*
BECKWITH, Actg. Asst. Prof. ETHELWYNN R., A.M. (Western Reserve). Coll. for Women, Western Reserve Univ., Cleveland, Ohio. *2032 E. 115th St.*
BECKWITH, W. S., A.M. (Harvard). Grad. Student, Univ. of Chicago, Chicago, Ill. *52 Snell Hall.*
BEETLE, Prof. R. D., Ph.D. (Princeton). Dartmouth Coll., Hanover, N. H.
BEGLEY, KATHLEEN, A.B. (Oklahoma). Teacher, High School, Henryetta, Okla. *901 W. Gentry.*
BELCHER, Asst. Prof. R. T., B.A. (Queen's Univ., Ireland). Math. and Eng., Pomona Coll., Claremont, Calif. *452 W. 6th St.*
BELL, Prof. E. T., Ph.D. (Columbia). Univ. of Wash., Seattle, Wash. *5026 11th Ave., N.E.*
BELL, Prof. TALMON, A.M. (Cooper). Cooper Coll., Sterling, Kan.
BELLAMY, B. C., B.S., C.E. (Wyoming). Civil Eng., Laramie, Wyoming.
BENANDER, C. A., A.B. (Harvard). *9 Kearsarge Ave.*, Boston, Mass.
BENANDER, V. I., A.B. (Harvard). Instr., Case Sch. of Appl. Sc., Cleveland, Ohio. *Dept. of Math.*
BENDER, H. A., Ph.D. (Illinois). Instr., Univ. of Illinois, Urbana, Ill. *403 N. Mathews.*
BENEDICT, Prof. H. Y., Ph.D. (Harvard). Univ. of Texas, Austin, Texas.
BENEDICT, Prof. SUSAN R., Ph.D. (Michigan). Smith Coll., Northampton, Mass. *12 Barrett Pl.*
BENNER, J. A., A.B. (Penn. State). Instr., Lafayette Coll., Easton, Pa. *333 McCartney St.*
BENNETT, Asso. Prof. A. A., Ph.D. (Princeton). Pure Math., Univ. of Texas, Austin, Texas.
BENNETT, J. C., M.S. (Iowa). Instr., Purdue Univ., W. LaFayette, Ind. *820 N. Main St.*
BENNETT, Prof. J. L., A.B. (Kansas). Physics, Ottawa Univ., Ottawa, Kans.
BENNETT, Pres. J. N., A.M. (Nebraska). Doane Coll., Crete, Nebr.
BENSON, L. F., B.S. (E. Tex. State Teachers Coll.). Forney, Texas.
BERGER, Prof. ALLEN, A.M. (Peabody). S. W. State Teachers Coll., Durant, Okla. *407 N. 9th Ave.*
BERGER, EDLA G., A.M. (Minnesota). Cedar Terrace, Beechhurst, L. I., N. Y.
BERGSTRESSER, C. A., A.M. (Lafayette), M.S. (Pennsylvania). Head of Dept., DeWitt Clinton High School, New York, N. Y. *156 Macon St., Brooklyn, N. Y.*

BERKELEY, L. M., A.M. (Virginia). *36 W. 91st St.*, New York, N. Y.
BERNADITA, Sister M., A.M. (Univ. of Colorado). Head of Dept., Loretto Heights Coll., Loretto, Colo. *In the Orient 1924-*
BERNSTEIN, Asso. Prof. B. A., Ph.D. (California). Univ. of California, Berkeley, Calif. *40 Shasta St.*
BERRY, E. M., Ph.D. (Iowa). Instr., Purdue Univ., W. LaFayette, Ind. *423 Vine St.*
BERRY, Asst. Prof. GRACE E., A.M. (Mt. Holyoke). Dean of Women, Pomona Coll., Claremont, Calif.
BERRY, Asso. Prof. W. J., A.M. (Harvard). Poly. Inst. of Brooklyn, Brooklyn, N. Y. *224 St. John's Pl.*
BERT, Prof. O. F. H., A.M. (Geneva). Washington and Jefferson Coll., Washington, Pa. *28 N. Lincoln St.*
BETZ, WILLIAM, A.M. (Rochester). Vice-Prin., East High School, Rochester, N. Y. *301 Hazelwood Terrace.*
BIANCHI, Prof. L. Univ. of Pisa, Pisa, Italy. *Via Manzoni No. 3.*
BIBB, S. F., M.S. (Chicago). Instr., Univ. of North Dakota, Grand Forks, N. D. *Box 102, Univ. Station.*
BIGBEE, JOHN A., B.S. (Drury). High School, Little Rock, Ark. *3110 Battery St.*
BIGELOW, W. W., A.B. (Beloit). Jr. Mathematician, U. S. Coast and Geod. Surv., Washington, D. C. *205 New Jersey Ave., S.W.*
BIGGERSTAFF, JOHN, A.B. (Reed). Teaching Fellow, Univ. of Washington, Seattle, Wash. *4263 9th Ave., N.E.*
BILL, Prof. E. G., Ph.D. (Yale). Dean of Freshmen, Dartmouth Coll., Hanover, N. H.
BINGLEY, Asso. Prof. G. A., A.M. (Princeton). St. John's Coll., Annapolis, Md. *240 Prince George St.*
BIRCHBY, W. N., A.M. (Colo. Coll.). *177 S. Mentor Ave.*, Pasadena, Calif.
BIRCHENOUGH, Prof. HARRY, A.M. (Columbia). State Coll. for Teachers, Albany, N. Y. *687 Hudson Ave.*
BIRKHOFF, Prof. G. D., Ph.D. (Chicago). Harvard Univ., Cambridge, Mass. *22 Craigie St.*
BISHOP, Dean F. L., Ph.D. (Chicago). Eng. and Mines, Univ. of Pittsburgh, Pittsburgh, Pa.
BIXBY, Brig. Gen. W. H., U. S. Army, retired, Washington, D. C. *1709 Lanier Pl., N.W.*
BLACK, FLORENCE, A.M. (Kansas). Instr. Univ. of Kansas, Lawrence, Kans.
BLAIR, Prof. HAROLD, A.M. (Michigan). Western State Normal Coll., Kalamazoo, Mich. *1220 Academy St.*
BLAIR, R. V., A.M. (Peabody). Instr., Vanderbilt Univ., Nashville, Tenn.
BLAIR, Miss VEVIA, Horace Mann School, New York, N. Y. *430 W. 119th St.*
BLESSING, P. N., A.B. (Swarthmore). Instr., Univ. of Michigan, Ann Arbor, Mich. *302 E. Liberty.*
BLICHFELDT, Prof. H. F., Ph.D. (Leipzig). Stanford University, Calif.
BLISS, Prof. G. A., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.
BLOM, Prof. E. C., A.M. (Missouri). Des Moines Univ., Des Moines, Iowa.
BLUMBERG, A. A. Instr., A. and M. Coll. of Texas, College Station, Tex.
BLUMBERG, Asst. Prof. HENRY, Ph.D. (Göttingen). Univ. of Illinois, Urbana, Ill. *1810 W. Oregon St.*
BLUMENTHAL, L. M., M.S. (Chicago). Instr., Univ. of Michigan, Ann Arbor, Mich. *Dept. of Math.*
BOGARD, Prof. A., A.M. (Chicago). Coll. of St. Teresa, Winona, Minn. *1151 W. 5th St.*
BOHANNAN, Prof. R. D., C.E., E.M. (Virginia). Ohio State Univ., Columbus, Ohio. *226 E. 16th Ave.*
BOND, Asso. Prof. J. D., Ph.D. (Mich.). Univ. of Tenn., Knoxville, Tenn.
BOND, Supt. O. J., LL.D. (South Carolina). The Citadel, The Milit. Coll. of South Carolina, Charleston, S. C.
BOND, W. M., A.M. (Columbia). Instr., Coll. of the City of N. Y., New York, N. Y. *523 W. 136th St.*
BOOTHROYD, Prof. S. L., M.S. (Colorado State Coll.). Astr. and Geodesy, Cornell Univ., Ithaca, New York.
BORDEN, Asst. Prof. R. F., Ph.D. (Illinois). George Washington Univ., Washington, D. C.
BOREL, ÉMILE, Docteur ès Sc. (Paris). Prof. à la Faculté des Sciences de Paris, Paris, France. *32 rue du Bac, Paris VII.*

BORGER, Prof. R. L., Ph.D. (Chicago). Ohio Univ., Athens, Ohio. *70 Univ. Terrace.*
BORTOLOTTI, Prof. ETTORE, Dottore in mat. (Bologna). Univ. of Bologna, Bologna, Italy. *Via Maggiore 100.*
BOSE, A. C., M.A. Deputy Magistrate, Bengal Prov. Civil Service, Shambazar Post Office, Calcutta, India. *5 Mahenda Bose Lane.*
BOWDEN, Prof. JOSEPH, Ph.D. (Yale). Adelphi Coll., Brooklyn, N. Y. *24 Clifton Pl.*
BOWERS, Prof. W. R., A.M. (Columbia). Radford State Normal School, East Radford, Va.
BOWLES, C. F., A.M. (Harvard). Instr., S. Dak. State School of Mines, Rapid City, S. D.
BOYCE, Prof. M. G., A.B. (Erskine). Bryson Coll., Fayetteville, Tenn.
BOYD, Dean P. P., Ph.D. (Cornell). Univ. of Kentucky, Lexington, Ky.
BRACKETT, Prof. F. P., A.M. (Dartmouth). Pomona Coll., Claremont, Calif.
BRADLEY, Asso. Prof. H. C., B.S. (Mass. Inst.). Mass. Inst. of Tech., Cambridge, Mass.
BRADSHAW, Prof. J. W., Ph.D. (Strassburg). Univ. of Michigan, Ann Arbor, Mich. *1304 Cambridge Rd.*
BRAHDY, MARIE, A.M. (Columbia). Teacher, Seward High School, New York, N. Y. *55 Central Pk. W.*
BRAMBLE, Asso. Prof. C. C., Ph.D. U. S. Naval Acad., Annapolis, Md. *Fifth St.*
BRAND, Prof. LOUIS, Ph.D. (Harvard). Univ. of Cincinnati, Cincinnati, Ohio.
BRANDEBERRY, Asso. Prof. J. B., A.M. (Ohio State). Toledo Univ., Toledo, Ohio.
BRANDNER, F. A., M.S. (Chicago). Instr., Iowa State Coll., Ames, Ia.
BRATTON, Prof. W. A., A.B. (Williams). Whitman Coll., Walla Walla, Wash.
BRAY, Asst. Prof. H. E., Ph.D. (Rice). Rice Inst., Houston, Tex.
BRECKENRIDGE, W. E., A.M. (Yale). Asso., Teachers Coll., Columbia Univ., New York, N. Y. *21 Park Ave., Mt. Vernon, N. Y.*
BREIT, GREGORY, Ph.D. (Johns Hopkins). Dept. of Terr. Magn., Washington, D. C. *36th St. and Broad Branch Rd.*
BRENKE, Prof. W. C., Ph.D. (Harvard). Univ. of Nebraska, Lincoln, Nebr. *1250 S. 21st. St.*
BRENNAN, Prof. M. S., Sc.D. (Christian Brothers). Astr. and Geol., St. Louis, Mo. *6304 Minnesota Ave.*
BREWSTER, Asst. Prof. J. A., A.B. (Harvard). Coll. of the City of New York, New York, N. Y.
BREZLER, W. J., A.M. (Grove City). Instr., Univ. of Tennessee, Knoxville, Tenn. *712 W. Main Ave.*
BRIGETTA, Sister M. Instr., Coll. of St. Scholastica, Duluth, Minn.
BRIGHAM, Asst. Prof. L. A., A.M. (Boston Univ.). Astr. and Math., Boston Univ., Boston, Mass.
BRINDLE, Prin. G. W., M.E. High School, Surrency, Ga.
BRINK, Asso. Prof. R. W., Ph.D. (Harvard). Univ. of Minnesota, Minneapolis, Minn. *On leave of absence 1924–25, Edinburgh Univ.*
BRODIE, Prof. W. M., M.E. (Va. Poly. Inst.); A.M. (Columbia). Va. Poly. Inst., Blacksburg, Va.
BROGGI, Prof. UGO, Ph.D. (Göttingen). Buenos Aires and La Plata Univs., Buenos Aires, Arg. *53 Maipú.*
BROOKE, Prof. W. E., A.M. (Nebraska). Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.
BROWN, Asst. Prof. B. H., Ph.D. (Harvard). Dartmouth Coll., Hanover, N. H.
BROWN, E. C., A.M. (Maine). Instr., Worcester Poly. Inst., Worcester, Mass. *73 Institute Rd.*
BROWN, E. L., A.M. (Cornell). Asst. Supt. of Schools, Denver, Colo. *484 Lafayette St.*
BROWN, Prof. E. W., A.M. (Yale). Yale Univ., New Haven, Conn. *116 Everit St.*
BROWN, Prof. H. S., M.S. (Lafayette). Hamilton Coll., Clinton, N. Y.
BROWN, J. S., A.M. (Texas). Head of Dept., State Teachers Coll., San Marcos, Tex.
BROWN, Prof. LILLIAN, A.M. (Columbia). Hood Coll., Frederick, Md.
BROWN, MYRTLE C., A.M. (Texas). Teacher, N. Texas State Normal Coll., Denton, Tex. *1415 W. Oak St.*
BROWN, Asst. Prof. T. H., Ph.D. (Yale). Foreign Trade, Columbia Univ. *27 Strawberry Hill Ave., Stamford, Conn.*
BROWNE, Asst. Prof. E. T., A.M. (Virginia). Univ. of North Carolina, Chapel Hill, N. C.
BRUCE, Prof. R. E., Ph.D. (Boston Univ.). Boston Univ., Boston, Mass.
BRYAN, Asso. Prof. N. R., Ph.D. (Columbia). Univ. of Maine, Orono, Me. *8 Juniper St.*
BRYANT, E. S. High School, Everett, Mass. *55 Lexington St.*
BUCHANAN, Prof. DANIEL, Ph.D. (Chicago). Head of Dept., Univ. of British Columbia, Vancouver, B. C.

BUCHANAN, Prof. H. E., Ph.D. (Chicago). Tulane Univ., New Orleans, La.
BUCHANAN, Asst. Prof. MARGARET, Ph.D. (Bryn Mawr). West Virginia Univ., Morgantown, W. Va.
BULLARD, Asso. Prof. J. A., Ph.D. (Clark). U. S. Naval Acad., Annapolis, Md. *262 King George St.*
BULLARD, Prof. W. G., Ph.D. (Clark). Syracuse Univ., Syracuse, N. Y. *117 Redfield Pl.*
BUMER, C. T., A.M. (Harvard). Instr., Ohio State Univ. *68 W. Church St., Newark, Ohio.*
BUNYAN, L. H., A.M. (Wisconsin). Instr., Univ. of Wisconsin, Madison, Wis.
BURDICK, R. D., A.M. (Columbia). Instr., Coll. of the City of New York, New York, N. Y. *6083 Broadway.*
BURGESS, H. T., Ph.D. (Yale). *37 Beers St.*, New Haven, Conn.
BURGESS, Asst. Prof. R. W., Ph.D. (Cornell). Brown Univ. *On leave of absence, with Western Elec. Co., 22 E. 38th St., New York, N. Y.*
BURKE, J. G., A.M. (Mt. St. Mary's). Vice-Pres., Mt. St. Mary's Coll., Emmitsburg, Md.
BURKETT, Asst. Prof. E. J., A.M. (Pittsburgh). Trinity Coll., Hartford, Conn.
BURLEY, J. F., Civil Engineer, Pittsburgh, Pa. *630 Hastings St., E.S.*
BURNAM, Prof. J. E., A.M. (Texas). Simmons Coll., Abilene, Texas. *1846 Orange St.*
BURRELL, JULIA, A.B. (Rice Inst.). *510 Fifteenth St.*, Galveston, Texas.
BURROWS, W. R. *Box 114*, Mass. Inst. of Tech., Cambridge, Mass.
BURWELL, Asst. Prof. W. R., Ph.D. (Oxford). Brown Univ., Providence, R. I.
BUSH, EDITH L., A.B. (Tufts). Instr., Tufts Coll., Tufts College, Mass. *17 Latin Way.*
BUSHYAGER, G. R., A.M. (Penn. State). Instr., Penn. State Coll., State College, Pa. *University Club.*
BUSSEY, Asst. Dean and Prof. W. H., Ph.D. (Chicago). Univ. of Minnesota, Minneapolis, Minn.
BUTLER, L. G., A.B. (Oregon). Prin., High School, LaConner, Wash.
BUTLER, Dr. PIERCE, Ph.D. (Hartford Theol. Sem.). Senior Asst., The Newberry Library, Chicago, Ill.
BUTLER, UNDINE, A.B. (Okla.). Teacher, Clossen High School, Oklahoma City, Okla.

CAFFREY, GENEVIEVE E., A.M. (Washington). *R. F. D. No. 5*, Seattle, Wash.
CAIN, Prof. WILLIAM, A.M. (N. C. M. & P. I.). Univ. of North Carolina, Chapel Hill, N. C.
CAIRNS, Prof. W. D., Ph.D. (Göttingen). Oberlin Coll., Oberlin, Ohio.
CAJORI, Prof. FLORIAN, Ph.D. (Tulane). Univ. of Calif., Berkeley, Calif.
CALDERWOOD, H. F. Stockman, Glasgow, Mont.
CALDWELL, MINNIE W., A.M. (Missouri). Chowan Coll., Murfreesboro, N. C.
CALKINS, Asst. Prof. HELEN, A.M. (Columbia). Knox Coll., Galesburg, Ill.
CALLAWAY, Mrs. THEODOSIA T., B.S. (Columbia). Stephens Coll., Columbia, Mo. *1 Watson Place.*
CAMP, Prof. B. H., Ph.D. (Yale). Wesleyan Univ., Middletown, Conn.
CAMP, C. C., Ph.D. (Cornell). Asso., Univ. of Illinois, Urbana, Ill.
CAMPBELL, Asst. Prof. A. D., Ph.D. (Cornell). Univ. of Arkansas, Fayetteville, Ark.
CAMPBELL, Prof. D. F., Ph.D. (Harvard). Armour Inst. of Tech., Chicago, Ill. *1209 Hinman Ave., Evanston, Ill.*
CAMPBELL, G. A., Ph.D. (Harvard). Research Engineer, Amer. Tel. and Tel. Co., New York, N. Y.
CAMPBELL, H. A., A.B. (Kingfisher). Teacher, Tech. School, Omaha, Nebr. *Tech. School, 17th and Leavenworth.*
CAMPBELL, J. R., A.B. (Syracuse). Instr., Hollywood Jr. Coll., Hollywood, Calif.
CAMPBELL, Asso. Prof. J. W., Ph.D. (Chicago). Univ. of Alberta, Edmonton South, Alberta, Can.
CAMPBELL, MARY A., A.M. (Texas). Head of Dept., South Park Coll., Beaumont, Texas. *390 Emile St.*
CANDY, Prof. A. L. Acting Dean, Univ. of Nebraska, Lincoln, Nebr. *Station A.*
CAPARÓ, Prof. J. A., Ph.D. (Notre Dame). Notre Dame Univ., Notre Dame, Ind. *P. O. Box 54.*
CAPRON, Prof. P. A., A.M. (Harvard). U. S. Naval Acad., Annapolis, Md. *203 Duke of Gloucester St.*
CARDIN, C. J., M.E. (Lehigh). Instr., Washington and Jefferson Coll., Washington, Pa. *85 Highland Ave.*
CAREY, Asst. Prof. E. F. A., M.S. (California). Univ. of Montana, Missoula, Mont. *310 McLeod Ave.*

CARIS, Pres. A. G., A.M. (Defiance). Defiance Coll., Defiance, Ohio.
CARIS, P. A., A.M. (Bucknell). Teacher, West. Phila. High School for Boys, Philadelphia, Pa.
CARIS, V. B., A.M. (Defiance). Instr., Ohio State Univ., Columbus, Ohio. *72 W. Maynard Ave.*
CARLEN, MILDRED E., Ph.B. (Brown). Asst., Connecticut Coll., New London, Conn.
CARLSON, R. C., A.M. (Columbia). Head of Dept., Sen. and Jr. High Schools, New Brunswick, N. J. *135 College Ave.*
CARMICHAEL, F. L., A.M. (Princeton). *2036 E. 17th Ave.*, Denver, Colo.
CARMICHAEL, Prof. R. D., Ph.D. (Princeton). Univ. of Illinois, Urbana, Ill. *207 W. Washington Blvd.*
CARPENTER, Prof. D. R., A.M. (Princeton). Roanoke Coll., Salem, Va.
CARPENTER, R. H., A.M. (Kansas). Jr. Coll., Iola, Kans. *301 East St.*
CARR, E. L., A.B. (Ewing Coll.). *6109 Dorchester Ave.*, Chicago, Ill.
CARR, Asst. Prof. F. E., Ph.D. (Chicago). Oberlin Coll., Oberlin, Ohio.
CARROLL, I. S., A.M. (Columbia). Instr., Syracuse Univ., Syracuse, N. Y. *419 E. Colvin St.*
CARRUTH, Prof. W. M., A.B. (Cornell). Hamilton Coll., Clinton, N. Y. *P. O. Box 25.*
CARSCALLEN, G. E., A.M. (Illinois). Wabash Coll., Crawfordsville, Ind.
CARTER, Asst. Prof. B. E., A.M. (Harvard). Colby Coll., Waterville, Me. *3 Center Pl.*
CARTER, C. C. *R. D. 2*, Bluffs, Ill.
CARTER, Asso. Prof. MAY B., A.B. (Meredith). Western Coll. for Women, Oxford, Ohio.
CARUS, E. H., Ph.D. (Chicago). With Open Court Pub. Co., Chicago, Ill. *La Salle, Ill.*
CARVER, Asst. Prof. W. B., Ph.D. (Johns Hopkins). Cornell Univ., Ithaca, N. Y.
CASTER, MARY E., A.M. (Columbia). Teacher, High School, Paterson, N. J.
CASE, Prof. J. E., A.M. (Gonzaga Univ.). St. Mary's Coll., St. Mary's, Kans.
CEDERBERG, Prof. W. E., Ph.D. (Wisconsin). Augustana Coll., Rock Island, Ill. *2542–22½ Ave.*
CHACE, ARNOLD B., Sc.D. Chancellor, Brown Univ., Providence, R. I. *99 Power St.*
CHAMBERS, Prof. G. G., Ph.D. (Pennsylvania). Univ. of Pa., Lansdowne, Pa. *79 Drexel Ave.*
CHANDLER, H. W., M.S. (Iowa). Univ. of Florida, Gainesville, Fla.
CHAPMAN, MARGARET F., A.M. (Wisconsin). State Coll., Mankato, Minn.
CHARLES, Prof. R. L., A.M. (Lehigh). Physics, Franklin & Marshall Coll., Lancaster, Pa. *510 Race St.*
CHELLBORG, JULIA R. S., B.S. (Hunter). Hunter Coll., New York, N. Y.
CHEPMELL, Maj. C. H. *85 Wilbury Crescent*, Hove, Sussex, England.
CHIN, Prof. JUNG LU, Graduate Peking Teachers Coll. Nankai Univ., Tientsin, China.
CHITTENDEN, Asso. Prof. E. W., Ph.D. (Chicago). Univ. of Iowa, Iowa City, Iowa.
CHURCH, EARL, C.E. (Syracuse). Parish, N. Y.
CIPOLLA, Prof. MICHELE, Dottore in mat. Prof. Ord. di. Analisi Algebrica, R. Universita di Catania, Catania, Sicily, Italy. *Via umberto I, 242.*
CLACK, Prof. R. W., A.M. (Grinnell). Alma Coll., Alma, Mich. *209 W. Downie St.*
CLARK, ANNIE C. (Mrs. Theron Clark), Ph.B. (Brown). Univ. of Southern California, Los Angeles, Calif.
CLARK, Miss PEARL E., A.M. (Montana). Chaffey Jr. Coll., Ontario, Calif. *223 West J St.*
CLARK, R. F., A.B. (Williams), Pd.B. (Albany State Normal). Dewitt Clinton High School, New York, N. Y. *194 Christie Hts. St., Leonia, N. J.*
CLARKE, Prof. E. H., Ph.D. (Chicago). Math. and Astr., Hiram Coll., Hiram, Ohio.
CLARKE, J. A., A.M. (Princeton). West Phila. High School for Boys, Philadelphia, Pa. *Devon, Pa.*
CLARKE, Sister M. BORGIA, A.M. (Catholic Univ.). Prof., Loretto Coll., Webster Groves, Mo.
CLAWSON, Prof. J. W., M.A. (New Brunswick). Ursinus Coll., Collegeville, Pa.
CLAYTOR, E. M., B.S. (The Citadel). Instr., Univ. of the South, Sewanee, Tenn.
CLELAND, Prof. W. E., Ph.D. (Princeton). Geneva Coll., Beaver Falls, Pa.
CLEMENT, F. C., A.B. (St. Olaf). Instr., St. Olaf Coll., Northfield, Minn.
CLEMENTS, Asso. Prof. G. R., Ph.D. (Harvard). U. S. Naval Acad., Annapolis, Md. *7 Thompson St.*
CLEVENGER, C. H., M.S. (Chicago). Route 7, Hartford City, Ind.
COAR, Prof. H. L., Ph.D. (Illinois). Math. and Astr., Marietta Coll., Marietta, Ohio.
COATES, W. M., A.B. (Williams). Student of Aeronautics, Univ. of Göttingen, Göttingen, Germany. *Pension Creuznacher, Nikolausbergerweg 59.*
COBB, Prof. H. E., A.M. (Wesleyan). Lewis Inst., Chicago, Ill.

COBLE, Prof. A. B., Ph.D. (Johns Hopkins). Univ. of Illinois, Urbana, Ill. *702 W. Washington Blvd.*

COE, Asst. Prof. C. J., A.M. (Harvard). Univ. of Michigan, Ann Arbor, Mich. *1924–25, 3 rue de l'Estrapade (5), Paris, France.*

COFFIN, J. G., Ph.D. (Clark). Math. Physics. Genl. Baking Co., Hempstead, L. I., N. Y. *199 Elizabeth Ave.*

COFFIN, Prof. L. M., A.M. (Michigan). Coe Coll., Cedar Rapids, Ia. *1027 2d Ave.*

COHEN, Asso. Prof. ABRAHAM, Ph.D. (Johns Hopkins). Johns Hopkins Univ., Baltimore, Md.

COIT, Prof. W. A., Ph.D. (Boston Univ.). Acadia Univ., Wolfville, N. S.

COKER, Prof. E. C., A.B. (Virginia). Astr. and Math., Univ. of South Carolina, Columbia, S. C. *1730 College St.*

COLAW, J. M., A.M. (Dickinson). High School, Monterey, Va.

COLE, J. P., B.S. (La. State). Instr., La. State Univ., Baton Rouge, La.

COLE, Prof. LENA R., A.M. (Missouri). Central Normal Coll., Danville, Ind.

COLEMAN, Prof. J. B., A.M. (Columbia). Univ. of South Carolina, Columbia, S. C.

COLEMAN, Prof. R. H., A.B. (Charleston). Coll. of Charleston, Charleston, S. C.

COLLIER, Asst. Prof. MYRTIE, B.S. (Chicago). So. Branch, Univ. of California, Los Angeles, Calif. *320½ N. Alexandria Ave.*

COLLITON, J. W., C.E., E.M. (Lafayette). Teacher, High School, Trenton, N. J. *223 Highland Ave.*

COLPITTS, Asso. Prof. JULIA T., Ph.D. (Cornell). Iowa State Coll., Ames, Ia.

COLWELL, Prof. R. C., Ph.D. (Princeton). Physics, West Virginia Univ., Morgantown, W. Va. *332 Demain Ave.*

COLYER, Prof. E. E., A.M. (Kansas). State Teachers Coll., Hays, Kans. *406 W. 4th St.*

COMSTOCK, Prof. C. E., A.M. (Knox). Bradley Poly. Inst., Peoria, Ill.

CONDIT, Prof. I. S., A.M. (Parsons). State Teachers Coll., Cedar Falls, Ia. *115 E. 11th St.*

CONGDON, Asso. Prof. A. R., A.M. (Nebraska). Pedagogy of Math., Univ. of Nebraska, Lincoln, Nebr. *Station A.*

CONKLING, R. P., A.B. (Cornell). Head of Dept., Newark Tech. School and Central High School, Newark, N. J. *Newark Tech. School.*

CONNELLY, MARION. *828 Simpson St.*, Evanston, Ill.

CONWELL, Asst. Prof. G. M., Ph.D. (Princeton). N. Y. State Coll. for Teachers, Albany, N. Y.

CONWELL, Prof. H. H., M.S. (Kansas). Beloit Coll., Beloit, Wis. *921 Park Ave.*

COOK, Prof. E. C., A.M. (Dartmouth). St. Stephen's Coll., Annandale-on-Hudson, N. Y. *Absent 1924–25, Hanover, N. H.*

COOLIDGE, Prof. J. L., Ph.D. (Bonn). Harvard Univ., Cambridge 38, Mass. *27 Fayerweather St.*

COOPER, A. E., E.E. (Texas). Instr., Applied Math., Univ. of Texas, Austin, Tex. *105 W. 26th St.*

COPE, T. F., B.S. (Tulane). Instr., Tulane Univ., New Orleans, La.

COPELAND, Asst. Prof. LENNIE P., Ph.D. (Penna.). Wellesley Coll., Wellesley, Mass. *14 Waban St.*

CORBIN, Prof. C. E., A.M. (Northwestern). Coll. of the Pacific, Stockton, Calif. *117 W. Euclid Ave.*

COREY, S. A. *1079 23d St.*, Des Moines, Ia.

CORNELL, Miss ODYNE O., A.B. (Nebraska). *505 E. 13th St.*, Oklahoma City, Okla.

CORNETET, R. L., B.S. (Otterbein). Teacher, High School, Chillicothe, Ohio.

COSBY, Prof. BYRON, A.M. (Missouri). State Teachers Coll., Kirksville, Mo.

COTTINGHAM, Mrs. KENNETH, A.M. (Ohio State). *384 17th Ave.*, Columbus, Ohio.

COULTRAP, Prof. M. W., A.M. (Ohio Wesleyan). North-Western Coll., Naperville, Ill. *95 E. Franklin St.*

COURT, Asso. Prof. N. A., D.Sc. (Ghent, Belgium). Univ. of Oklahoma, Norman, Okla.

COWLEY, Asso. Prof. ELIZABETH B., Ph.D. (Columbia). Vassar Coll., Poughkeepsie, N. Y.

COWLING, Prof. A. H., A.M. (Texas). E. Texas State Normal Coll., Commerce, Tex.

CRAGWALL, Prof. J. A., M.S. (Vanderbilt). Head of Dept., Wabash Coll., Crawfordsville, Ind. *Kennedy Pl.*

CRAIG, C. C., A.M. (Indiana). Fellow of Amer.-Scandinavian Soc., Univ. of Lund, 1924–25. *M. Stenbocksgatan 6, Lund, Sweden.*

CRAMBLET, Prof. W. H., Ph.D. (Yale). Bethany Coll., Bethany, W. Va. *Box 631.*

CRAMLET, C. M., M.S. (Washington). Asso., Univ. of Washington, Seattle, Wash. *Dept. of Math.*

CRANE, Asst. Prof. RUFUS, A.B. (Middlebury), B.S. (Mass. Inst.). Math. and Eng., Ohio Wesleyan Univ., Delaware, Ohio. *407 W. William St.*

CRATHORNE, Asso. Prof. A. R., Ph.D. (Göttingen). Math. Statistics, Univ. of Illinois, Urbana, Ill. *802 Penna. Ave.*

CRAWLEY, Prof. E. S., Ph.D. (Pennsylvania). Univ. of Pennsylvania, Philadelphia, Pa.

CREELMAN, G. W., A.B. (Harvard). Hotchkiss School, Lakeville, Conn.

CRENSHAW, Prof. B. H., M.E. (Ala. Poly.). Alabama Poly. Inst., Auburn, Ala.

CRESSE, Prof. G. H., Ph.D. (Chicago). Univ. of Arizona, Tucson, Ariz. *Box 125, Univ. Sta.*

CROCKETT, Prof. C. W., C.E. (Rensselaer), A.M. (Mercer). Math. and Astr., Rensselaer Poly. Inst., Troy, N. Y. *221 Stow Ave.*

CROMWELL, J. W., Jr., A.M. (Dartmouth). M St. High School, Washington, D. C. *1815 13th St., N.W.*

CROOKS, Prof. C. G. Centre Coll., Danville, Ky.

CROWE, Asst. Prof. S. E., A.B. (Ohio State). Michigan Agric. Coll., East Lansing, Mich. *137 Univ. Drive.*

CRUM, C. W. R., M.D. Brunswick, Md.

CRUM, Asst. Prof. W. L., Ph.D. (Yale). Statistics, Harvard Univ., Cambridge 38, Mass. *Box 93.*

CRUSE, Asst. Prof. S. R., M.S. (Arizona). Univ. of Arizona, Tucson, Ariz. *725 E. Speedway.*

CULMER, ORPHA A., A.M. (Michigan). State Normal School, Florence, Ala.

CUMMINGS, Asso. Prof. LOUISE D., Ph.D. Vassar Coll., Poughkeepsie, N. Y.

CUNHA, Prof. PEDRO JOSÉ DA. Univ. of Lisbon, Lisbon, Portugal. *Rua de S. Bento, No. 706.*

CURRIER, Asst. Prof. C. H., A.M. (Brown). Brown Univ., Providence, R. I.

CURTIS, H. B., Ph.D. (Cornell). Instr., Northwestern Univ., Evanston, Ill. *2015 Orrington Ave.*

CURTISS, Prof. D. R., Ph.D. (Harvard). Northwestern Univ., Evanston, Ill. *2023 Sherman Ave.*

CUTTING, L. H., B.S. (Chicago). Teacher, Westport High School, Kansas City, Mo. *3917 Summit St.*

DADOURIAN, Prof. H. M., Ph.D. (Yale). Trinity Coll., Hartford, Conn.

DAHLENE, Prof. OSCAR, M.S. (Kansas). Eng. and Math., Univ. of Alabama, University, Ala.

DALAKER, Prof. H. H. Univ. of Minnesota, Minneapolis, Minn. *523 Walnut St., S.E.*

DALTON, Prof. J. P., D.Sc. (St. Andrews; Cape). Univ. of the Witwatersrand, Johannesburg, S. Afr. *P. O. Box 1176.*

DALY, WINIFRED D., A.M. (Tulane). Teacher, Sophia B. Wright High School, New Orleans, La. *Napoleon Ave. and Prytania.*

DANCER, C. W., A.M. (Ohio State). Instr., Toledo Univ., Toledo, Ohio. *325 Ontario St.*

DANIELLS, Asst. Prof. MARIAN E., A.B. (Chicago). Iowa State Coll., Ames, Ia. *307 Ash Ave.*

DANTZCHER, W. F., Student, Columbia Univ., New York, N. Y.

DAPPERT, J. W., Civil Eng., Lock Box 141, Taylorville, Ill.

DARLING, F. W., A.B. (Cornell). U. S. Coast and Geod. Surv., Washington, D. C.

DARNELL, ALBERTUS, Ph.B. (Michigan). Detroit Jr. Coll., Detroit, Mich. *1216 Edison Ave.*

DARRACOTT, W. H. *574 Pine St.*, Winnipeg, Man., Can.

DASPIT, Mrs. ARMAUD P., A.B. (La. State Univ.). Instr., State Univ., Baton Rouge, La. *759 Boyd Ave.*

DAUGHERTY, Prof. R. D., Ph.B. (Iowa Wesleyan). Iowa State Teachers Coll., Cedar Falls, Ia. *2218 Franklin St.*

DAUS, Asst. Prof. P. H., Ph.D. (California). Univ. of California, So. Branch, Los Angeles, Calif. *855 N. Vermont Ave.*

DAVIS, ALFRED. Head Asst. Math., Soldan High School, St. Louis, Mo.

DAVIS, C. H. Travelers Ins. Co., Hartford, Conn.

DAVIS, H. A., A.B. (W. Va.). Instr., West Virginia Univ., Morgantown, W. Va.

DAVIS, Asst. Prof. H. T. Univ. of Indiana, Bloomington, Ind.

DAVIS, Asst. Prof. J. E., A.M. (Wisconsin). Drexel Inst., Philadelphia, Pa.

DAVIS, Prof. J. M., A.M. (Kentucky). Univ. of Kentucky, Lexington, Ky. *340 Madison Place.*

DAVISSON, Prof. S. C., Sc.D. (Tübingen). Indiana Univ., Bloomington, Ind.

DEAN, ALICE C., A.M. (Rice). Acting Libn. and Fellow in Math., Rice Inst., Houston, Texas.

DECHERD, MARY E., A.M. (Texas). Univ. of Texas, Austin, Texas. *2313 Nueces St.*

DECKER, Prof. F. F., Ph.D. (Syracuse). Syracuse Univ., Syracuse, N. Y. *312 Marshall St.*

DECLEENE, Rev. L. A. V., A.M. (Catholic Univ.). Prof., St. Norbert's Coll., West De Pere, Wis.

DE COU, Prof. E. E., M.S. (Chicago). Univ. of Oregon, Eugene, Ore. *606 East 13th Ave.*

DEDERICK, Prof. L. S., Ph.D. (Harvard). Univ. of British Columbia, Vancouver, B. C. *1154 W. 12th Ave.*

DEESZ, Asst. Supt. L. A., B.S. (Carnegie). Elec. Dept., Colo. Fuel and Iron Co., Pueblo, Colo. *412 Minnequa Ave.*

DE LA, J. W. R. H., B.S. (S. Dak. State Coll.). Tutor, S. Dak. State Coll. *Minot, N. Dak.*

DE LA GARZA, E. *601 St. Charles St.*, Brownsville, Texas.

DE LONG, Prof. I. M., A.M. (Simpson). Univ. of Colorado, Boulder, Colo. *1341 Broadway.*

DE LURY, Prof. A. T., M.A. (Toronto). Univ. of Toronto, Toronto, Ont., Can.

DEMAREE, R. G., M.S. (Chicago). *1801 N. Madison Ave.*, Pasadena, Calif.

DEMAUD, L. S., B.S. (Missouri). Teacher, High School, Oklahoma City, Okla. *1615 W. 37th.*

DEMING, Prof. R. M., B.C.E. (Iowa State). Math. and Astr., Upper Iowa Univ., Fayette, Iowa.

DENNY, Prof. A. K., M.S. (Lincoln). Prof. and Registrar, Lincoln Coll., Lincoln, Ill. *221 Oglesby Ave.*

DENSTON, Asst. Prof. WALTER, B.A. (Cambridge). Kenyon Coll., Gambier, Ohio.

DENTON, Asst. Prof. W. W., Ph.D. (Illinois). Univ. of Michigan, Ann Arbor, Mich. *1014 Cornwell Pl.*

DEY, L. M., Govt. Official, Rangoon, Burma. *25/2 Mohan Bagan Row, Calcutta, Shambazar Post-office, India.*

DICE, ELIZABETH, A.M. (Texas). North Dallas High School, Dallas, Texas.

DICK, Prof. F. J. Raja-Yoga Coll., Point Loma, Calif.

DICKEY, Prin. H. E., A.B. (Cornell Coll.). High School, Sac City, Ia.

DICKINSON, Prof. C. N. Hollins Coll., Hollins, Va.

DICKSON, Prof. L. E., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill. *5535 University Ave.*

DICKSTEIN, Prof. S., Ph.D. Math. and History of Math., Univ. of Warsaw, Warsaw, Poland. *Marszaskowska St. 117.*

DIEDERICH, E. F., A.B. (Butler). Head of Dept., Cedarville Coll., Cedarville, Ohio.

DILLINGHAM, Asso. Prof. ALEX. U. S. Naval Acad., Annapolis, Md. *12 Thompson St.*

DIMICK, Prof. C. E., A.M. (Penna.). U. S. Coast Guard Academy, New London, Conn. *Sales Ferry, Conn.*

DINWIDDIE, Pres. A. B., Ph.D. (Virginia). Tulane Univ., New Orleans, La. *Station 20.*

DOAK, Asso. Prof. ELEANOR C., Ph.B. (Chicago). Mount Holyoke Coll., South Hadley, Mass.

DOAN, Asst. Prof. C. S. Purdue Univ., W. LaFayette, Ind. *1020 3d St.*

DOBBIN, Sister MARIOLA, A.M. (Wisconsin). Rosary Coll., River Forest, Ill.

DOBELL, H. A., A.B. (Syracuse). Grad. Student, Univ. of Pa. *111 Horace St., Woodbury, N. J.*

DODD, Prof. E. L., Ph.D. (Yale). Univ. of Texas, Austin, Texas. *3012 West Ave.*

DONAGHHO, Prof. J. S., A.M. (Marietta). Univ. of Hawaii, Honolulu, T. H. *961 Alewa Drive.*

DONAHUE, Asst. Prof. J. E., A.M. (Harvard). Univ. of Vermont, Burlington, Vt.

DONNELLY, L. E. Teacher, Deep Springs School, Deep Springs, Calif.

DOSTAL, B. F., A.M. (Indiana). Instr., Univ. of Michigan, Ann Arbor, Mich. *1318 Forest Court.*

DOTTERER, Prof. J. E., A.B. (Blue Ridge Coll.). Math. and Physics, Manchester Coll., North Manchester, Ind.

DOUGHERTY, Miss LUCY T., B.S. (Washburn). Teacher, Jr. Coll., Kansas City, Kan. *The Gould Hotel.*

DOUGLAS, C. H., A.M., Litt.D. (Colgate). Editor-in-Chief, D. C. Heath & Co., New York, N. Y.

DOUGLAS, J. L., A.M. (Davidson). Davidson Coll., Davidson, N. C.

DOUGLASS, R. D., A.M. (Maine). Instr., Mass. Inst. of Tech., Cambridge, Mass. *76 Drew Rd., Belmont, Mass.*

DOUSHKESS, Asst. Prof. V. H., A.M. (Lafayette). Lafayette Coll., Easton, Pa. *216 McCartney St.*

DOWLING, Asso. Prof. L. W., Ph.D. (Clark). Univ. of Wisconsin, Madison, Wis.

DOWNEY, W. F., Ed.M. (Harvard). Headmaster, English High School, Boston, Mass. *77 Hammond Rd., Belmont, Mass.*

DOWNING, Asso. Prof. H. H., M.S. (Chicago). Univ. of Kentucky, Lexington, Ky.

DRESDEN, Asso. Prof. ARNOLD, Ph.D. (Chicago). Univ. of Wisconsin, Madison, Wis. *2114 Vilas St.*

DROKE, Prof. G. W., A.M. (Arkansas). Univ. of Arkansas, Fayetteville, Ark.

DROZ, CORALIE, A.M. (La. State). Teacher, High School, Baton Rouge, La. *214 St. Louis St.*

DUBE, Prof. L. H., Ph.D. (Gregorian Univ., Rome). Higher Math., Ottawa Univ., Ottawa, Can.

DUERNER, LAURA, B.S. (Oregon). Instr., N. Dakota Agric. Coll., Fargo, N. D. *1113 13th St. N.*

DUNCAN, BERTHA K., A.M. (Texas). Head of Dept., Grenada Coll., Grenada, Miss.
DUNKEL, Asso. Prof. OTTO, Ph.D. (Harvard). Washington Univ., St. Louis, Mo.
DU PASQUIER, L. G., Ph.D. (Zurich). Prof. ordinaire de math. super., Univ. de Neuchâtel, Neuchâtel, Switzerland. *33 Sablons.*
DURELL, FLETCHER, Ph.D. (Princeton). Head of Dept., Lawrenceville School, Lawrenceville, N. J.
DURFEE, Asst. Prof. W. H., M.C.E. (Harvard). Hobart Coll., Geneva, N. Y. *645 Main St.*
DURFEE, Prof. W. P., Ph.D. (Johns Hopkins). Hobart Coll., Geneva, N.Y. *639 Main St.*
DURHAM, ANNIE D., A.B. (Tex. Presbyter. Coll.). Instr., Texas Presbyterian Coll., Milford, Tex.
DUSTHEIMER, Prof. O. L., A.M. (Clark). Baldwin-Wallace Coll., Berea, Ohio. *272 Beech St.*

EAGLES, Prof. T. R., A.M. (North Carolina). Howard Coll., Birmingham, Ala. *8016 Underwood Ave.*
EARHART, Dean FRANC C., B.S. (Lenox). Lenox Coll., Hopkinton, Ia.
EARL, J. M., A.B. (Carleton). Instr., St. Lawrence Univ., Canton, N. Y. *10 Harrison St.*
EARLE, Prof. M. D., A.M. (Furman). Furman Univ., Greenville, S. C.
ECHOLS, Col. C. P. Prof., U. S. Milit. Acad., West Point, N. Y.
ECHOLS, Prof. W. H., C.E. (Virginia). Univ. of Virginia, University, Va.
ECKERSLEY, J. O., C.E. (Cooper Union). Ch. Engr., Dept. of Markets, New York, N. Y. *4269 White Plains Ave.*
EDGECOMB, Mrs. A. L. *921 7th St.*, Boulder, Colo.
EDINGTON, Asst. Prof. W. E., Ph.D. (Illinois). Purdue Univ., W. LaFayette, Ind. *633 Russell St.*
EDMONDSON, Prof. T. W., Ph.D. (Clark). New York Univ., New York, N. Y. *Univ. Heights.*
EELLS, Prof. W. C., A.M. (Chicago). Appl. Math., Whitman Coll., Walla Walla, Wash. *1134 Alvarado Terr.*
EICHELBERGER, J. H. *821 S. Oak Ave.*, Freeport, Ill.
EIESLAND, Prof. J. A., Ph.D. (Johns Hopkins). West Virginia Univ., Morgantown, W. Va.
EISENHART, Prof. L. P., Ph.D. (Johns Hopkins). Princeton Univ., Princeton, N. J. *22 Alexander St.*
ELSTON, J. S., A.B. (Cornell). Asst. Actuary, Travelers Ins. Co., Hartford, Conn.
ELY, Prof. J. A. St. John's Univ., Shanghai, China.
EMCH, Asso. Prof. ARNOLD, Ph.D. (Kansas). Univ. of Illinois, Urbana, Ill. *1002 S. Orchard St.*
EMERY, STEPHEN, A.M. (Boston Univ.). Head of Dept., Erasmus Hall High School, Brooklyn, N. Y. *317 E. 21st St.*
EMMONS, Prof. C. W., A.M. (Illinois). Simpson Coll., Indianola, Ia.
EMMONS, Prof. L. C., B.S. (Central Normal), A.B. (Indiana). Michigan Agric. Coll., East Lansing, Mich.
ENGLISH, HARRY, A.B. (Johns Hopkins), LL.M. (Columbian Univ.). Head of Dept., High Schools, Washington, D. C. *2907 P St., N.W.*
ENGLISH, H. W. Master in Chancery, Morgan Co., Ill. *Jacksonville, Ill.*
ENRIQUES, Prof. FEDERIGO. Univ. of Bologna, Bologna, Italy.
EPPERSON, C. A., A.M. (Missouri). With Natl. Cash Register Co., Kansas City, Mo.
EPPES, Prof. J. B., A.M. (Johns Hopkins). U. S. Naval Acad., Annapolis, Md.
ERICSON, HENRY, B.S. (Chicago). Instr., High School, Milwaukee, Wis. *3414 Chestnut St.*
ERNSBERGER, IVA, A.M. (Nebraska). Instr., Iowa State Coll., Ames, Ia. *Cranford Apt.*
ERWIN, Prof. J. T., A.M. (Vanderbilt). George Washington Univ., Washington, D. C. *2023 G St., N.W.*
ESCOTT, E. B., M.S. (Chicago). Walton School of Commerce, Chicago, Ill. *1019 S. East Ave., Oak Park, Ill.*
ESHLEMAN, J. D., Ph.D. (Chicago). Instr., Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*
ESTES, J. G., A.B. (Tex. Chr. Univ.). Instr., Texas Christian Univ., Fort Worth, Tex.
ESTY, Prof. T. C., A.M. (Amherst). Amherst Coll., Amherst, Mass.
ETTLINGER, Asso. Prof. H. J., Ph.D. (Harvard). Univ. of Texas, Austin, Tex. *2910 Harris Park Ave.*
EVANS, Prof. G. C., Ph.D. (Harvard). Pure Math., Rice Inst., Houston, Tex.
EVANS, G. W., A.B. (Harvard). Headmaster, Charlestown High School, Boston, Mass.
EVANS, Prof. H. B., Ph.D. (Pennsylvania). Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*

EVANS, P. H. Chief Actuary, Northwestern Mut. Life Ins. Co., Milwaukee, Wis. *210 Wisconsin St.*

EVERETT, Prof. H. S., Ph.D. (Chicago). Appl. Math. and Astr. and Dir. of the Observ., Bucknell Univ., Lewisburg, Pa. *28 Univ. Ave.*

EVERETT, Prof. J. P., A.M. (Michigan). Western State Normal School, Kalamazoo, Mich. *903 W. South St.*

EVERETT, Asso. Prof. J. R., A.M. (Wisconsin). Colo. School of Mines, Golden, Colo.

EWING, W. M. Junior Student, Rice Inst., Houston, Tex. *Lockney, Tex.*

FAIRCHILD, Prof. J. T., A.M., C.E. (Ohio Northern); Ph.M. (Carnegie Inst.). Ohio Northern Univ., Ada, Ohio.

FARNER, Prin. E. F., A.M. (Kansas). Jr. Coll. and High School, Parsons, Kans. *1613 Partridge.*

FARNUM, FAY, A.M. (Cornell). *612 E. State St.*, Ithaca, N. Y.

FAUGHT, Prof. J. B., Ph.D. (Penna.). Yankton Coll., Yankton, S. D.

FEAGIN, RUTH M., B.S. (Alabama). Head of Dept., Martin Coll., Pulaski, Tenn.

FEDERICO, P. J., B.S. (Case School). Asst. Examiner, U. S. Patent Office, Washington, D. C. *1812 G St., N.W.*

FEEMSTER, Prof. H. C., A.M. (Nebraska). York Coll., York, Nebr.

FEHN, Asso. Prof. A. R., Ph.B. (Baldwin-Wallace). Centre Coll., Danville, Ky. *337 W. Broadway.*

FEHR, Dr. HENRI. Editor L'Enseignment de Math., Geneva, Switzerland. *110 Florissant.*

FELTGES, EDNA M., A.M. (Wisconsin). Instr., Eng. Math., Univ. of Wisconsin, Madison, Wis. *419 Sterling Pl.*

FERGUSON, Prof. C. E., A.M. (Missouri). Stephen F. Austin State Teachers Coll., Nacogdoches, Tex.

FERGUSON, Miss ZOE. Crane Jr. High School, Chicago, Ill. *2317 Commonwealth Ave.*

FERRY, Pres. F. C., Ph.D. (Clark). Hamilton Coll., Clinton, N. Y.

FICKEN, H. W., B.S. (C. C. N. Y.). Mathematician, U. S. Coast and Geod. Surv., Washington, D. C.

FIELD, FLORENCE E., A.M. (Michigan). Teacher, High School, Jackson, Mich. *304 S. Hannah, Albion, Mich.*

FIELD, Prof. FLOYD, A.M. (Harvard). Georgia School of Tech., Atlanta, Ga. *Route 1, Decatur, Ga.*

FIELD, Prof. PETER, Ph.D. (Cornell). Univ. of Michigan, Ann Arbor, Mich. *904 Olivia Ave.*

FIELD, S. E., A.M. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich. *513 Thompson St.*

FIELDS, Prof. J. C., Ph.D. (Johns Hopkins). Univ. of Toronto, Toronto, Can.

FINDLAY, Prof. WILLIAM, Ph.D. (Chicago). McMaster Univ., Toronto, Can. *153 Westminster Ave.*

FINE, Prof. H. B., Ph.D. (Leipzig). Princeton Univ., Princeton, N. J. *Library Pl.*

FINKEL, Prof. B. F., Ph.D. (Penna.). Drury Coll., Springfield, Mo. *1227 Clay St.*

FINLEY, Prof. G. W., B.S. (Kansas State). Colo. State Teachers Coll., Greeley, Colo. *1933 9th Ave.*

FISHER, CLAIRE B., A.M. (West Virginia). Potomac State Coll., Keyser, W. Va. *152 Overton Pl.*

FISKE, Prof. T. S., Ph.D. (Columbia). Columbia Univ., New York, N. Y.

FITCH, ANNIE L. MACKINNON (Mrs. EDWARD FITCH), Ph.D. (Cornell). Clinton, N. Y.

FITCH, PHILIP, A.B. (Colo. Coll.). Asst. Prin., North High School, Denver, Colo. *3647 Clay St.*

FITE, Prof. W. B., Ph.D. (Cornell). Columbia Univ., New York, N. Y.

FITTERER, Prof. J. C., C.E. (Colorado). Actg. Dean of Coll. of Eng. and Prof. of Civil and Irrig. Eng., Univ. of Wyoming, Laramie, Wyo. *511 S. 11th St.*

FLANDERS, R. L., B.S. (Norwich). Instr., Norwich Univ., Northfield, Vt.

FLEET, Prof. R. R., Ph.D. (Heidelberg, Germany). William Jewell Coll., Liberty, Mo.

FLEISIG, A. J. Instr., St. Procopius Coll., Lisle, Ill.

FLEMING, Asst. Prof. ANNIE W., A.M. (California). Iowa State Coll., Ames, Ia. *719 Douglas Ave.*

FLYNN, J. D., A.B. (Trinity). Travelers Ins. Co., Hartford, Conn.

FOBERG, J. A., B.S. (Illinois). State Dir. of Math. Instruction, Camp Hill, Pa.

FOCKE, Dean T. M., Ph.D. (Göttingen). Case School of Appl. Sc., Cleveland, Ohio.

FORAKER, Asst. Prof. F. A., M.S. (Ohio Northern). Univ. of Pittsburgh. *1313 Macon Ave., Swissvale, Pa.*

FORD, Asst. Prof. L. R., Ph.D. (Harvard). Rice Inst., Houston, Tex.
FORD, Prof. W. B., Ph.D. (Harvard). Univ. of Michigan, Ann Arbor, Mich. *204 Mason Hall.*
FORSYTH, Asst. Prof. C. H., Ph.D. (Michigan). Dartmouth Coll., Hanover, N. H.
FORSYTH, DAVID, B.A. (Toronto). Prin. Emeritus, Kitchener and Waterloo Colleg. Inst., Kitchener, Ont. *31 Margaret Ave.*
FORT, Prof. TOMLINSON, Ph.D. (Harvard). Hunter Coll., New York, N. Y. *Park Ave. and 68th St.*
FOSTER, R. M., B.S. (Harvard). Dept. of Dev. and Research, Amer. Tel. and Tel. Co., New York, N. Y. *195 Broadway.*
FOX, Asso. Prof. HAL, B.S. (Miss. A. & M.). Mississippi A. & M. Coll., A. & M. College, Miss.
FOX, J. F. Mathematician, U. S. Coast and Geod. Surv., Washington, D. C. *2748 Woodley Pl.*
FRANKEL, E. T., B.S. (C. C. N. Y.). Discolog Business Service Co., New York, N. Y. *Box 164, Sta. D.*
FRANKLIN, PHILIP, Ph.D. (Princeton). Instr., Mass. Inst. of Tech., Cambridge 39, Mass. *46 Mass. Ave.*
FRARY, Asst. Prof. R. W., B.S. (Va. Milit. Inst.). La. State Normal Coll., Natchitoches, La.
FREAS, ELIZABETH, A.B. (Lake Erie). Lake Erie Coll., Painesville, Ohio.
FRÉCHET, MAURICE. Prof. d'Analyse Supér., Univ. de Strasbourg, Strasbourg, France. *141ª R. Principale, Oberhausbergen, par Strasbourg.*
FREEMAN, GLADYS H., A.M. (Chicago). Instr., State Coll. of Washington, Pullman, Wash. *1401 Star Route.*
FREEMAN, NEDA B., A.M. (Boston Univ.). Teacher, Irving Coll., Mechanicsburg, Pa.
FRENCH, A. T., A.M. (Columbia). Head of Dept., State Teachers Coll., Winona, Minn. *363 Winona St.*
FRUMVELLER, Prof. A. F. Marquette Univ., Milwaukee, Wis.
FRY, T. C., Ph.D. (Wisconsin). Math. Physics, Western Elec. Co., Inc., New York, N. Y. *463 West St.*
FUBINI, Prof. GUIDO. Prof. di Analisi Mat., R. Scuola di Ingegneria, Turin, Italy. *Via Pietro Micca 12.*
FULMER, Asst. Prof. H. K., A.M. (Columbia). Georgia School of Tech., Atlanta, Ga.
FUNK, Asso. Prof. J. C., A.M. (Columbia). Jr. Coll., Santa Maria, Calif.
FURREY, MARGARET L., B.S. (Northwestern). Teacher, New Trier High School, Kenilworth, Ill.

GABA, Asso. Prof. M. G., Ph.D. (Chicago). Univ. of Nebraska, Lincoln, Nebr. *1425 S. 22d St.*
GAFAFER, W. M., A.M. (Columbia). Instr., Barnard Coll., Columbia Univ. *Leonia, N. J.*
GAINES, Prof. R. E., A.M. (Furman). Richmond Coll., Richmond College, Va.
GALE, Prof. A. S., Ph.D. (Yale). Univ. of Rochester, Rochester, N. Y. *11 Thayer St.*
GARABEDIAN, Asst. Prof. C. A., Ph.D. (Harvard). Northwestern Univ., Evanston, Ill. *904 Foster St.*
GARDNER, Prof. R. W., A.M. (Boston Univ.). Eastern Nazarene Coll., Wollaston, Mass. *218 Beach St.*
GARLOUGH, Prof. C. D., A.M. (Illinois). Prof. and Registrar, Wheaton Coll., Wheaton, Ill.
GARNER, C. L., B.E. (N. C. State Coll.). Hydrogr. and Geod. Engr., U. S. Coast and Geod. Surv., Washington, D. C.
GARRETSON, Asst. Prof. W. V. N., Ph.D. (Michigan). Rutgers Coll., New Brunswick, N. J.
GARRETT, Prof. W. H., A.M. (Illinois Coll.). Vice-Pres., Math. and Astr., Baker Univ., Baldwin City, Kans.
GARRISON, Prof. D. M., Grad. U. S. Naval Acad. Head of Dept., St. John's Coll., Annapolis, Md. *236 Prince George St.*
GAULT, Asst. Prof. A. E., M.S. (Chicago). Bradley Poly. Inst., Peoria, Ill.
GAVER, Asst. Prof. H. H., A.M. (Virginia). U. S. Naval Acad., Annapolis, Md. *160 Prince George St.*
GAY, Asst. Prof. H. J., A.M. (Clark). Worcester Poly. Inst., Worcester, Mass. *62 Wachusett St.*
GAYDEN, Prof. W. J., A.M. (Baylor). Head of Dept., Howard Payne Coll., Brownwood, Tex. *900 Booker St.*
GAYLORD, Asst. Prof. LESLIE J., M.S. (Chicago). Agnes Scott Coll., Decatur, Ga.
GECKELER, Prof. O. T., A.B. (Indiana). Head of Dept., Carnegie Inst. of Tech., Pittsburgh, Pa.
GEE, E. F., Ph.B. (Michigan). Instr., Coll. of City of Detroit. *325 Wallace Blvd., Ypsilanti, Mich.*
GEHMAN, H. M., A.M. (Penna.). Instr., Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*

George, O. A., A.M. (Minnesota). Instr., Jr. Coll., Mason City, Ia. *19 6th St., N.W.*
Georges, J. S., M.S. (Chicago). Univ. High School, Chicago, Ill. *921 E. 56th St.*
Gérardin, André. Editor *Sphinx-Oedipe*, Nancy, France. *32 Quai Claude le Lorrain.*
Gerst, Rev. F. J., Ph.D. (Johns Hopkins). St. Louis Univ., St. Louis, Mo.
Ghormley, L. O., A.M. (Chicago). Instr., Univ. of Tennessee, Knoxville, Tenn.
Gibbens, Gladys E. C., Ph.D. (Chicago). Instr., Univ. of Minnesota, Minneapolis, Minn. *122 Folwell Hall.*
Gibson, Falka M., A.B. (California). *3034 Clapp St.*, Fruitvale, Calif.
Gibson, Prof. J. L., Ph.D. (Vienna). Univ. of Utah, Salt Lake City, Utah.
Giesecke, Mildred, A.M. (Northwestern). Teacher, High School, Johnston City, Ill. *Box 384.*
Gilbert, Prof. G. A., A.M. (Gonzaga). Physics, Loyola Coll., Los Angeles, Calif. *1901 W. 16th St.*
Gillespie, Asst. Prof. D. C., Ph.D. (Göttingen). Cornell Univ., Ithaca, N. Y. *Cayuga Hts. On leave 1924–25.*
Gillespie, Prof. William, Ph.D. (Chicago). Princeton Univ., Princeton, N. J. *Grad. College.*
Gilman, Asst. Prof. R. E., Ph.D. (Princeton). Brown Univ., Providence, R. I. *654 Hope St.*
Gingrich, Prof. C. H., Ph.D. (Chicago). Math. and Astron., Carleton Coll., Northfield, Minn.
Ginnings, R. M., M.S. (Chicago). Head of Dept. of Math., Western Ill. State Teachers Coll., Macomb, Ill. *314 N. Ward St.*
Glazier, Asst. Prof. Harriet E., A.M. (Chicago). So. Branch, Univ. of California, Los Angeles, Calif.
Glenn, Prof. O. E., Ph.D. (Pennsylvania). Univ. of Pa., Philadelphia, Pa. *127 McKinley Ave., Lansdowne, Pa.*
Glover, Prof. B. C., B.S. (Northwestern). Otterbein Coll., Westerville, Ohio.
Glover, Prof. J. W., Ph.D. (Harvard). Univ. of Michigan, Ann Arbor, Mich. *620 Oxford Rd.*
Gokhale, Asso. Prof. V. D., Ph.D. (Chicago). Univ. of Philippines, Manila, P. I.
Gold, J. S., B.S. (Bucknell). Instr., Bucknell Univ., Lewisburg, Pa. *Box 191.*
Gorrell, Asso. Prof. G. W., A.M. (Ohio State). Univ. of Denver, Denver, Colo. *2236 S. Milwaukee St.*
Gossard, Asso. Prof. H. C., Ph.D. (Johns Hopkins). Univ. of Wyoming, Laramie, Wyoming.
Gould, Alice B., A.B. (Bryn Mawr). Care W. W. Vaughan, Exchange Bldg., Boston, Mass.
Gouwens, Asst. Prof. Cornelius, Ph.D. (Chicago). Iowa State Coll., Ames, Ia. *109 Hyland Ave.*
Graber, Prof. M. E., Ph.D. (Iowa). Physics, Morningside Coll., Sioux City, Ia.
Graesser, R. F., A.M. (Illinois). Univ. of Illinois, Urbana, Ill. *401 W. Nevada St.*
Graham, Maria D., B.S. (Teachers Coll., Columbia Univ.). Head of Dept. of Math., Teachers Training School, Greenville, N. C.
Graham, Asst. Prof. P. H., A.M. (Virginia). Washington Square Coll., New York Univ., New York, N. Y. *136 Brewster Ave., Flushing, N. Y.*
Grant, Alice A., A.M. (Brown). Teacher, Oldfields School, Glencoe, Md.
Grant, Prof. E. D., Ph.D. (Chicago). Math. and Registrar, Earlham Coll., Richmond, Ind.
Granville, W. A., Ph.D. (Yale). Educ. Director, U. S. National Life and Casualty Co., Chicago, Ill. *29 S. LaSalle St.*
Graustein, Asst. Prof. W. C., Ph.D. (Bonn). Harvard Univ., Cambridge, Mass. *44 Langdon St.*
Gravatt, T. E. Instr., Pennsylvania State Coll., State College, Pa.
Graves, Asst. Prof. G. H., Ph.D. (Columbia). Purdue Univ., W. LaFayette, Ind. *829 Main St.*
Green, Prof. R. L., A.M. (Indiana). Stanford University, Calif.
Greenleaf, Asst. Prof., B.S. (Boston). DePauw Univ., Greencastle, Ind. *9 N. College Ave.*
Griffin, Prof. F. L., Ph.D. (Chicago). Reed Coll., Portland, Ore.
Grove, C. C., Ph.D. (Johns Hopkins). Baldwin, L. I., N. Y.
Grove, Asst. Prof. V. G., A.M. (Kentucky). Michigan Agric. Coll., East Lansing, Mich. *Box 507.*
Gudheim, Asso. Prof. H. E., M.E. (Royal Univ. of Tech., Stockholm, Sweden). Graphics, Va. Poly. Inst., Blacksburg, Va.
Guilbeau, Lucye, A.B. (La. State). Teacher, St. Gabriel High School, Baton Rouge, La. *911 West St.*
Guilliams, J. M., A.B. (Central Normal Coll.). Teacher, Berea Normal School, Berea, Ky.
Gummer, Prof. C. F., Ph.D. (Chicago). Queen's Univ., Kingston, Ont. *143 Collingwood St.*
Gummere, Prof. H. V., A.M. (Harvard). Drexel Inst., Philadelphia, Pa. *418 S. Wycombe Ave., Lansdowne, Pa.*
Gundersen, Prof. Carl, Ph.D. (Columbia). Oklahoma A. and M. Coll., Stillwater, Okla.

GWALTNEY, J. D., A.B. (Harvard). Teacher, Jefferson Milit. Coll., Washington, Miss.
GWINN, I. J., A.B. (Morningside). Grad. Asst., Physics, Univ. of Iowa, Iowa City, Ia. *Dept. of Physics.*
GWINNER, Prof. HARRY, M.E. (Maryland). Univ. of Maryland, College Park, Md.

HACKER, J. A., B.S. (Valparaiso). Sioux Falls Coll., Sioux Falls, S. D. *1411 Norton Ave.*
HACKNEY, LILIAN, A.B. (W. Virginia). Marshall Coll., Huntington, W. Va.
HADAMARD, Prof. JACQUES, LL.D. (Yale). École Polytech. et Coll. de France, Paris, France. *25 rue Humboldt, Paris, XIV.*
HADLEY, Prof. LAURENCE, Ph.D. (Mich.). Purdue Univ., W. LaFayette, Ind. *121 Lutz Ave.*
HADLEY, Prof. S. M., Ph.D. (Wisconsin). Penn. Coll., Oskaloosa, Ia. *427 N. 3d St.*
HADLOCK, E. H. Student Asst., Univ. of Maine, Orono, Me. *25 Myrtle St.*
HAGELSTEIN, E. L. *213 W. Harris Ave.*, San Angelo, Texas.
HAIGLER, C. E., M.S. (Harvard). Math. and Physics, Manter Hall School, Cambridge, Mass. *51 Franklin St., Watertown, Mass.*
HALDEMAN, C. B. Ross, Butler Co., Ohio.
HALE, W. R., A.M. (Alabama). Instr., Univ. of Fla., Gainesville, Fla.
HALL, Prof. A. G., Ph.D. (Leipzig). Registrar, Univ. of Michigan, Ann Arbor, Mich. *1036 Oakland Ave.*
HALL, BLANCHE, A.M. (Illinois). Instr., Georgetown Coll., Georgetown, Ky.
HALL, ELIZABETH L. (Mrs. J. J.), Ph.B. (Syracuse). Instr., West High School, Rochester, N. Y. *268 Alexander St.*
HALL, RUTH H., A.B. (Brown). Rosemary Hall, Greenwich, Conn.
HALL, Prof. W. S., M.E., M.S. (Lafayette). Lafayette Coll., Easton, Pa. *College Campus.*
HALPERIN, Asso. Prof. HILLEL, E.E. (Liége), A.M. (Columbia). A. and M. Coll. of Texas, College Station, Tex.
HAMILTON, Prof. W. A., Ph.D. (Chicago). Antioch Coll., Yellow Springs, Ohio. *101 Walnut St.*
HAMILTON, W. M., A.M. (Michigan). Asst., U. S. Nautical Almanac Office, U. S. Naval Observ., Washington, D. C.
HAMMER, H. H., A.M. (Texas). With the Amer. Life Re-Insurance Co., Dallas, Tex.
HAMMOND, Asst. Prof. E. S., Ph.D. (Princeton). Bowdoin Coll., Brunswick, Me. *84 Federal St.*
HANAWALT, Prof. F. W., A.M.(DePauw). Coll. of Puget Sound, Tacoma, Wash. *826 N. Steele St.*
HANCOCK, CLARA L., A.M. (Iowa). Teacher, Jr. Coll., Virginia, Minn.
HANCOCK, Prof. HARRIS, Ph.D. (Berlin). Univ. of Cincinnati, Cincinnati, Ohio.
HANNA, Prof. U. S., Ph.D. (Penna.). Indiana Univ., Bloomington, Ind. *828 Atwater Ave.*
HANSON, H. O., A.B. (Columbia). Mut. Life Ins. Co., East Elmhurst, L. I., N. Y. *Ditmars Ave.*
HANTHORN, EMMA E., A.B. (Nebraska). Teacher, State Teachers Coll., Kearney, Nebr. *714 W. 23d St.*
HARDIN, Prof. J. A., A.M. (Chicago). Acting Dean, Centenary Coll. of La., Shreveport, La.
HARDING, Prof. A. M., Ph.D. (Chicago). Math. and Astr., Univ. of Arkansas, Fayetteville, Ark. *537 Leverett St.*
HARDING, HOWARD, B.M.E. (Michigan). With Rochester Rwy. and Light Co., Rochester, N. Y. *29 Kingston St.*
HARDY, Prof. G. H. Savilian Prof. of Geometry, Oxford Univ., Oxford, Eng. *New College.*
HARDY, Prof. J. G., Ph.D. (Johns Hopkins). Williams Coll., Williamstown, Mass.
HARGETT, Prof. A. J., A.M. (Transylvania). Texas Christian Univ., Fort Worth, Tex.
HARMOUNT, G. P., A.M. (Ohio State). Head of Dept., East High School, Columbus, Ohio. *2290 Indianola Ave.*
HARPER, H. D. Instr., Murray Hill Voc. School, New York, N. Y. *1 S. 20th St., Elmhurst, N. Y.*
HARRELL, Asso. Prof. J. W., A.M. (Brown). Appl. Math., Baylor Univ., Waco, Tex.
HARRINGTON, C. E., M.E. (Cornell). Instr., Univ. of Buffalo, Buffalo, N. Y. *52 Winter St.*
HARRIS, Asso. Prof. ISABEL, A.M. (Columbia). Westhampton Coll., University of Richmond, Va.
HARRY, S. C., A.B. (Johns Hopkins). Head of Dept., City Coll., Baltimore, Md. *1530 Linden Ave.*
HARSHBARGER, Prof. W. A., B.S. Washburn Coll., Topeka, Kans. *1401 College Ave.*
HART, Asso. Prof. BERTHA I., A.B. (Western Md.). Western Maryland Coll., Westminster, Md.
HART, Prof. J. N., C.E. (Maine), M.S. (Chicago). Univ. of Maine, Orono, Me.
HART, Asso. Prof. W. L., Ph.D. (Chicago). Univ. of Minnesota, Minneapolis, Minn.
HART, Asst. Prof. W. W., A.B. (Chicago). Univ. of Wisconsin, Madison, Wis. *R. F. D. 7.*

HARTER, Prof. G. A., Ph.D. (St. John's). Delaware Coll., Newark, Del.
HARTIG, Asst. Prof. H. E., Ph.D. (Minnesota). Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.
HARTSFIELD, J. M. Junior Student, Rice Inst., Houston, Tex. *4942 Reiger Ave., Dallas, Tex.*
HASEMAN, Prof. CHARLES, Ph.D. (Göttingen). Univ. of Nevada, Reno, Nev.
HASEMAN, MARY G., Ph.D. (Bryn Mawr). Instr., Univ. of Illinois, Urbana, Ill. *433 Nat. Hist. Bldg.*
HASKELL, Prof. M. W., Ph.D. (Göttingen). Univ. of California, Berkeley, Calif. *P. O. Box 3.*
HASKELL, R. N., B.S. (Chicago). Instr., Michigan Agric. Coll., East Lansing, Mich. *Box 631.*
HASSLER, Prof. J. O., Ph.D. (Chicago). Univ. of Oklahoma, Norman, Okla. *425 Lahoma Ave.*
HATCH, Asst. Prof. D. A., Eng. of Mines (Lafayette). Lafayette Coll., Easton, Pa. *705 High St.*
HATHAWAY, Prof. A. S., B.S. (Cornell). Retired, Rose Poly. Inst. *Boerne, Tex.*
HAWKES, Dean H. E., Ph.D. (Yale). Columbia Univ., New York, N. Y.
HAWKINS, JULIA L., B.S. (Chicago). Dean of Women, Oklahoma Coll. for Women, Chickasha, Okla.
HAYNES, Prof. E. S., Ph.D. (California). Univ. of Missouri, Columbia, Mo. *505 Hitt St.*
HAYS, W. H., A.B. (Missouri). Supt. of Schools, Connell, Wash.
HAZARD, Asst. Prof. C. T., A.M. (Indiana). Purdue Univ., W. LaFayette, Ind. *344 N. Western Ave.*
HAZELTINE, B. A., B.S. (Tufts). Instr., Middlebury Coll., Middlebury, Vt. *38 South St.*
HAZLETT, Asso. Prof. OLIVE C., Ph.D. (Chicago). Mount Holyoke Coll., South Hadley, Mass.
HEAL, W. E. U. S. Coast and Geod. Surv., Washington, D. C. *3747 Huntington St., Chevy Chase, D. C.*
HEDRICK, Prof. E. R., Ph.D. (Göttingen). So. Branch, Univ. of California, Los Angeles, Calif. *523 N. Mariposa Ave.*
HEINZ, ALBERT, A.M. (Missouri). Head of Dept., Tsing Hua Coll., Peking, China.
HELWIG, MARY E. Teacher, High School, Kansas City, Kans. *736 State Ave.*
HEMKE, P. E., Ph.D. (Johns Hopkins). Research Mathematician, Langley Aeronaut. Lab., Langley Field, Va.
HENDERSON, Prof. ARCHIBALD, Ph.D. (North Carolina; Chicago). Head of Dept., Univ. of North Carolina, Chapel Hill, N. C.
HENDERSON, ROBERT, B.A. (Toronto). Actuary, Equitable Life Assur. Co., New York, N. Y. *120 Broadway.*
HENNA, JEANETTE, A.B. (Westhampton). Instr., Marion Coll., Marion, Va. *Box 142.*
HENNEL, Asso. Prof. CORA B., Ph.D. (Indiana). Indiana Univ., Bloomington, Ind. *822 E. Atwater Ave.*
HENROTEAU, F. C., Dr. of Physics and Math. (Brussels). Head of Astrophys. Dept., Dominion Observ., Ottawa, Can.
HENRY, Prof. T. B., A.B. (Kansas). Highland Coll., Highland, Kans.
HEREN, Asst. Prof. MABEL M., M.S. (Northwestern). Knox Coll., Galesburg, Ill.
HERRICK, Asst. Prof. C. A., M.E. (Minnesota). Eng. Math., Univ. of Minnesota, Minneapolis, Minn. *4120 Sheridan Ave. S.*
HERRON, Prof. C. L., A.M. (Hillsdale). Hillsdale Coll., Hillsdale, Mich. *188 Hillsdale St.*
HESS, Prof. G. W., Ph.D. (Michigan). Union Univ., Jackson, Tenn. *203 Pleasant Ave.*
HICKOX, Prof. E. J., A.M. (Columbia). Math. and Physics, Internatl. Y. M. C. A. Coll., Springfield, Mass.
HICKS, H. C., M.S. (Chicago). Fellow, Calif. Inst. of Tech., Pasadena, Calif.
HICKS, R. L., B.S. (Lebanon Univ.), A.B. (Oklahoma). Dale, Pott. Co., Okla.
HICKSON, A. O., B.A. (Acadia). Instr., Brown Univ., Providence, R. I.
HIGGINS, Prof. W. S., M.E.E. (Harvard). Univ. of Florida, Gainesville, Fla.
HIGH, M. DE TURK, A.B. (Franklin and Marshall). Head of Dept. and Dean of Men, Central State Normal School, Lock Haven, Pa. *Normal School.*
HIGHTOWER, Prof. RUBY U., A.M. (Georgia). Shorter Coll., Rome, Ga. *On leave of absence 1924–25, Univ. of Missouri.*
HILDEBRANDT, MARTHA, Ph.B. (Chicago). Teacher, Proviso High School, Maywood, Ill. *109 27th Ave., Bellwood, Ill.*
HILDEBRANDT, Prof. T. H., Ph.D. (Chicago). Univ. of Michigan, Ann Arbor, Mich. *1930 Cambridge Rd.*
HILL, A. L., A.B. (Doane). Math. and Phys., State Normal and Teachers Coll., Peru, Nebr.

HILL, GRACE H., A.B. (N. J. Coll. for Women). Lock Box 148, Clinton, N. J.
HILL, J. P., A.B. (Washington and Lee). *1733 N St., N.W.*, Washington, D. C.
HILL, M. A., Jr., A.M. (North Carolina). *Box 215*, Chapel Hill, N. C.
HILL, Asso. Prof. W. H., A.M. (Colorado). State Teachers Coll., Pittsburg, Kans. *1012 S. Broadway.*
HILLARD, Prof. C. R., A.M. (Columbia). Head of Dept., Coll. of the Ozarks, Clarksville, Ark. *406 College Ave.*
HILLE, C. E., Ph.D. (Stockholm, Sweden). Instr., Princeton Univ., Princeton, N. J. *64 Nassau St.*
HIMWICH, A. A., M.D. (New York Univ.). Physician, New York, N. Y. *1871 Madison Ave.*
HINTON, Prof. J. C., A.M. (Georgia). Wesleyan Coll., Macon, Ga. *105 Lamar St.*
HIRSCH, BLANCHE, B.S. (Columbia). Prin., Alcuin Prep. School, New York, N. Y. *15 W. 86th St.*
HIRSCHLER, Prof. E. J., A.B. (Kansas). Bluffton Coll., Bluffton, Ohio.
HITCHCOCK, Prof. R. R., A.M. (Northwestern). Univ. of N. Dakota, University, N. D.
HITT, Prof. J. R., B.S. (Mississippi). Mississippi Coll., Clinton, Miss.
HOAR, R. S., A.M. (Harvard). Asst. to Secy., Bucyrus Co., S. Milwaukee, Wis. *715 Hawthorne Ave.*
HOARE, Prof. A. J., A.M. (Michigan). Fairmount Coll., Wichita, Kans. *1717 Holyoke Ave.*
HOBBS, Asso. Prof. A. W., Ph.D. Univ. of North Carolina, Chapel Hill, N. C.
HODGDON, F. C., A.B. (Tufts). With Ginn and Co., New York, N. Y. *70 Fifth Ave.*
HODGE, Asst. Prof. F. H., A.M. (Boston). Purdue Univ., W. LaFayette, Ind. *820 N. Main St.*
HODGKINS, Dean H. L., Ph.D. (George Washington). Prof. of Math. and Dean of the Univ., George Washington Univ., Washington, D. C.
HOFMANN, Prof. ADAM, M.S. (St. Mary). Univ. of Dayton, Dayton, Ohio.
HOLDER, Prof. F. J., Ph.D. (Yale). Mercer Univ., Macon, Ga. *912 College St.*
HOLGATE, Prof. T. F., Ph.D. (Clark). Northwestern Univ., Evanston, Ill. *617 Library Pl.*
HOLLCROFT, Prof. T. R., Ph.D. (Cornell). Wells Coll., Aurora, N. Y. *On leave first sem., 1924–25, Univ. of Rome.*
HOLLIS, ELINOR V., M.S. (Chicago). *11 Boynton St.*, Worcester, Mass.
HOLMES, HELMA L., A.M. (Nebraska). Instr., Pure Math., Univ. of Texas, Austin, Tex. *2623 Univ. Ave.*
HOLMES, J. T. Farmer, Orleans, Ill.
HOOPER, Dean F. F., A.M. (Wisconsin). Univ. of Chattanooga, Chattanooga, Tenn.
HOOVER, B. P., A.M. (Colorado). Asst., Univ. of Illinois, Urbana, Ill. *Dept. of Math.*
HOPKINS, Asst. Prof. L. A., Ph.D. (Chicago). Univ. of Michigan, Ann Arbor, Mich.
HORN, MARVEL C., A.M. (Ohio State). Teacher, Wells High School, Steubenville, Ohio. *1003 La Belle Ave.*
HORNE, Dean C. E., Ph.D. (Chicago). Coll. of A. and M. Arts, Univ. of Porto Rico, Mayagüez, P. R.
HORSBURGH, E. M., D.Sc. (Edinburgh). Lecturer in Technical Math., Univ. of Edinburgh, Edinburgh, Scotland. *11 Granville Terr.*
HORSFALL, I. O., A.B. (Utah). Head of Dept., Latter-Day Saints Univ., Salt Lake City, Utah. *533 E. 6th So. St.*
HORTON, GOLDIE P., Ph.D. (Texas). Instr., Pure Math., Univ. of Texas, Austin, Tex. *504 W. 32d St.*
HOSKINS, Prof. L. M., C.E. (Wisconsin). Appl. Math., Stanford Univ., Palo Alto, Calif. *1240 Waverley St.*
HOUSEHOLDER, Asso. Prof. F. C., A.B. (Kansas). N. Dakota Agric. Coll., Fargo, N. D. *1129 College St.*
HOWARD, Prof. C. M., E.Mines (Ala. Poly. Inst.). Texas Woman's Coll., Fort Worth, Tex. *3318 Ave. I.*
HOWE, Asst. Prof. ANNA M., Ph.D. Sophie Newcomb Memorial Coll., New Orleans, La.
HOWE, Dean H. A., Sc.D. (Boston). Dean of Coll. of Liberal Arts and Prof. of Astr., Univ. of Denver, Denver, Colo.
HOWIE, Prof. J. M., A.M. (Columbia). Nebraska Wesleyan Univ., University Place, Nebr. *420 E. 14th St.*
HOWLAND, Prof. L. A., Ph.D. (Munich). Fisk Prof. of Math., Wesleyan Univ., Middletown, Conn. *34 Howe Ave.*

HUBBS, H. N., B.E. (Union). Instr., Hobart Coll., Geneva, N. Y. *72 Hamilton St.*
HUBERT, Asst. Prof. W. G., Sc.D. (New York Univ.). Coll. of the City of New York. *125 Lee Ave., Yonkers, N. Y.*
HUFFER, Asst. Prof. R. C., A.M. (Illinois). Math. and Astr., Beloit Coll., Beloit, Wis. *1108 Central Ave.*
HUGHES, H. K., A.M. (Iowa). Instr., Univ. of Iowa, Iowa City, Ia. *24 Physics Bldg.*
HUGHES, Asst. Prof. JEWELL C., Ph.D. (Chicago). Univ. of Arkansas, Fayetteville, Ark.
HULBURT, Prof. L. S., Ph.D. (Johns Hopkins). Johns Hopkins Univ., Baltimore, Md.
HULL, Prof. DANIEL, M.S. (Notre Dame). Univ. of Notre Dame, Notre Dame, Ind.
HULSE, J. F., A.B. (Coll. of Ozarks). *705 W. 22½ St.*, Austin, Tex. (?)
HUME, Chancellor ALFRED, D.Sc. (Vanderbilt). Univ. of Mississippi, University P. O., Miss.
HUNT, Asst. Prof. G. H., C.E. (Cornell). So. Branch, Univ. of California, Los Angeles, Calif. *855 N. Vermont Ave.*
HUNT, Asst. Prof. MILDRED, Ph.D. (Chicago). Illinois Wesleyan Univ., Bloomington, Ill. *1211 Fell Ave.*
HUNTER, Asst. Prof. H. E., M.S. (Illinois). State Teachers Coll., Pittsburg, Kans.
HUNTINGTON, A. H., A.M. (Columbia). Asst. Prin., Yeatman High School, St. Louis, Mo.
HUNTINGTON, Prof. E. V., Ph.D. (Strassburg). Mech., Harvard Univ., Cambridge, Mass. *27 Everett St.*
HUNTLEY, H. B., A.B. (Harvard). Instr., New Hampshire Coll. *770 Central Ave., Dover, N. H.*
HURWITZ, Prof. W. A., Ph.D. (Göttingen). Cornell Univ., Ithaca, N. Y. *White Hall 8.*
HUSSEY, L. W., A.M. (Harvard). Instr., Union Coll., Schenectady, N. Y.
HUTCHINS, MABEL, A.B. (Blue Mountain). Head of Dept., Blue Mountain Coll., Blue Mountain, Miss.
HUTCHINSON, Prof. C. A., A.M. (Wittenberg). Eng. Math., Univ. of Colorado, Boulder, Colo. *837 15th St.*
HYDE, Asst. Prof. EMMA, A.M. (Chicago). State Agric. Coll., Manhattan, Kans.

ILOFF, P. M., A.M. (Michigan). Instr., State Teachers Coll., Chico, Calif. *315 Orient St.*
INGELS, NELLE L., A.M. (Illinois). Interstate Commerce Commission, Washington, D. C. *E.-F. Bldg. Govt. Hotels.*
INGOLD, Asso. Prof. LOUIS, Ph.D. (Chicago). Univ. of Missouri, Columbia, Mo. *206 Thilly Ave. On leave 1924–25.*
INGRAHAM, M. H., A.B. (Cornell). Instr., Univ. of Wisconsin, Madison, Wis.
IRWIN, Asso. Prof. FRANK, Ph.D. (Harvard). Univ. of California, Berkeley, Calif. *2921 Regent St.*
ISAACS, Prof. C. A., A.M. (Columbia). State Coll. of Washington, Pullman, Wash.

JABLONOWER, JOSEPH, Pd.M. (N. Y. Univ.). Teacher, Ethical Culture School, New York, N. Y. *133 Central Pk. W.*
JACKSON, Prof. DUNHAM, Ph.D. (Göttingen). Univ. of Minnesota, Minneapolis, Minn. *100 Folwell Hall.*
JACKSON, Prof. FRANCES H., A.M. (Columbia). Salem Coll., Winston-Salem, N. C.
JACKSON, Asso. Prof. J. B., A.M. (Columbia). Univ. of South Carolina, Columbia, S. C. *227 S. Waccamaw Ave.*
JACKSON, MARY F., B.S. (Nebraska). Teacher, High School, Lincoln, Nebr. *1546 E. St.*
JACKSON, ROSA L., A.B. (Western Coll.). Instr., Northwestern Univ., Evanston, Ill.
JACKSON, Prof. T. W., A.B. (Westminster). Jamestown Coll., Jamestown, N. D.
JACOBS, JESSIE M., Ph.D. (Illinois). Instr., Univ. of Texas, Austin, Tex.
JAEGER, C. G., A.B. (Missouri). Instr., Univ. of Missouri, Columbia, Mo. *Eng. Bldg.*
JAMES, Asst. Prof. GLENN, Ph.D. (Columbia). So. Branch, Univ. of California, Los Angeles, Calif.
JAMISON, Prof. G. H., A.M. (Chicago). State Teachers Coll., Kirksville, Mo. *Box 116.*
JANES, W. C., A.M. (Nebraska). Instr., State Agric. Coll., Manhattan, Kans.
JARRETT, ETHEL L., A.B. (Cornell). Chicago Latin School for Girls, Chicago, Ill. *1218 N. Dearborn St.*
JEFFERS, H. M., Ph.D. (California). Asst. Astron., Lick Observ., Mt. Hamilton, Calif.
JENISON, J. R., A.B. (Tarkio; Chicago). Tarkio Coll., Tarkio, Mo.
JERBERT, A. R., M.S. (Washington). Assoc., Univ. of Washington, Seattle, Wash. *707 E. 55th.*

JOFFE, S. A., M.S. (New York Univ.). Asst. Actuary, Mut. Life Ins. Co., New York, N. Y. *32 Nassau St.*

JOHN, Bro. F., A.M. (Catholic Univ.). La Salle Coll., Philadelphia, Pa. *1240 N. Broad St.*

JOHN, F. W., M.E. (Cornell). Instr., Washington Square Coll., New York Univ. *182 Nepperhan Ave., Yonkers, N. Y.*

JOHNSON, ABIGAIL E., B.S. (Columbia). Teacher, High School, Morristown, N. J. *Box 234. Morris Plains, N. J.*

JOHNSON, A. FRANCES, Ph.D. (Minnesota). Constantinople Coll., Constantinople, Turkey, *Box 39, Galata P. O.*

JOHNSON, Prof. B. F., A.B. (Central Coll.). State Normal School, Cape Girardeau, Mo.

JOHNSON, C. L., B.S. (Ore. Agric.). Head of Dept., Oregon Agric. Coll., Corvallis, Ore. *1001 Jefferson St.*

JOHNSON, Prof. E. N., A.M. (Drake), M.S. (Kansas). Butler Coll., Indianapolis, Ind. *304 Downey Ave.*

JOHNSON, M. F., C.E. (Mich. Agric.), M.S. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich. *R. D. 8.*

JOHNSON, MARIE M., M.S. (Iowa). Instr., Lake Forest Coll., Lake Forest, Ill. *Lois Durand Hall.*

JOHNSON, MYRA I., Ph.B. (Syracuse). Grad. Student, Columbia Univ., New York, N. Y. *58 W. 90th St.*

JOHNSON, Prof. R. A., Ph.D. (Harvard). Hamline Univ., St. Paul, Minn.

JOHNSON, Asst. Prof. R. P., A.M. (Virginia; Harvard). Carnegie Inst. of Tech., Pittsburgh, Pa. *2348 Sherbrook St., Squirrel Hill.*

JOHNSON, W. W. Instr. in Appl. Math., Y. M. C. A. Night Schools, Cleveland, Ohio. *1078 E. 99th St.*

JOHNSTON, F. E., A.M. (Illinois). Asst., Univ. of Illinois, Urbana, Ill. *1005 W. Nevada St.*

JOHNSTON, Asst. Prof. K. P., B.A., B.S. (Queen's). Queen's Univ., Kingston, Ont., Can. *Dept. of Math.*

JOHNSTON, Asst. Prof. L. S. Penn. State Coll., State College, Pa.

JOHNSTON, NELLIE M., A.B. (Kearney State Teachers Coll.). Prin., High School, Gibbon, Nebr.

JONES, AGNES A. Teacher, Norwin High School, Irwin, Pa.

JONES, B. W., A.M. (Harvard). Instr., Adelbert Coll., Western Reserve Univ., Cleveland, Ohio.

JONES, C. H., A.M. (Northwestern). New Trier Twp. High School, Kenilworth, Ill.

JONES, Prof. E. H., A.M. (Harvard). Southern Methodist Univ., Dallas, Tex.

JONES, G. S., B.S. (Ohio Univ.). Prin., High School, Harriettsville, Ohio.

JONES, J. H. With Allyn and Bacon, Chicago, Ill. *1006 S. Michigan Ave.*

JONES, Prof. J. S. W., D.Sc. (Washington Coll.). Washington Coll., Chestertown, Md. *404 Washington Ave.*

JONES, S. I., B.S. (N. Tex. Normal Coll.). Asst. Treas., Life and Casualty Ins. Co., Nashville, Tenn.

JORDAN, Asst. Prof. E. E., M.A. (Dalhousie). Univ. of British Columbia. *4195 15th Ave. W., Point Grey, B. C., Can.*

JORDAN, Asst. Prof. H. E., Ph.D. (Chicago). Univ. of Kansas, Lawrence, Kans. *1600 Kentucky St.*

JORDAN, M. F., A.M. (Maine). Instr. in Astr., Harvard Univ., Cambridge, Mass. *11 Jarvis St.*

JURDAK, Prof. M. H., M.A. (Amer. Univ. of Beirut). Amer. Coll. of Beirut, Beirut, Syria.

JUSTIN, E. M., M.S. (Case School). Instr., Case School of Appl. Sc., Cleveland, Ohio.

KARPINSKI, Prof. L. C., Ph.D. (Strassburg). Univ. of Michigan, Ann Arbor, Mich. *1315 Cambridge Rd.*

KARR, LOIS, A.M. (Wisconsin). Instr., Lindenwood Coll., St. Charles, Mo.

KASNER, Prof. EDWARD, Ph.D. (Columbia). Columbia Univ., New York, N. Y. *22 W. 119th St.*

KAZARINOFF, D. K., Ancien élève de Univ. de Moscow. Instr., Univ. of Michigan, Ann Arbor, Mich. *On leave 1924–25 for study in Europe.*

KEITH, G. W., B.A. (Toronto). Math. Master, Parkdale Colleg. Inst., Toronto, Can. *53 Thorold St.*

KELLOGG, Asso. Prof. O. D., Ph.D. (Göttingen). Harvard Univ., Cambridge 38, Mass. *20 Craigie St.*

KEMPNER, Asst. Prof. A. J., Ph.D. (Göttingen). Univ. of Illinois, Urbana, Ill. *909 W. Calif. Ave.*

KENDALL, Asst. Prof. CLARIBEL, Ph.D. (Chicago). Univ. of Colorado, Boulder, Colo. *1091 14th St.*

KENDRIGAN, Prof. J. T., B.S. (Maine). Northland Coll., Ashland, Wis.

KENNEDY, KATHRYN M., A.B. (Indiana State Normal). Normal Training High School, Terre Haute, Ind.

KENNELLY, Prof. A. E., Sc.D. (Pittsburgh; Toulouse). Elec. Eng., Harvard Univ. and Mass. Inst. of Tech., Cambridge, Mass. *Harvard Univ.*

KEULEGAN, G. H., A.M. (Ohio State). Asso. Physicist, Bur. of Standards, Washington, D. C. *Room 311, N. W. Bldg.*

KIEFER, Prof. E. C., M.S. (Michigan). James Millikin Univ., Decatur, Ill. *1299 W. Macon St.*

KIESS, Prof. H. A., A.M. (Albright). Albright Coll., Myerstown, Pa.

KILLEBREW, C. D., M.S. (Ala. Poly.). Alabama Poly. Tech., Auburn, Ala.

KIMBALL, Prof. R. S., A.M. (Brown). Prof. and Asst. to Prin., State Normal School, Worcester, Mass.

KIME, D. O., M.S. (Illinois). Western Union Coll., Le Mars, Ia.

KINDLE, Asso. Prof. J. H., A.M. (Ohio State). Univ. of Cincinnati, Cincinnati, Ohio.

KING, Prof. G. E., M.S. (Wisconsin). Iowa Wesleyan Coll., Mt. Pleasant, Ia.

KINGERY, Prof. D. N., A.M. Macalester Coll., St. Paul, Minn. *135 Amherst St.*

KINGSTON, Prof. H. R., Ph.D. (Chicago). Math. and Astr., Univ. of Western Ontario, London, Ont.

KINNEAR, JENNIE E., B.A. (Queen's). Teacher, High School, Dunnville, Ont. *Box 143, Port Colborne, Ont.*

KINNEY, J. M., Ph.D. (Chicago). Instr., Crane Jr. Coll., Chicago, Ill. *8058 Bennett St.*

KIRCHNER, Prof. W. H., B.S. (Worcester Poly.). Drawing and Descr. Geom., Univ. of Minnesota, Minneapolis, Minn. *208 Main Eng. Bldg.*

KIRK, E. B., B.S. (Louisville). Address unknown.

KIRKER, STELLA B., A.M. (Nebraska). Head of Dept., High School, Lincoln, Nebr. *636 S. 17th St.*

KLAUBER, L. M., A.B. (Stanford). San Diego Cons. Gas and Elec. Co., San Diego, Calif. *256 Front St.*

KLINE, Asst. Prof. J. R., Ph.D. (Penna.). Univ. of Pennsylvania, Philadelphia, Pa. *Box 6, College Hall.*

KNAPP, Prof. G. A., A.M. (Hamilton). Maryville Coll., Maryville, Tenn.

KNISELY, ALEXANDER, B.S. (Valparaiso). Columbia City, Ind.

KNORR, H. V., A.B. (Susquehanna Univ.). Head of Dept., Central Wesleyan Coll., Warrenton, Mo.

KNOX, J. J., B.S. (Chicago). Purdue Univ., W. LaFayette, Ind. *319 W. Thornell St.*

KOCH, E. H., Jr., B.S. in E.E. (Penna.). Brooklyn Tech. High School. *208 N. Maple Ave., East Orange, N. J.*

KONANTZ, Prof. EMMA L., A.M. (Ohio Wesleyan). Peking Univ., Kwei Chia Chiang, Peking, China.

KOVARIK, Asso. Prof. A. F., Ph.D. (Minnesota). Physics, Yale Univ., New Haven, Conn. *Sloane Lab.*

KRATHWOHL, Asso. Prof. W. C., Ph.D. (Chicago). Armour Inst. of Tech., Chicago, Ill.

KREINES, G. A., B.S. (New York Univ.). *1001 Lafayette Ave.*, Brooklyn, N. Y.

KRETH, DANIEL, C.E. Surveyor and Civil Engr., Wellman, Ia.

KRYLOFF, Prof. NICOLAS, Doctor of Math. *honoris causa* (Kieff.). U.S.S.R., Kieff, Ukraine. *Street Bolchaia Vladimirskaia N 54.*

KUHN, Prof. H. W., Ph.D. (Cornell). Ohio State Univ., Columbus, Ohio. *1145 Fairview Ave.*

KUNTE, HELEN L., A.M. (Columbia). Instr., Hunter Coll., New York, N. Y. *Park Ave. and 68th St.*

KURZIN, W. H., M.S. (Chicago). Teacher, Crane Jr. Coll., Chicago, Ill. *1631 W. Roosevelt Rd.*

LAMBERT, Prof. P. A., Head of Dept., Math. and Astr., Lehigh Univ., S. Bethlehem, Pa. *215 S. Center St.*

LAMBERT, W. D., A.M. (Harvard). Mathematician, U. S. Coast and Geod. Surv., Washington, D. C.

LAMPEN, Prof. A. E., A.M. (Michigan). Hope Coll., Holland, Mich. *86 E. 14th St.*

LAMPLAND, C. O., A.M. (Indiana). Astronomer, Lowell Observ., Flagstaff, Ariz.

LANDIS, Prof. W. W., A.M. (Dickinson). Dickinson Coll., Carlisle, Pa.
LANDRY, Prof. A. E., Ph.D. (Johns Hopkins). Catholic Univ. of America, Washington, D. C. *3624 13th St., Brookland, D. C.*
LANE, Asst. Prof. E. P., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.
LANGE, Prof. LUISE, Ph.D. (Göttingen). Oxford Coll., Oxford, Ohio.
LANGELLOTTI, F. S., A.B. (George Washington). Actuary, M. M. Dawson and Son, New York, N. Y. *57 Kenmare St.*
LANGMAN, HARRY, Ph.D. (Columbia). Prop., Solar Chem. Mfg. Co., Brooklyn, N. Y. *35 W. 61st St., New York, N. Y.*
LAREW, Prof. GILLIE A., Ph.D. (Chicago). Randolph-Macon Woman's Coll., Lynchburg, Va.
LARSON, Asst. Prof. OLGA, A.M. (Missouri). Fla. State Coll. for Women. *Box 84, Apopka, Fla.*
LASLEY, Prof. J. W., Jr., Ph.D. (Chicago). Pure Math., Univ. of North Carolina, Chapel Hill, N. C.
LASTRAPES, ODESSA, A.B. (Tulane). Teacher, High School, Napoleonville, La.
LATIMER, Asst. Prof. C. G., Ph.D. (Chicago). Tulane Univ., New Orleans, La.
LATSHAW, ELMER. *1839 N. 60th St.*, West Philadelphia, Pa.
LAURENTINE, Sister MARIE, A.B. (Trinity Coll., Washington). Instr., Emmanuel Coll., Boston, Mass.
LAVES, Asso. Prof. KURT, Ph.D. (Berlin). Univ. of Chicago, Chicago, Ill. *5611 Kenwood Ave.*
LAZOTT, H. E. A., A.B. (Boston Univ.). Instr., South High School, Worcester, Mass.
LEAVENS, D. H., A.M. (Yale). Yale in China, Changsha, China.
LEBEUF, Prof. A. V. Prof. of Astr. and Dir. of Observatory, Univ. of Besançon, Besançon, France.
LEE, G. FARLEY, A.M. (Columbia). Judson Coll., Marion, Ala.
LEFSCHETZ, Prof. SOLOMON, Ph.D. (Clark). Univ. of Kansas, Lawrence, Kans. *Visiting Professor Princeton Univ. 1924–25.*
LEHMAN, Prof. D. A., A.M. (Western Reserve). Math. and Astr., Goshen Coll., Goshen, Ind.
LEHMANN, C. H., M.E. (Cooper). Engineer, N. Y. Telephone Co. *1502 Pacific St., Brooklyn, N. Y.*
LEHMER, Prof. D. N., Ph.D. (Chicago). Univ. of California, Berkeley, Calif. *2736 Regent St.*
LEIB, Prof. D. D., Ph.D. (Johns Hopkins). Connecticut Coll., New London, Conn.
LE LACHEUR, E. A., A.B. (Valparaiso). Computer, U. S. Coast and Geod. Surv., Washington, D. C. *Riverdale, Md.*
LEMON, H. B., M.S. (Denison). Instr., Denison Univ., Granville, Ohio.
LENNES, Prof. N. J., Ph.D. (Chicago). Univ. of Montana, Missoula, Mont.
LEONARD, Prof. H. B., Ph.D. (Colorado). Univ. of Arizona, Tucson, Ariz.
LEONARD, KATHERINE, A.M. (Vermont). State Teachers Coll., Moorhead, Minn.
LESTER, Prof. O. C., Ph.D. (Yale). Physics, Univ. of Colorado, Boulder, Colo. *1061 Eleventh St.*
LE STOURGEON, Asst. Prof. ELIZABETH, Ph.D. (Chicago). Univ. of Kentucky, Lexington, Ky.
LEVÉILLÉ, Prof. ARTHUR, B.A. (London). Univ. of Montreal, Montreal, Can. *1072 Rachel East.*
LEVITA, MAURICE M., A.M. (Univ. of Pa.). Instr., Temple Univ., Philadelphia, Pa. *c/o P. Miller, 743 S. 60th St.*
LEWIS, Prof. ANNA D., Ph.D. (Carleton). Math. and Physics, Lake Erie Coll., Painesville, Ohio.
LEWIS, Asst. Prof. C. F., A.B. (Denver). State Agric. Coll., Manhattan, Kans.
LEWIS, Prof. FLORENCE P., Ph.D. (Hopkins). Goucher Coll., Baltimore, Md.
LIBBY, B. B., A.M. (Wisconsin). Lowell High School, San Francisco, Calif. *1350 27th Ave.*
LIBMAN, E. E., Ph.D. (Illinois). Instr., Univ. of Illinois, Urbana, Ill.
LIGHT, Prof. G. H., Ph.D. (Yale). Univ. of Colorado, Boulder, Colo. *801 Base Line.*
LINDEMANN, Prof. C. A., A.M. (Bucknell). Pure Math. and Secy. of Faculty, Bucknell Univ., Lewisburg, Pa.
LINDQUIST, Prof. THEODORE, Ph.D. (Chicago). Mich. State Normal Coll., Ypsilanti, Mich. *103 Elm St.*
LINDSEY, Asso. Prof. LOUIS, Ph.D. (Syracuse). Syracuse Univ., Syracuse, N. Y. *706 Allen St.*
LINEHAN, Prof. P. H., Ph.D. (Columbia). Coll. of the City of New York, New York, N. Y. *346 Convent Ave.*
LINFIELD, B. Z., Ph.D. (Harvard). Address unknown.
LING, Prof. G. H., Ph.D. (Columbia). Univ. of Saskatchewan, Saskatoon, Can.
LINTON, M. ALBERT, A.M. (Haverford). Vice-Pres., Actuary, The Provident Mut. Life Ins. Co., Philadelphia, Pa. *409 Chestnut St.*
LIVINGSTON, G. R., A.M. (California). Math. and Surveying, State Teachers Coll., San Diego, Calif.

LOCKE, L. L., A.M. (Grove City Coll.). Teacher, Brooklyn Tr. School for Teachers, Brooklyn, N. Y. *950 St. Johns Place.*
LOEWEN, Prof. O. B., A.M. (Kansas). Ottawa Univ., Ottawa, Kans.
LOGSDON, Mrs. MAYME I., Ph.D. (Chicago). Instr. in Math. and Dean in Colleges, Univ. of Chicago, Chicago, Ill.
LONG, Asst. Prof. FLORENCE, M.S. (Illinois). Earlham Coll., Earlham, Ind.
LONG, T. R., A.B. (Rochester). Asst., Univ. of Rochester, Rochester, N. Y. *75 Cypress St.*
LONG, Prof. W. F., A.B. (F. and M.). Franklin and Marshall Coll., Lancaster, Pa.
LONGLEY, Prof. W. R., Ph.D. (Chicago). Sheffield Scientific School, New Haven, Conn.
LORD, MARGARET, A.B. (Mt. Holyoke). *198 Jackson St.*, Lawrence, Mass.
LOUD, Prof. F. H., Ph.D. (Haverford). Astr., Emeritus, Colorado Coll., Colorado Springs, Colo. *P. O. Box 1006.*
LOVE, Asso. Prof. C. E., Ph.D. (Michigan). Univ. of Michigan, Ann Arbor, Mich.
LOVETT, Pres. E. O., Ph.D. (Virginia; Leipzig). Rice Inst., Houston, Tex.
LOVITT, Prof. W. V., Ph.D. (Chicago). Colorado Coll., Colorado Springs, Colo.
LOWRY, G. L., A.M. (Bucknell). Vice-Prin., High School, Olean, N. Y. *7 Edwards Ct.*
LUBBEN, R. S., A.B. (Texas). Instr. in Pure Math., Univ. of Texas, Austin, Texas. *708 W. 22½ St.*
LUBIN, C. I., Ch.E. (Cincinnati). Instr., Coll. of Eng., Univ. of Cincinnati, Cincinnati, Ohio. *Coll. of Eng., 5501 Holmes.*
LUBY, WILLIAM, A.M. (Kansas). Jr. Coll., Kansas City, Mo. *5501 Holmes.*
LUCK, Prof. J. J., Ph.D. (Virginia). Univ. of Virginia, University, Va. *Colonnade Club.*
LUFKIN, H. M., S.T.B. (Phila. Divinity School). Teacher, High School, Dunkirk, N. Y. *627 Washington Ave.*
LUNN, Prof. A. C., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.
LUNN, L. E. Supt. of Schools, Heron Lake, Minn.
LUTZ, Asst. Prof. JUNA M., A.M. (Chicago). Butler Coll., Indianapolis, Ind. *2524 College Ave.*
LYLE, G. A., B.S. (Hampden-Sidney). Instr., Lehigh Univ., Bethlehem, Pa. *527 Montclair Ave.*
LYMAN, Prof. E. A., A.B. (Michigan). Michigan State Normal Coll., Ypsilanti, Mich. *126 N. Washington St.*
LYNCH, INEZ D., A.B. (Southwestern). *811 Harvey Ave.*, Wellington, Kans.
LYONS, Asst. Prof. W. H., A.M. (Denver). State Agric. Coll., Manhattan, Kans. *358 N. 15th St.*
LYTLE, Asst. Prof. E. B., Ph.D. (Yale). Teaching of Math., Univ. of Illinois, Urbana, Ill. *903 Busey Ave.*

MACCREADIE, W. T., A.M. (Harvard). Instr., Cornell Univ., Ithaca, N. Y. *133 Linden Ave.*
MACDONALD, Prof. S. L., M.S. (Columbia). Math. and Astr., State Agric. Coll., Fort Collins, Colo.
MACDONALD, Prof. W. E., A.M. (Harvard). Canton Christian Coll., Canton, China.
MACDUFFEE, Asst. Prof. C. C., Ph.D. (Chicago). Ohio State Univ., Columbus, Ohio.
MACKENZIE, Prof. M. A., M.A. (Cantab.). Univ. of Toronto, Toronto, Ont., Can.
MACLEAN, Prof. N. B., Ph.D. (Chicago). Math. and Astr., Univ. of Manitoba, Winnipeg, Can.
MACMILLAN, Prof. W. D., Ph.D. (Chicago). Astr., Univ. of Chicago, Chicago, Ill.
MACNEISH, Asst. Prof. H. F., Ph.D. (Chicago). Coll. of the City of New York. *Scarsdale, N. Y.*
MACNUTT, Prof. BARRY, M.S. (Lehigh). Lehigh Univ., South Bethlehem, Pa. *Dept. of Physics.*
MCALISTER, Prof. E. H., A.M. (Oregon). Mech. and Astr., Univ. of Oregon, Eugene, Ore. *334 Pearl St.*
MCBANE, L. J., B.S. (Case School). *1598 Addison Rd.*, Cleveland, Ohio.
MCBEE, INA L., A.B. (Oklahoma). Teacher, Jr. High School, Duncan, Okla. *1015 Ash Ave.*
MCCAIN, Prof. GERTRUDE I., Ph.D. (Indiana). Westminster Coll., New Wilmington, Pa.
MCCARTHY, E. D., A.B. (Cornell). Instr., Penn. State Coll., State College, Pa. *331 W. College Ave.*
MCCARTHY, J. J., B.S. (Tufts). Instr., New York Univ., New York, N. Y. *University Heights.*
MCCARTY, A. L., A.M. (California). Instr., Lowell High School, San Francisco, Calif. *2137 Parker St., Berkeley, Calif.*
MCCLELLAND, Prof. C. W., C.E. (Texas). Daniel Baker Coll., Brownwood, Tex. *Route 3, Box 62.*
MCCLENON, Prof. R. B., Ph.D. (Yale). Grinnell Coll., Grinnell, Ia. *1512 4th Ave.*

McCormick, Prof. Clarence. Southwestern State Teachers Coll., Weatherford, Okla.
McCoy, W. C., A.B. (Ohio State). Grad. Asst., Ohio State Univ., Columbus, Ohio. *130 E. Lakeview Ave.*
McCue, Dean M. J., C.E., M.S. (Notre Dame). Coll. of Eng., Univ. of Notre Dame, Notre Dame, Ind.
McDill, Prof. R. M., A.M. (Indiana). Hastings Coll., Hastings, Nebr. *119 E. 7th St.*
McEwen, Asst. Prof. G. F., Ph.D. (Stanford). Oceanographer, Scripps Inst., Univ. of California. *Box 68, La Jolla, Calif.*
McEwen, W. H., A.M. (Minnesota). Asst., Univ. of Minnesota, Minneapolis, Minn. *126 Folwell.*
McFarland, Dora, A.B. (Monmouth). Instr., Univ. of Oklahoma, Norman, Okla.
McFarland, Elsie J., Ph.D. (California). Instr., Univ. of California, Berkeley, Calif. *200 Panoramic Way.*
McGavock, Martha P., A.M. (Chicago). Rockford Coll., Rockford, Ill.
McGaw, Prof. F. M., A.M. (Phila.). Math. and Eng. Drawg., Cornell Coll., Mt. Vernon, Ia.
McGilvray, Hugh, A.B. (Oklahoma). *734 De Barr,* Norman, Okla.
McGuire, Mrs. Margaret H., B.S. (Minnesota). Instr., Univ. High School, Minneapolis, Minn.
McKee, J. R. Instructing Federal Students, College Station, Tex.
McKelvey, Asso. Prof. J. V., Ph.D. (Cornell). State Coll., Ames, Ia. *2117 Graeber St.*
McKelvey, Martha M. (Mrs. J. V.), M.S. (Iowa). Instr., State Coll., Ames, Ia. *2117 Graeber St.*
McKinney, Prof. T. E., Ph.D. (Chicago). Math. and Astr., Univ. of South Dakota, Vermilion, S. D. *222 N. Univ. St.*
McKnight, Nannie J., A.M. (George Washington). Teacher, Western High School, Washington, D. C. *3037 Dumbarton Ave.*
McLatchey, Nina M., A.M. (Kansas). Teacher, High School, Topeka, Kans. *917 Morris Ave.*
McLaury, Prof. H. L., A.M. (Harvard). State School of Mines, Rapid City, S. D.
McMackin, F. J., Ph.D. (Columbia). Instr., Cooper Union Inst. of Tech., New York, N. Y. *49 Vreeland Ave., Rutherford, N. J.*
McMichael, Ellen M., A.B. (Rockford). Head of Dept., Blackburn Coll., Carlinville, Ill.
McMillan, Mary B., A.M. (Wisconsin). Head of Dept., State Normal School, River Falls, Wis.
McNatt, J. Q. Div. Engr., Colo. Fuel and Iron Co., Canon City, Colo.
McNelly, Sister Mary of the Presentation, A.M. (Catholic Univ.). Prof., Our Lady of the Lake Coll., San Antonio, Tex.
McSweeny, Prof. A. A., A.M. (Montana). Head of Dept., John Tarleton Agric. Coll., Stephenville, Tex.
Maddox, A. C., A.M. (Columbia). La. State Normal Coll., Natchitoches, La.
Mahoney, J. O., M.S. (Vanderbilt). Dir., Dallas Evening Schools, Dallas, Tex.
Maizlish, Israel, Ph.D. (Minnesota). *53 Warren St.,* Lynn, Mass.
Mallory, V. S., A.M. (Columbia). Head of Dept., High School, East Orange, N. J. *351 Springdale Ave.*
Manchester, Prof. R. E., A.M. (Michigan). State Teachers Coll., Kent, Ohio.
Mangold, Prof. M. Cecilia, M.S. (Trinity Coll., Washington). Trinity Coll., Washington, D. C.
Manning, Dr. H. M. U. S. Public Health Service, Astoria, Ore.
Manning, Asso. Prof. H. P., Ph.D. (Johns Hopkins). Retired. Brown Univ., Providence, R. I.
Manning, J. A., A.B. (Louisiana). Teacher of Science, High School, Vivian, La. *Box 311.*
Manson, Prof. E. S., M.S. (Mass. Inst. of Tech.). Astr., Ohio State Univ., Columbus, Ohio.
Many, Asst. Prof. Anna E., A.M. (Tulane). Sophie Newcomb Memorial Coll., New Orleans, La. *1325 Henry Clay Ave.*
Markley, Prof. J. L., Ph.D. (Harvard). Univ. of Michigan, Ann Arbor, Mich. *1816 Geddes.*
Marm, Anna, A.M. (Univ. of Kansas). Instr., Univ. of Kansas, Lawrence, Kans. *741 N. 2d St., Lindsborg, Kans.*
Marriott, Asst. Prof. R. W., Ph.D. (Pennsylvania). Swarthmore Coll., Swarthmore, Pa.
Marsh, H. B., A.M. (Amherst). Head of Dept., Tech. High School, Springfield, Mass. *Technical High School.*
Marshall, Frances L., B.S. (Ottawa Univ.). Teacher, High School, Iola, Kans. *520 E. Jackson.*
Marshall, Prof. William, Ph.D. (Zürich). Purdue Univ., LaFayette, Ind.
Marthakis, P. S., M.S. (Univ. of Utah). West High School, Salt Lake City, Utah.
Martin, Prof. A. V., A.B. (Hampden-Sidney). Presbyterian Coll. of S. C., Clinton, S. C.
Martin, Asso. Prof. Emilie N., Ph.D. (Bryn Mawr). Mount Holyoke Coll., South Hadley, Mass.

MARTINSON, E. P., A.M. (Nebraska). Instr., Colo. School of Mines, Golden, Colo.
MARTYN, W. J., M.A. (New Zealand). Mathematical Master, Otago Boys High School, Dunedin, New Zealand.
MASKELL, Sister BLANCHE MARIE, A.M. (Fordham). Coll. of St. Elizabeth, Convent Station, N. J.
MASON, Asso. Prof. T. E., Ph.D. (Ind.). Purdue Univ., LaFayette, Ind. *103 Waldron St.*
MATHESON, Prof. J., M.A. (Queen's Univ.). Queen's Univ., Kingston, Ont., Canada.
MATHEWS, JANE H., B.S. (Columbia). Teacher, High School, Pittsburgh, Pa. *5456 Penn Ave., E.E.*
MATHEWS, R. M., A.B. (Butler). Grad. Student, Univ. of Illinois, Urbana, Ill. *602 W. California St.*
MATHEWSON, Asst. Prof. L. C., Ph.D. (Illinois). Dartmouth Coll., Hanover, N. H.
MATTESON, Asso. Prof. JANE L., Michigan State Normal Coll., Ypsilanti, Mich. *318 Cross St.*
MAULDIN, Prof. JANICE, A.M. (Chicago). Gulf Park Coll., Gulfport, Miss.
MAURUS, Prof. E. J., M.S. (Notre Dame). Univ. of Notre Dame, Notre Dame, Ind. *L. B. 75.*
MAY, Prof. LIDA B., A.M. (Texas). Kidd-Key Coll., Sherman, Texas.
MAYER, Asst. Prof. E. S., C.E. (Rensselaer). U. S. Naval Acad., Annapolis, Md.
MAYNE, Mrs. GRACE K., A.B. (Texas). Teacher, High School, Austin, Texas. *1707 Brackenridge.*
MEADOR, Prof. G. E., A.M. (Oklahoma). Oklahoma City Coll., Oklahoma City, Okla. *1538 W. 24th.*
MEDER, A. E., Jr., A.M. (Columbia). Asst., Columbia Univ., New York, N. Y.
MEEK, Prof. H. B., A.M. (Maine). Hotel Administration, Cornell Univ., Ithaca, N. Y.
MEEK, KATE M. High School, South Pasadena, Calif.
MELVILLE, Asso. Prof. C. E., A.B. (Northwestern). Clark Univ., Worcester, Mass.
MENDENHALL, GERTRUDE W., B.S. (Wellesley). State Normal Coll., Greensboro, N. C.
MENDENHALL, Pres. W. O., Ph.D. (Michigan). Friends Univ., Wichita, Kans.
MENSENKAMP, L. E., A.M. (Illinois). Instr., High School, Freeport, Ill. *521 W. Cottonwood St.*
MERCEDES, Sister, M.S. (Notre Dame). Instr., Mary Manse Coll., Toledo, Ohio. *Ursuline Rd.*
MERGENDAHL, T. E., M.S. (Tufts). Tufts Coll., Tufts College, Mass.
MERRILL, Prof. A. S., Ph.D. (Chicago). Univ. of Montana, Missoula, Mont. *541 Beckwith Ave.*
MERRILL, Prof. HELEN A., Ph.D. (Yale). Wellesley Coll., Wellesley, Mass.
MERRILL, J. E., A.B. (Boston Univ.). Instr., Case School of Appl. Sc., Cleveland, Ohio. *Dept. of Math.*
MERRIMAN, MANSFIELD, Ph.D. (Yale). Consulting Engineer, New York, N. Y. *1071 Madison Ave.*
MERRISS, A. A. *913 East 23d St. North*, Portland, Ore.
MESSICK, Prof. J. F., Ph.D. (Johns Hopkins). Emory Univ., Emory University, Ga.
MEYER, J. B., M.S. (Purdue). Head of Dept., State Teachers Coll., Valley City, N. D. *814 5th Ave.*
MEYER, Rev. J. H., S.J. *140 Bavonne St.*, New Orleans, La.
MICHAL, A. D., Ph.D. (Rice). Instr., Rice Inst., Houston, Texas.
MICKELSON, E. L., A.M. (Minnesota). Hamline Univ., St. Paul, Minn.
MICKLE, M. KATHERINE, A.B. (Alabama). Roanoke, Ala.
MIKAMI, YOSHIO. Research Assoc., Imperial Acad., Tokyo, Japan. *29, Uyeno Sakuragicho.*
MIKESH, J. S. Instr., Sheffield Sc. School, Yale Univ., New Haven, Conn. *1072 Yale Sta.*
MILES, Asst. Prof. E. J., Ph.D. (Chicago). Yale Univ., New Haven, Conn. *87 Marvel Rd.*
MILLER, A. L., Ph.D. (Harvard). *25 Clinton Rd.*, Brookline 46, Mass.
MILLER, Prof. BESSIE I., Ph.D. (Johns Hopkins). Rockford Coll., Rockford, Ill.
MILLER, E. B. Instr., Univ. of Wisconsin, Madison, Wis.
MILLER, Prof. G. A., Ph.D. (Cumberland). Univ. of Illinois, Urbana, Ill. *1203 W. Illinois St.*
MILLER, Prof. I. L., A.M. (Indiana). State Coll. of A. and M. Arts, Brookings, S. D. *811 12th Ave.*
MILLER, Prof. J. A., Ph.D. (Chicago). Swarthmore Coll., Swarthmore, Pa.
MILLER, Prof. J. S., Sc.D. (Virginia). Emory and Henry Coll., Emory, Va.
MILLER, Asso. Prof. NORMAN, Ph.D. (Harvard). Queen's Univ., Kingston, Ont., Can.
MILLINGTON, Asst. Prof. H. G., C.E. (Rensselaer). Coll. of Eng., Univ. of Vermont, Burlington, Vt.
MILLS, Prof. C. N., A.M. (Indiana). South Dakota State Normal, Aberdeen, S. D.
MILNE, Prof. W. E., Ph.D. (Harvard). Univ. of Oregon, Eugene, Ore.
MILNE, T. H., M.A. (Toronto). Instr., Univ. of Buffalo, Buffalo, N. Y.

MINISTER, J. E., B.S. (Ohio State Univ.). Chauncey, Ohio.
MIRICK, G. R., A.M. (Michigan). Teacher, Lincoln School, Teachers Coll., New York, N. Y. *425 W. 123d St.*
MISER, Asso. Prof. W. L., Ph.D. (Chicago). Armour Inst. of Tech., Chicago, Ill.
MITCHELL, Prof. B. E., Ph.D. (Columbia). Millsaps Coll., Jackson, Miss. *727 Arlington Ave.*
MITCHELL, Mrs. DECIMA E., M.A. *Box 1550*, Univ. Station, Austin, Tex.
MITCHELL, FANNIE S., A.B. (N. C. Coll. for Women). Teacher, High School, Greensboro, N. C. *115 W. Bessemer Ave.*
MITCHELL, Prof. H. B., A.M. (Columbia). Columbia Univ., New York, N. Y. *80 Washington Square.*
MITCHELL, Prof. U. G., Ph.D. (Princeton). Univ. of Kansas, Lawrence, Kans. *1313 Mass. St.*
MITRA, Prof. N. B., M.A. (Calcutta Univ.). Ewing Christian Coll., Allahabad City, U. P., India.
MIZE, CLARA P., B.S. (Peabody). *Box 432*, Pulaski, Tenn.
MODE, E. B., A.M. (Harvard). Instr., Boston Univ., Boston, Mass.
MODESITT, R. L., A.M. (Indiana). Teacher, Eastern Illinois State Teachers Coll., Charleston, Ill.
MOLINA, E. C. Tel. Engr., Tel. and Tel. Co., 15 Dey St., New York, N. Y.
MOODY, Prof. W. A., A.M. (Bowdoin). Bowdoin Coll., Brunswick, Me. *60 Federal St.*
MOON, HELEN, M.S. (Iowa). Actuarial Dept., Central Life Ins. Co., Des Moines, Iowa.
MOONEY, MARY S. Dean of Women and Prof., Henderson-Brown Coll., Arkadelphia, Ark.
MOORE, Asso. Prof. C. L. E., Ph.D. (Cornell). Mass. Inst. of Tech., Cambridge, Mass.
MOORE, Prof. C. N., Ph.D. (Harvard). Univ. of Cincinnati, Cincinnati, Ohio. *219 Woolper Ave.*
MOORE, Prof. E. H., Ph.D. (Yale). Univ. of Chicago, Chicago, Ill. *Del Prado Hotel.*
MOORE, Asso. Prof. F. C., A.B. (Dartmouth). Mass. Agric. Coll., Amherst, Mass.
MOORE, Prof. G. E., M.S. (Illinois). Hanover Coll., Hanover, Ind.
MOORE, L. T., B.S. (Emory). Asst., Johns Hopkins Univ., Baltimore, Md. *3009 Guilford Ave.*
MOORE, Prof. L. W., C.E. (Cornell). Math. and Astr., Albany Coll., Albany, Ore. *528 W. 9th St.*
MOORE, Prof. R. L., Ph.D. (Chicago). Pure Math., Univ. of Texas, Austin, Texas.
MOORE, Prof. W. A., A.M. (Chicago). Birmingham Southern Coll., Birmingham, Ala.
MOOTS, E. E., M.S. (Maine), C.E. (Wisconsin). Head of Dept., Cornell Coll., Mt. Vernon, Iowa. *824 Summit Ave.*
MORENO, Prof. H. C., Ph.D. (Clark). Applied Math., Stanford University, Cal.
MORENUS, Prof. EUGENIE M., Ph.D. (Columbia). Sweet Briar Coll., Sweet Briar, Va.
MORGAN, F. M., Ph.D. (Cornell). Asst. Dir., Clark School, Hanover, N. H.
MORIARTY, M. M. S., A.B. (Holy Cross). Instr., Holyoke High School, Holyoke, Mass.
MORITZ, Prof. R. E., Ph.D. (Strassburg). Univ. of Washington, Seattle, Wash.
MORLEY, Prof. FRANK, A.M., Sc.D. (Cambridge, Eng.). Johns Hopkins Univ., Baltimore, Md. *2026 Park Ave.*
MORLEY, Prof. R. K., Ph.D. (Clark). Worcester Poly. Inst., Worcester, Mass. *43 Laconia Rd.*
MORRIL, J. S., A.B. (Southwestern). *607 W. Illinois St.*, Urbana, Ill.
MORRIS, Prof. C. C., A.M. (Harvard). Ohio State Univ., Columbus, Ohio.
MORRIS, Asso. Prof. F. R., Ph.D. (California). State Coll., Fresno, Calif.
MORRIS, MAX, A.M. (Harvard). Instr., Case School of Appl. Sc., Cleveland, Ohio.
MORRIS, Prof. RICHARD, Ph.D. (Cornell). Rutgers Coll., New Brunswick, N. J. *12 Johnson St.*
MORRISSY, M. E., A.M. (Dubuque). Instr., Marquette Univ., Milwaukee, Wis. *103 31st St.*
MORROW, E. B., A.M. (Princeton). Asst. Headmaster, Gilman Country School, Roland Park, Md.
MORSE, Prof. D. S., Ph.D. (Cornell). Union Coll., Schenectady, N. Y. *1508 Nott St.*
MORSE, Asst. Prof. MARSTON, Ph.D. (Harvard). Cornell Univ., Ithaca, N. Y. *410 Stewart Ave.*
MORTIMORE, R. H., A.B. (Iowa). Instr., Graceland Coll., Lamoni, Ia.
MORTON, Asso. Prof. A. B., A.M. (Brown). Pure Math., Georgia School of Tech., Atlanta, Ga.
MOSSMAN, THIRZA A., A.B. (Nebraska). Instr., State Agric. Coll., Manhattan, Kans.
MOULTON, Prof. E. J., Ph.D. (Chicago). Northwestern Univ., Evanston, Ill. *909 Colfax St.*
MOULTON, Prof. F. R., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.
MOURAD, Lieut. Commander SALIH (Turkish Navy). Head of Dept. of Math., Naval Coll., Prof. of Physics, Civil Eng. Coll., Constantinople, Turkey. *Naval College, Halki, Constantinople.*
MOURHESS, C. A. *1413 Ames Pl., N.E.*, Washington, D. C.
MOUZON, E. D., Jr., A.B. (Southern Meth. Univ.). Asst., Univ. of Illinois, Urbana, Ill. *Dept. of Math.*

MUEHLMAN, Rev. PAUL, A.M. (St. Louis Univ.). Instr., Loyola Univ., Chicago, Ill. *Sheridan Rd. and Loyola Ave.*

MUIR, Sir THOMAS, D.Sc. (Univ. of Cape Town). Late Supt.-Genl. of Educ., Cape Colony. *Elmcote, Sandown Rd., Rondebosch, S. Afr.*

MULLEMEISTER, Miss HERMANCE, Ph.D. (Utrecht, Holland). Instr., Univ. of Washington, Seattle, Wash. *1900 E. 68th St.*

MULLEN, Mrs. S. W., B.S. (Michigan). N. E. High School, Detroit, Mich. *286 E. Grand Blvd.*

MULLINS, Asso. Prof. G. W., Ph.D. (Columbia). Barnard Coll., Columbia Univ., New York, N. Y. *Barnard Coll., 119th and Broadway.*

MUNROE, FLORENCE L., A.B. (Wellesley). Teacher, High School, Northampton, Mass. *5 Franklin St.*

MURNAGHAN, Asso. Prof. F. D., Ph.D. (Johns Hopkins). Johns Hopkins Univ., Baltimore, Md. *702 Evesham Ave., Govans, Baltimore.*

MURRAY, Prof. D. A., Ph.D. (Johns Hopkins). Appl. Math., McGill Univ., Montreal, Can.

MURRAY, Asst. Prof. F. H., Ph.D. (Harvard). Dalhousie Univ., Halifax, N. S.

MUSSELMAN, Asso. J. R., Ph.D. (Johns Hopkins). Johns Hopkins Univ., Baltimore, Md.

MYERS, Prof. G. W., Ph.D. (Munich). Teaching of Math. and Astr., Univ. of Chicago, Chicago, Ill. *6639 Kimbark Ave., Jackson Pk. Sta.*

MYERS, Prof. H. S., A.M. (Chicago). Southwestern Coll., Winfield, Kans.

NASSAU, Asso. Prof. J. J., Ph.D. (Syracuse). Math. and Astr., Case School of Appl. Sc., Cleveland, Ohio.

NAUER, A. R. Mech. Engr., 300 Lynch St., St. Louis, Mo.

NEFF, Prof. I. F., M.S. (Drake; Chicago). Drake Univ., Des Moines, Ia. *2801 Brattleboro Ave.*

NEIKIRK, Asst. Prof. L. I., Ph.D. (Penna.). Univ. of Washington, Seattle, Wash. *4723 21st Ave., N.E.*

NELSON, Asst. Prof. A. L., Ph.D. (Chicago). Univ. of Michigan, Ann Arbor, Mich. *1208 Olivia Ave.*

NELSON, C. A., Ph.D. (Chicago). Asso., Johns Hopkins Univ., Baltimore, Md.

NELSON, W. A., A.B. (Texas). In charge of Math., High School, Waco, Tex. *1808 Avondale Ave.*

NELSON, Asst. Prof. W. K., M.S. (Colorado). Eng. Math., Univ. of Colorado, Boulder, Colo. *925 Grandview Ave.*

NEWELL, M. J., A.M. (Michigan). Teacher, High School, Evanston, Ill. *2017 Sherman Ave.*

NEWKIRK, B. L., Ph.D. (Munich). Research Dept., Genl. Elec. Co., Schenectady, N. Y.

NEWLIN, Prof. R. L., M.S. (Chicago). Guilford Coll., Guilford College, N. C.

NEWLIN, Prof. W. J., A.M. (Amherst; Harvard). Amherst Coll., Amherst, Mass.

NEWSON, Prof. MARY W., Ph.D. (Göttingen). Eureka Coll., Eureka, Ill.

NICHOLS, Prof. I. C., Ph.D. (Michigan). Appl. Math., La. State Univ., Baton Rouge, La.

NICOLET, JUSTIN, C.E. *8723 Parnell Ave.,* Auburn Pk. Sta., Chicago, Ill.

NOBLE, Prof. C. A., Ph.D. (Göttingen). Univ. of California, Berkeley, Calif. *2224 Piedmont Ave.*

NORDGAARD, Prof. M. A., Ph.D. (Columbia). St. Olaf Coll., Northfield, Minn. *1406 St. Olaf Ave.*

NORTON, Asso. Prof. ADA A., Ph.M. (Michigan). Michigan State Normal Coll., Ypsilanti, Mich. *510 Pearl St.*

NORWOOD, C. E., A.B. (Boston). *2 Greenwood Sq.,* Hyde Park, Mass.

NYBERG, J. A., M.S. (Chicago). Instr., Hyde Park High School, Chicago, Ill. *1039 E. Marquette Rd.*

ODELL, LETITIA R., A.M. (Denver). Teacher, North High School, Denver, Colo.

OGLESBY, Asso. Prof. E. J., A.M. (Virginia). New York Univ. *202 S. 18th St., Flushing, N. Y.*

OLDS, Pres. G. D., A.M., LL.D. (Rochester). Amherst Coll., Amherst, Mass. *3 Orchard St.*

OLSEN, LEANDER, B.S. (Utah Agric.). Teacher, High School, Moroni, Utah.

OLSON, EMMA J., A.B. (S. Dak.). *220 N. Grant St.,* Canton, S. D.

OLSON, Asst. Prof. H. L., Ph.D. (Chicago). Michigan Agric. Coll., East Lansing, Mich. *214 Albert Ave.*

O'NEILL, Prof. E. J., A.M. (Catholic Univ.). Mount St. Charles High School, Helena, Mont.

ONO, YEISUKE. *10, Kobinata Dai-Machi,* Koishikawa, Tokyo, Japan.

OPP, J. E., B.S. (Nebraska). Asst. Instr., Univ. of Nebraska, Lincoln, Nebr. *2736 F St.*

O'QUINN, R. L., A.B. (Louisiana). Instr., La. State Univ., Baton Rouge, La.

ORDWAY, FRANK, B.S. (Ala. Poly.). Head of Dept., Phillips High School, Birmingham, Ala.

OSBORN, JESSE, Ph.D. (Cornell). Harris Teachers Coll., St. Louis, Mo.
OSBORNE, Prof. G. A. Emeritus, Mass. Inst. of Tech. *249 Berkeley St., Boston, Mass.*
OSGOOD, Prof. W. F., Ph.D. (Erlangen). Harvard Univ., Cambridge, Mass. *74 Avon Hill St.*
O'SHAUGHNESSY, Prof. LOUIS, Ph.D. (Penna.). Appl. Math., Virginia Poly. Inst., Blacksburg, Va. *Box 177.*
OSTERHOUT, A. W., A.B. (Cotner). Cotner Coll., Bethany, Nebr. *550 E. Midway.*
OVERMAN, Prof. J. R., A.M. (Columbia). State Normal Coll., Bowling Green, Ohio.
OVERSTREET, P. C., A.B. (Asbury). Math. and Physics, John Fletcher Coll., University Park, Ia.
OWENS, Asst. Prof. F. W., Ph.D. (Chicago). Cornell Univ., Ithaca, N. Y. *110 Westbourne Lane.*
OWENS, HELEN B. (Mrs. F. W.), Ph.D. (Cornell). Private Teacher, Ithaca, N. Y. *110 Westbourne Lane.*

PAASWELL, GEORGE, C.E. (Cornell). Consulting Engr., New York, N. Y. *441 E. Tremont Ave.*
PACKARD, Asst. Prof. A. A. St. Stephen's Coll., Annandale-on-Hudson, N. Y.
PACKER, MARGARET C., A.M. (Brown). Instr., Math. and Physics, Hood Coll., Frederick, Md.
PALMER, Asso. Prof. C. I., A.B. (Michigan). Armour Inst. of Tech., Chicago, Ill. *6440 Greenwood Ave.*
PALMER, E. S., Ph.B. (Yale). Ballistic Engr., Winchester Repeating Arms Co., New Haven, Conn. *494 Whitney Ave.*
PALMIÉ, Prof. ANNA H., Ph.B. (Cornell). Coll. for Women, Western Reserve Univ., Cleveland, Ohio. *On leave 1924–25.*
PANCOAST, Prof. J. W., B.S. (Swarthmore). Guilford Coll., Guilford College, N. C.
PARKER, Prof. W. P., A.B. (Davidson). Union Christian Coll., Pyengyang, Korea, Japan.
PARSON, S. F. Head of Dept., No. Ill. State Normal School, De Kalb, Ill. *305 College Ave.*
PARSONS, F. W., B.S. in E.E. (Ohio Northern). Franklin High School, Los Angeles, Calif. *5353 Abbott Pl.*
PARTRIDGE, E. A., Ph.D. (Penna.). Science, West Phila. High School for Boys, Philadelphia, Pa. *48th and Walnut Sts.*
PASSANO, Asso. Prof. L. M., A.B. (Johns Hopkins). Mass. Inst. of Tech., Cambridge, Mass.
PATTEN, Prof. W. E., C.E. (Cornell). Hydraulic Eng., Tangshan Univ., Tangshan, N. China.
PATTENGILL, Asso. Prof. E. A., B.S. (Cornell). State Coll., Ames, Ia. *504 Ash St.*
PATTERSON, Prof. W. J., M.A. (Queen's). Western Univ., London, Ont. *856 Hellmuth Ave.*
PATTILLO, Dean N. A., Ph.D. (Johns Hopkins). Randolph-Macon Woman's Coll., Lynchburg, Va.
PAULA, Sister MARY, M.S. (Notre Dame). Prof., St. Mary's Coll., Monroe, Mich.
PEARCE, Prof. W. H., A.M. (Michigan). Central State Normal School, Mt. Pleasant, Mich.
PECK, Prof. J. N., B.S. (Fremont). Berea Coll. Acad., Berea, Ky.
PEDERSEN, Asso. Prof. F. M., Sc.D. (New York Univ.). Coll. of the City of New York, New York, N. Y. *452 W. 144th St.*
PEED, Prof. M. T. Emory Coll., Emory University, Ga.
PEGRAM, Prof. ANNIE M., A.M. (Trinity). Greensboro Coll., Greensboro, N. C.
PEHRSON, Asst. Prof. E. W. Univ. of Utah, Salt Lake City, Utah.
PELL, Mrs. ANNA J., Ph.D. (Chicago). Asso. Prof., Bryn Mawr Coll., Bryn Mawr, Pa.
PELLETIER, Prof. ARTHUR. Higher Alg., École Poly., Montreal, Can. *228 St. Denis St.*
PENN, S. S. *834 Hewitt Pl.*, New York, N. Y.
PEPPER, Mrs. PAULA H. (Mrs. H. C.), A.M. (Texas). Grad. Student, Univ. of Missouri, Columbia, Mo. *1209 Walnut St.*
PERKINS, F. W., A.M. (Harvard). Grad. Student, Harvard Univ., Cambridge 38, Mass. *249 Huron Ave.*
PERKINS, Prof. L. R., A.M. (Tufts). Middlebury Coll., Middlebury, Vt.
PERRY, EDITH M., A.M. (Oklahoma). American Boarding School, Lovetch, Bulgaria.
PERRY, R. D., B.S. (Southwest Tex. Tea. Coll.). Instr., Teachers Coll., Greeley, Colo.
PETERS, Prin. W. A., A.B. (La. State Univ.). High School, Zachary, La.
PETERSON, C. A. V. Electrician, Minneapolis, Minn. *4341 Minnehaha Ave.*
PETERSON, O. J., A.M. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich. *515 N. Division St.*
PETTERSEN, C. A., Ph.B. (Northwestern). Asst. Prin., Schurz High School, Chicago, Ill. *3922 Lowell Ave.*
PETTIT, Prof. H. P., Ph.D. (Illinois). Head of Dept., Illinois Wesleyan Univ., Bloomington, Ill.
PETTY, Pres. L. E., A.M. (Peabody). Silliman Coll., Clinton, La.

PHALEN, Asso. Prof. H. R., M.S. (Chicago). Armour Inst. of Tech., Chicago, Ill. *1656 Farwell Ave., Rogers Pk. Sta.*
PHELPS, Prof. E. B. U. S. Publ. Health Service, Ridgewood, N. J. *46 Phelps Rd.*
PHENIX, J. D., A.B. (Texas). Teacher, High School, Austin, Tex.
PHILLIPS, Prof. E. C., Ph.D. (Johns Hopkins). Math. and Astr., Woodstock Coll., Woodstock, Md.
PHILLIPS, Asso. Prof. H. B., Ph.D. (Johns Hopkins). Mass. Inst. of Tech., Cambridge, Mass.
PHILLIPS, Prin. H. M., B.S. (Syracuse; St. Lawrence). High School, De Kalb Junction, N. Y.
PHINNEY, W. L., B.S. (Dartmouth). Instr., Worcester Poly. Inst., Worcester, Mass.
PIERCE, Prof. JESSE, M.S. (Chicago). Heidelberg Univ., Tiffin, Ohio.
PIERCE, Asst. Prof. T. A., Ph.D. (California). Univ. of Nebraska, Lincoln, Nebr. *1718 Pepper Ave.*
PIERSON, A. D., B.S. in Ed. (Missouri). Jr. Coll., Kansas City, Mo. *437 S. Denver.*
PI MU EPSILON FRATERNITY, Secy. of, Syracuse Univ., Syracuse, N. Y. *Box 13, Faculty Post-Office, Hall of Languages.*
PINCHERLE, Prof. SALVATORE, Sc.D. (Bologna). Univ. of Bologna, Bologna, Italy.
PLANT, Prof. L. C., M.S. (Chicago). Michigan Agric. Coll., East Lansing, Mich.
PLAPP, E. MARIE, M.S. (Chicago). Teacher, Marshall High School, Chicago, Ill. *4140 N. Keeler Ave.*
POBANZ, J. F., Ph.D. (California). Head of Dept., Jr. Coll., Modesta, Calif. *408 Magnolia Drive.*
POLLARD, H. S., M.S. (Iowa). Instr., Univ. of Wisconsin, Madison, Wis.
POOLER, L. G., A.M. (Columbia). Physics, Columbia Univ., New York, N. Y.
POOR, Asst. Prof. V. C., Ph.D. (Chicago). Univ. of Michigan, Ann Arbor, Mich. *930 Packard St.*
POPE, MARY G., Grad. Winthrop Coll. Instr., Winthrop Coll., Rock Hill, S. C.
PORITSKY, HILLEL, A.B. (Cornell). Instr., Cornell Univ., Ithaca, N. Y. *White Hall.*
PORTER, C. S., A.M. (Clark). Instr., Amherst Coll., Amherst, Mass. *Box 675.*
PORTER, Prof. HUGH, A.M. (Texas). N. Tex. State Normal Coll., Denton, Tex.
PORTER, Prof. M. B., Ph.D. (Harvard). Univ. of Texas, Austin, Tex. *310 E. 21st St.*
PORTER, T. I., A.B. (Missouri). Instr., Univ. of Omaha, Omaha, Nebr.
PORTER, W. R., B.S. (Carnegie Inst.). Instr., Mass. Agric. Coll., Amherst, Mass. *R. D. 1, Box 33.*
POST, E. L., Ph.D. (Columbia). *280 Manhattan Ave.*, New York, N. Y.
POUND, V. E., Ph.D. (Toronto). Instr., Univ. of Buffalo, Buffalo, N. Y. *Ridgeway, Ont.*
POUNDER, Asst. Prof. I. R., M.A. (Toronto). Univ. of Toronto, Toronto, Ont.
POWELSON, INEZ D., A.M. (California). Teacher, Bakersfield, Calif. *Colonial Apts., 1705 B St.*
PRESTON, AMY F., A.M. (Columbia). City Schools, Columbus, Ohio. *1445½ E. Main St.*
PRIDE, H. H., M.S. (N. Y. Univ.). Instr., New York Univ., New York, N. Y.
PRIESTER, Prof. G. C., M.S. (Minnesota). Math. and Mech., Univ. of Minnesota, Minneapolis, Minn. *Coll. of Eng. and Archit.*
PROSSER, JULIA P., A.M. (South Carolina). Head of Dept., St. Mary's School, Raleigh, N. C.
PUGH, V. S., B.S. (La. Poly.). Instr., La. Poly. Inst., Ruston, La.
PUTNAM, R. G., Ph.D. (Chicago). Instr., New York Univ., New York, N. Y. *86 W. 179th St.*
PUTNAM, Prof. T. M., Ph.D. (Chicago). Univ. of California, Berkeley, Calif.

QUARLES, LOUIS, A.B. (Michigan). Lawyer, Milwaukee, Wis. *Lines, Spooner and Quarles, 68 Wisconsin St.*

RAGSDALE, Asso. Prof. VIRGINIA, Ph.D. State Normal Coll., Greensboro, N. C. *Jamestown, N. C.*
RALLS, Mrs. EFFIE M., A.B. (Tri State Ind.). Head of Dept., High School, Coalgate, Okla.
RAMAGE, C. J., Ph.D. (Grove City). Lawyer, Saluda, S. C.
RAMANA-SASTRIN, V. V., Ph.D. Vedaraniam, Tanjore Dt., South India.
RAMBO, Asso. Prof. SUSAN M., Ph.D. (Michigan). Smith Coll., Northampton, Mass. *12 Barrett Pl.*
RAMLER, Prof. O. J., Ph.D. (Catholic Univ.). Catholic Univ., Washington, D. C. *12 Girard St., N. E.*
RAMSDELL, Prof. G. E., A.M. (Harvard). Bates Coll., Lewiston, Me. *40 Mountain Ave.*
RAMSEY, Prof. ARTHUR, A.B. (Grove City). Grove City Coll., Grove City, Pa. *Box 31.*
RANKIN, J. M. Instr., Coll. of Idaho, Caldwell, Ida. *Box 97, R. F. D. 4.*
RANKIN, Prof. W. W., Jr., A.M. (North Carolina). Head of Dept., Agnes Scott Coll., Decatur, Ga.

Ransom, Prof. W. R., A.M. (Tufts; Harvard). Tufts Coll., Tufts College, Mass. *29 Sawyer Ave.*
Ranum, Prof. Arthur, Ph.D. (Chicago). Cornell Univ., Ithaca, N. Y. *3 Central Ave.*
Rasor, Prof. S. E., A.M., M.S. (Chicago). Ohio State Univ., Columbus, Ohio. *1594 Neil Ave.*
Rau, Prof. A. G., Ph.D. (Lehigh). Moravian Coll., Bethlehem, Pa. *38 W. Market St.*
Rawlins, Asst. Prof. C. H., Jr., Ph.D. (Johns Hopkins). Postgrad. School, U. S. Naval Acad., Annapolis, Md. *13 Franklin St.*
Raynor, Asst. Prof. G. E., Ph.D. (Princeton). Wesleyan Univ., Middletown, Conn. *32 Wyllys St.*
Rea, Asst. Prof. P. L., A.M. (Marietta). Marietta Coll., Marietta, Ohio.
Reagan, C. A., B.S. (Moores Hill). Asst. Instr. and Grad. Student, Univ. of Kansas, Lawrence, Kans.
Reaves, Prof. Caroline M., A.M. (Oklahoma). Coker Coll., Hartsville, S. C.
Reaves, Prof. S. W., Ph.D. (Chicago). Univ. of Oklahoma, Norman, Okla. *527 Chautauqua Ave.*
Rechard, Asst. Prof. O. H., A.M. (Gettysburg). Univ. of Wyoming, Laramie, Wyo. *269 N. 8th St.*
Redden, Asso. Prof. J. E., A.B. (Furman). John Tarleton Agric. Coll., Stephenville, Tex. *Box 568.*
Reddick, Prof. H. W., Ph.D. (Columbia). Cooper Union, New York, N. Y.
Redditt, B. H., A.M. (Johns Hopkins). Lebanon Valley Coll., Annville, Pa. *Box 219.*
Reece, Prof. R. H., A.M. (Colorado). Math. and Physics, N. Mexico School of Mines, Socorro, N. M.
Reed, Asso. Prof. F. W., Ph.D. (Virginia). Ohio Univ., Athens, Ohio.
Reed, Asso. Prof. L. J., Ph.D. (Penna.). Biometry and Vital Statistics, Johns Hopkins Univ., Baltimore, Md.
Rees, Asst. Prof. C. J., A.M. (Chicago). Univ. of Delaware, Newark, Del.
Rees, Asso. Prof. E. L., C.E. (Kentucky), A.M. (Chicago). Univ. of Kentucky, Lexington, Ky. *118 Park Ave.*
Rees, P. K., A.B. (Southwestern Univ.). Teacher, Jr. High School, Austin, Tex. *605 W. 9th St.*
Rees, W. A., A.B. (Southwestern Univ.). Teacher, High School, Austin, Tex. *605 W. 9th St.*
Reese, Minna E., B.S. (Northwestern). Teacher, High School, Hubbard, Ia.
Reeve, W. D., Ph.D. (Minnesota). Teachers Coll., Columbia Univ., New York, N. Y.
Reid, Clair, A.M. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich. *1511 Montclair.*
Reid, Prof. L. W., Ph.D. (Göttingen). Haverford Coll., Haverford, Pa.
Reilly, Asso. Prof. J. F., Ph.D. (Iowa). Univ. of Iowa, Iowa City, Ia. *Room 212, Physics Bldg.*
Remick, Prof. B. L., Ph.M. (Cornell Coll.). State Agric. Coll., Manhattan, Kans. *613 Houston St.*
Renner, Theresa M., M.S. (Illinois). Instr., Blackburn Univ., Carlinville, Ill.
Reuterdahl, Pres. Arvid. Ramsey Inst. of Tech., St. Paul, Minn. *222 Otis Ave.*
Reynolds, Asst. Prof. C. N., Ph.D. (Harvard). West Virginia Univ., Morgantown, W. Va. *317 Brockway Ave.*
Reynolds, Asso. Prof. J. B., Ph.D. (Moravian). Math. and Astr., Lehigh Univ., Bethlehem, Pa. *721 W. Broad St.*
Reynolds, Lena E., A.M. (California). Head of Dept., Jr. Coll., Fullerton, Calif.
Rice, Prof. Harris, A.M. (Harvard). Worcester Poly. Inst., Worcester, Mass.
Rice, Asso. Prof. J. N., Ph.D. (Catholic Univ.). Catholic Univ., Washington, D. C. *3326 13th St., N.E.*
Rice, L. H., A.B. (Syracuse). Instr., Mass. Inst. of Tech., Cambridge, Mass.
Richardson, Prof. A. V., M.A. (Cambridge, England). Bishops Coll., Lennoxville, Quebec, Canada.
Richardson, Prof. C. H., A.M. (Illinois). Georgetown Coll., Georgetown, Ky.
Richardson, M. R., A.M. (Trinity). *618 McMannen St.*, Durham, N. C.
Richardson, Prof. R. G. D., Ph.D. (Yale). Brown Univ., Providence, R. I.
Richert, Prof. D. H., A.B. (Oberlin). Bethel Coll., Newton, Kans.
Rickard, Asst. Prof. Hortense, A.M. (Ohio State). Ohio State Univ., Columbus, Ohio. *333 W. 10th Ave.*
Rider, Asso. Prof. P. R., Ph.D. (Yale). Washington Univ., St. Louis, Mo.
Rietz, Prof. H. L., Ph.D. (Cornell). Univ. of Iowa, Iowa City, Ia.
Riggs, Prof. N. C., M.S. (Harvard). Carnegie Inst. of Tech., Pittsburgh, Pa.
Riley, J. L., A.M. (Georgetown). Grad. Sch., Univ. of Chicago, Chicago, Ill. *5641 Drexel Ave.*
Risley, Prof. W. J., Sr., A.M. (Illinois; Harvard). Colorado School of Mines, Golden, Colo.

RITT, Asst. Prof. J. F., Ph.D. (Columbia). Columbia Univ., New York, N. Y.
RITTENHOUSE, W. H. *4133 N. Ninth St.*, Philadelphia, Pa.
ROACH, ORVIS A. *401 Cedar St.*, San Antonio, Tex.
ROBB, J. M., A.M. (Michigan). Head of Dept., High School, Everett, Wash.
ROBBINS, ALICE V., Ph.M. (Chicago). Teacher, State Teachers Coll., Mankato, Minn. *418 S. 4th St.*
ROBBINS, Asst. Prof. C. K., A.M. (Harvard). Purdue Univ., W. LaFayette, Ind. *418 Vine.*
ROBERT, Asso. Prof. H. M., Jr., A.M. (Yale). U. S. Nav. Acad., Annapolis, Md. *53 Southgate Ave.*
ROBERTS, Prof. MARIA M., B.L. (Iowa State). Dean, Jr. Coll., and Prof., Iowa State Coll., Ames, Ia. *219 Ash Ave.*
ROBERTSON, W. E., A.B. (Oklahoma). Student, Oxford Univ., Oxford, England.
ROBERTSON, PERCIVAL, Ph.B. (Yale). Instr., The Principia, St. Louis, Mo.
ROBINSON, A. C., A.B. (Pacific). Gilman School, Baltimore, Md.
ROBINSON, FANNIE H., A.B. (Smith). *96 N. Walnut St.*, East Orange, N. J.
ROBINSON, Asso. Prof. G. E., B.A. (Dalhousie). Univ. of British Columbia, Vancouver, B. C. *852 Thurlow St.*
ROBINSON, H. A. *505 Walker St.*, Augusta, Ga.
ROBINSON, Asst. Prof. L. V., A.M. (Texas). Oklahoma City Univ., Oklahoma City, Okla. *1538 W. 24th St.*
ROBISON, Prof. G. M., Ph.D. (Cornell). Trinity Coll., Durham, N. C.
RODGERS, Dean T. G., A.M. (Wisconsin). New Mexico Normal Univ., E. Las Vegas, N. Mex.
ROE, Prof. E. D., Jr., Ph.D. (Erlangen). John Raymond French Prof. of Math., and Dir. of the Observatory, Syracuse Univ., Syracuse, N. Y. *123 W. Ostrander Ave.*
ROESER, H. M., M.S. (Geo. Washington Univ.). Engineer, Physicist, U. S. Bur. of Standards, Washington, D. C.
ROEVER, Prof. W. H., Ph.D. (Harvard). Pure Math., Washington Univ., St. Louis, Mo.
ROGERS, B. A., M.S. (Chicago). Grad. Student, Iowa State Coll., Ames, Iowa. *2864 West St.*
ROHRBAUGH, FLORENCE C., A.B. (Oklahoma). Teacher, High School, Norman, Okla. *455 College Ave.*
ROMAN, Asso. Prof. IRWIN, Ph.D. (Chicago). Vanderbilt Univ., Nashville, Tenn. *309 32d Ave. S.*
ROMIG, L. V., A.M., M.S. (Michigan). Williamson, W. Va. *Genl. Delivery.*
ROOD, MARION B., M.S. (Michigan). Ann Arbor, Mich.
ROOT, Prof. R. E., Ph.D. (Chicago). U. S. Naval Acad., Annapolis, Md. *7 Franklin St.*
ROSANOFF, LILLIAN, Ph.D. (Clark). Teacher, Commercial High School, Brooklyn, N. Y. *328 Clinton Ave.*
ROSEBRUGH, Prof. T. R., M.A. (Toronto). Elec. Eng., Univ. of Toronto, Toronto, Can. *92 Walmer Rd.*
ROSENBACH, Asst. Prof. J. B., M.S. (Illinois). Carnegie Inst. of Tech., Pittsburgh, Pa. *5836 Alderson St., Suite No. 5.*
ROSENBAUM, JOSEPH, Ph.D. (Cornell). The Milford School, Milford, Conn.
ROSENGARTEN, Prof. GEORGE, Ph.D. (Penna.). W. Phila. Boys High School, Philadelphia, Pa. *414 Edgewood Rd., Upper Darby P. O.*
ROSS, G. A., A.M. (George Washington). Teacher, Central High School, Washington, D. C.
ROTH, W. E., A.M. (Wisconsin). Teacher, High School, West Allis, Wis. *671 71st Ave.*
ROTHROCK, Prof. D. A., Ph.D. (Leipzig). Dean, Indiana Univ., Bloomington, Ind. *1000 Atwater Ave.*
ROUSE, Asst. Prof. L. J., Ph.D. (Michigan). Univ. of Michigan, Ann Arbor, Mich. *1137 Michigan Ave.*
ROWE, Prof. J. E., Ph.D. (Johns Hopkins). Head of Dept., and Dir. of Extension, Coll. of William and Mary, Williamsburg, Va.
RUMBLE, Prof. D., A.M. (Harvard). Emory Univ., Emory University, Ga.
RUMNEY, ETHEL A., M.S. (Chicago). Instr., State Teachers Coll. of Emporia, Emporia, Kans. *1006 Market St.*
RUNNING, Prof. T. R., Ph.D. (Wisconsin). Univ. of Michigan, Ann Arbor, Mich. *1019 Michigan Ave.*
RUPP, C. A., A.M. (Harvard). Instr., Math. and Astr., Hamline Univ., St. Paul, Minn.
RUSK, Prof. W. J., M.A. (Toronto). Math. and Astr., Grinnell Coll., Grinnell, Ia. *1415 Park St.*
RUSSELL, Adj. Prof. BEULAH, A.M. (Chicago). Randolph-Macon Woman's Coll., Lynchburg, Va. *No. 310, R. M. W. Coll.*

RUSSELL, Prof. H. E. Univ. of Denver, Denver, Colo.
RUSSELL, Asso. Prof. W. P., A.M. (Cumberland). Pomona Coll., Claremont, Calif.
RUTLEDGE, Asst. Prof. GEORGE, Ph.D. (Illinois). Mass. Inst. of Tech. *29 Bellevue Rd., Belmont, Mass.*
RYAN, Prof. W. J., A.M., M.S. (St. Louis Univ.). St. Louis Univ., St. Louis, Mo.

SAFFORD, Prof. F. H., Ph.D. (Harvard). Univ. of Pennsylvania, Philadelphia, Pa. *College Hall.*
SAGE, Asso. Prof. J. R., M.S. (Rose Poly.). Registrar, State Coll., Ames, Ia. *2227 Knapp St.*
SAGEN, G. O., A.B. (California). Kern Co. Union High School, Bakersfield, Calif.
SANDERS, Prof. BERNICE, A.B. (Wilberforce). Wilberforce Univ., Wilberforce, Ohio. *Box 8.*
SANDERS, Prof. S. T. La. State Univ., Baton Rouge, La. *714 Mills Ave.*
SAUND, D. S., A.M. (California). *Box 225*, Berkeley, Calif.
SAUREL, Prof. PAUL, Sc.D. (Bordeaux, France). Coll. of the City of New York, New York, N. Y.
SAUTÉ, GEORGE, Ph.B. (Brown). Asst., Brown Univ., Providence, R. I.
SCARBOROUGH, Asst. Prof. J. B., Ph.D. (Johns Hopkins). U. S. Naval Acad., Annapolis, Md. *P. O. Box 332.*
SCARBOROUGH, Prof. J. H., Ph.D. (Vanderbilt). State Teachers Coll., Warrensburg, Mo.
SCHARF, Rev. C. G., A.B. (St. Cyril), B.S. (Chicago). Treas. and Instr. in Sc., St. Cyril High School, Chicago, Ill. *6413 Dante Ave.*
SCHMALL, C. N., A.B. (C. C. N. Y.). Publ. School No. 69, Borough of Manhattan, New York, N. Y. *103 W. 118th St.*
SCHMIEDEL, OSCAR, A.M. (Bethany, W. Va.). *2622 College Ave.*, Berkeley, Calif.
SCHOFIELD, MARY E., A.M. (California). Buena Park, Calif.
SCHOONMAKER, Prof. HAZEL E., A.M. (Radcliffe). Western Coll. for Women, Oxford, Ohio. *Box 122.*
SCHOONOVER, R. H., A.B. (Indiana). Instr., Detroit Jr. Coll. *606 Hawthorne Ave., Royal Oak, Mich.*
SCHORLING, Asso. Prof. RALEIGH, A.M. (Chicago). School of Educ., Univ. of Michigan, Ann Arbor, Mich. *School of Educ.*
SCHOTTENFELS, IDA M., A.M. (Chicago). *5525 Kimbark Ave.*, Chicago, Ill.
SCHREIBER, E. W., A.M. (Chicago). Head of Dept., Proviso Twp. High School, Maywood, Ill. *406 S. 2d St.*
SCHUB, PINCUS, formerly student in Turkey. Instr., Hebrew School, Newark, N. J. *179 E. Broadway, New York, N. Y.*
SCHUYLER, ELMER, M.S. (Lafayette). Instr., Bay Ridge High School, Brooklyn, N. Y. *87 71st St.*
SCHWEITZER, A. R., Ph.D. (Chicago). *452 Oakdale Ave.*, Chicago, Ill.
SCOTT, Prof. CHARLOTTE A., D.Sc. (London). Emeritus, Bryn Mawr Coll., Bryn Mawr, Pa. *233 Roberts Rd.*
SCOTT, Prof. G. H., A.M. (Harvard). Math. and Physics, Illinois Coll., Jacksonville, Ill. *830 Grove St.*
SEALE, Asst. Prof. R. Q., A.M. (Columbia). Southern Methodist Univ., Dallas, Tex.
SEARLE, MARY, B.S. (Wellesley). Instr., Sweet Briar Coll., Sweet Briar, Va.
SEE, Prof. T. J. J., Ph.D. (Berlin). Dir., Naval Observatory, Mare Island, Calif.
SEIDLIN, Prof. JOSEPH, A.M. (Columbia). Physics, Alfred Univ., Alfred, N. Y.
SEIVERLING, I. F., A.M. (Columbia). Head of Dept., State Normal School, Millersville, Pa.
SELLEW, Prof. G. T., Ph.D. (Yale). Knox Coll., Galesburg, Ill.
SENSENIG, Prof. WAYNE, Ph.D. (Penna.). *309 Bangor Rd.*, Cynwyd, Mont. Co., Pa.
SEUBERT, G. A., B.S. (Pittsburgh). *309 Weldon St.*, Latrobe, Pa.
SEWELL, Prof. S. M., A.M. (Texas), M.S. (Chicago). Southwest Texas State Normal Coll., San Marcos, Texas. *500 N. Guadalupe St.*
SHANNON, Prof. J. I., A.M. (St. Louis). Physics, St. Louis Univ., St. Louis, Mo.
SHARP, Prof. J. M., A.B. (Mississippi). Woman's Coll., Hattiesburg, Miss. *Sta. B.*
SHAUB, H. C., A.M. (Dartmouth). Instr., Cornell Univ., Ithaca, N. Y.
SHEETS, Asst. Prof. R. A., A.M. (Denison). Miami Univ., Oxford, Ohio. *111 N. Univ. Ave.*
SHELDON, Prof. E. W., Ph.D. (Yale). Univ. of Alberta, Edmonton South, Alb., Can.
SHELLENBERGER, R. C., A.M. (Michigan). Head of Dept., City Schools and Jr. Coll., Bay City, Mich. *2025 Fifth St.*
SHENTON, Asst. Prof. W. F., Ph.D. (Hopkins). U. S. Naval Acad., Annapolis, Md.
SHEPPARD, N. E., M.A. (Toronto). Univ. of Toronto, Toronto, Ont.

SHEPPARD, Prin. R. S., M.A. (Alberta). High School, Strathcona, Alb., Can.
SHERER, C. R., A.M. (Nebraska). Instr., Univ. of Nebraska, Lincoln, Nebr. *Bethany, Nebr.*
SHERK, Prof. W. H., A.M. (Chicago). Univ. of Buffalo, Buffalo, N. Y.
SHERWOOD, Prof. G. E. F., Ph.D. (Chicago). So. Branch, Univ. of California, Los Angeles, Calif.
SHIBLI, J., A.M. (Wisconsin). Instr., Penn. State Coll., State College, Pa.
SHIRK, Prof. J. A. G., M.S. (Kansas). Math. and Applied Mech., Kansas State Teachers Coll., Pittsburg, Kans.
SHIRLEY, Asst. Prof. H. L., A.B. (Indiana). Univ. of Nevada, Reno, Nev.
SHIVELY, Prof. C. S., Ph.D. (Denver). Juniata Coll., Huntington, Pa. *1805 Moore St.*
SHIVELY, Prof. L. S., Ph.D. (Chicago). Mount Morris Coll., Mt. Morris, Ill.
SHOHAT, J. A., Magister of Pure Math. (Petrograd). Instr., Univ. of Michigan, Ann Arbor, Mich.
SHOOK, C. A., Ph.D. (Johns Hopkins). Instr., Yale Univ., New Haven, Conn. *191 Mansfield St.*
SHOOK, Prof. G. A., Ph.D. (Illinois). Math. and Physics, Wheaton Coll., Norton, Mass. *Box 4.*
SHOPTAUGH, J. R., A.B. (Culver-Stockton). Univ. Demonstration High School, Baton Rouge, La.
SHORT, C. A., M.S. (Delaware). Continental Fibre Co., Newark, Del.
SHORT, JESSIE M., A.M. (Carleton). Instr., Reed Coll., Portland, Ore.
SHORT, Prof. W. T., A.B. (Okla. Baptist Coll.). Oklahoma Baptist Univ., Shawnee, Okla.
SHOWMAN, Asst. Prof. H. M., A.M. (Harvard). So. Branch, Univ. of Calif., Los Angeles, Calif.
SHULL, Prof. C. E., A.M. (Bridgewater). Bridgewater Coll., Bridgewater, Va.
SHUMAN, J. W., B.S. (Mass. Inst. of Tech.). Secy.-Treas., Power Eng. Co., Consulting Eng., Minneapolis, Minn. *621 Metropolitan Life Bldg.*
SHUMWAY, Asso. Prof. R. R., A.B. (Minnesota). Univ. of Minnesota, Minneapolis, Minnesota.
SICELOFF, Asst. Prof. L. P., Ph.D. (Columbia). Columbia Univ., New York, N. Y.
SILBERSTEIN, LUDWIK, Ph.D. (Berlin). Math. Physicist, Research Lab., Eastman Kodak Co., Rochester, N. Y.
SILVERMAN, Asst. Prof. L. L., Ph.D. (Missouri). Dartmouth Coll., Hanover, N. H.
SIMESTER, J. H., M.A. (Toronto). Instr., Carnegie Inst. of Tech., Pittsburgh, Pa. *5707 Walnut St.*
SIMMONS, Asst. Prof. H. A., B.S. (Chicago). Univ. of Pittsburgh. *Grad. Student, Univ. of Chicago, Chicago, Ill., 1924–25. 5721 Kenwood.*
SIMON, Asso. Prof. W. G., Ph.D. (Chicago). Adelbert Coll., Western Reserve Univ., Cleveland, Ohio.
SIMONS, Asso. Prof. LAO G., Ph.D. (Columbia). Hunter Coll., New York, N. Y. *180 W. 88th St.*
SIMPSON, Prof. C. G., A.M. (Columbia). Math. and Mech., Coll. of Elec. Eng., Milwaukee, Wis.
SIMPSON, Prof. T. M., Ph.D. (Wisconsin). Univ. of Florida, Gainesville, Fla.
SIMPSON, Prof. T. McN., Jr., Ph.D. (Chicago). Randolph Macon Coll., Ashland, Va.
SINCLAIR, Asso. Prof. MARY EMILY, Ph.D. (Chicago). Oberlin Coll., Oberlin, Ohio. *260 Oak St.*
SINGER, Prof. S. A., A.M. (Capital). Capital Univ., Columbus, Ohio. *2322 E. Main St.*
SISAM, Prof. C. H., Ph.D. (Cornell). Colorado Coll., Colorado Springs, Colo. *311 E. San Rafael St.*
SKARSTEDT, Prof. MARCUS, Ph.D. (California). Whittier Coll., Whittier, Calif.
SKILES, Prof. W. V., A.M. (Harvard). Georgia School of Tech., Atlanta, Ga.
SKINNER, Prof. E. B., Ph.D. (Chicago). Univ. of Wisconsin, Madison, Wis.
SKIRROW, W. A., M.A. (Queen's). Asst. Master, Parkdale Colleg. Inst., Toronto, Ont., Can.
SLAUGHT, Prof. H. E., Ph.D. (Chicago). Univ. of Chicago, Chicago, Ill.
SLAWTER, HARRY, A.B. (W. Va. Wesleyan). Math. and Physics, Union High School, Bridgeport, W. Va.
SLEIGHT, Prof. E. R., A.M. (Albion). Albion Coll., Albion, Mich.
SLICHTER, Prof. C. S., M.S., Sc.D. (Northwestern). Univ. of Wisconsin, Madison, Wis. *636 Frances St.*
SLOBIN, Prof. H. L., Ph.D. (Clark). Univ. of New Hampshire, Durham, N. H.
SMAIL, Asst. Prof. L. L., Ph.D. (Columbia). Univ. of Oregon, Eugene, Ore.
SMITH, Prof. A. W., Ph.D. (Chicago). Colgate Univ., Hamilton, N. Y.
SMITH, Prof. C. D., A.B. (Mississippi). Louisiana Coll., Pineville, La. *On leave at Univ. of Iowa.*
SMITH, Prof. CLARA E., Ph.D. (Yale). Wellesley Coll., Wellesley, Mass. *14 Waban St.*
SMITH, C. W., A.B. (Minnesota). State Normal School, Superior, Wis.
SMITH, Prof. DAVID EUGENE, Ph.D., LL.D. (Syracuse). Teachers Coll., Columbia Univ., New York, N. Y. *525 W. 120th St.*
SMITH, Asso. Prof. D. M., Ph.D. (Chicago). Ga. School of Tech., Atlanta, Ga.

SMITH, Prof. E. R., Ph.D. (Munich). Iowa State Coll., Ames, Ia.
SMITH, ELMER R., A.M. (Vanderbilt). Florida State Coll., Tallahassee, Fla. *79 College Ave.*
SMITH, Asso. Prof. E. S., Ph.D. (Virginia). Univ. of Cincinnati, Cincinnati, Ohio.
SMITH, F. L., A.B. (Drury Coll.). Instr., Physics, Univ. of Oklahoma, Norman, Okla.
SMITH, G. S., A.M. (Texas). Head of Dept., High School, Port Arthur, Texas. *2328 Procter.*
SMITH, Asst. Prof. G. W., M.S. (Colorado). Univ. of Kansas, Lawrence, Kans.
SMITH, H. B., A.B. (Kenyon). Civil and Mech. Eng., Wilmington, N. C.
SMITH, H. L., M.S. (Chicago). Instr., Univ. of Minnesota, Minneapolis, Minn.
SMITH, H. W., B.S., C.E. (Georgia). Head of Dept., High School, Gainesville, Ga. *P. O. Box 463.*
SMITH, Prof. I. W., A.M. (Illinois). N. Dak. Agric. Coll., Fargo, N. D.
SMITH, Asst. Prof. J. P., A.M. (Woodstock). Woodstock Coll., Woodstock, Md.
SMITH, Prof. L. W., Ph.D. (Washington and Lee). Washington and Lee Univ., Lexington, Va.
SMITH, Prof. M. G., Ph.D. (Illinois). Greenville Coll., Greenville, Ill.
SMITH, MARGARETHE M., A.M. (Radcliffe). *3 Wyoming St.*, Grove Hall, Boston, Mass.
SMITH, Prof. P. F., Ph.D. (Yale). Yale Univ., Sheffield Sc. Sch., New Haven, Conn. *330 Willow St.*
SMITH, P. K., A.M. (South Carolina). Dovesville, S. C.
SMITH, Asst. Prof. R. F., M.S. (New York Univ.). Coll. of the City of New York, New York, N. Y.
SMITH, R. R., A.B. (Yale). Mgr., College Dept., Macmillan Co., New York, N. Y. *64–66 Fifth Ave.*
SMITH, Prof. SARAH E., B.S. (Mount Holyoke). Mount Holyoke Coll., South Hadley, Mass.
SMITH, Prof. T. F., A.M. (Manhattan Coll.). Little Rock Coll., Little Rock, Ark.
SMITH, Prof. W. M., Ph.D. (Columbia). Lafayette Coll., Easton, Pa.
SNEDECOR, Asso. Prof. G. W., A.M. (Michigan). State Coll., Ames, Ia.
SNYDER, Asst. Prof. A. D., A.M. (Wisconsin). Union Coll., Schenectady, N. Y. *862 Union St.*
SNYDER, Prof. VIRGIL, Ph.D. (Göttingen). Cornell Univ., Ithaca, N. Y. *214 Univ. Ave.*
SOMERS, Capt. R. H., Graduate, U. S. Military Acad. U. S. Army, Riverdale, Md.
SOUSLEY, Prof. C. P., Ph.D. (Johns Hopkins). Rose Poly. Inst., Terre Haute, Ind. *426 S. 19th St.*
SPARKS, Asso. Prof. F. W., M.S. (Chicago). State Normal Coll., Natchitoches, La.
SPEAR, Prof. JOSEPH, A.B. (Harvard). Chairman of Dept., School of Eng., Northeastern Univ., Boston, Mass.
SPEEKER, Asso. Prof. G. G., A.M. (Indiana). Michigan Agric. Coll., East Lansing, Mich.
SPENCELEY, Asst. Prof. G. W., A.M. (Harvard). Miami Univ., Oxford, Ohio. *100 E. Walnut St.*
SPENCER, Prof. MARY C., M.S. (Cornell). Sophie Newcomb Memorial Coll., New Orleans, La.
SPERRY, MAY, A.M. (Brown). Instr., Syracuse Univ., Syracuse, N. Y.
SPERRY, Asst. Prof. PAULINE, Ph.D. (Chicago). Univ. of California, Berkeley, Calif.
SPIES, Prof. ANTOINETTE, A.M. (Louvain). Coll. of the Sacred Heart, Manhattanville, New York, N. Y.
SPINKS, M. J. Chief Eng., Champion Bridge Co., Wilmington, Ohio. *Box 594.*
SPOONER, Prof. C. C., A.M. (Amherst). State Normal School, Marquette, Mich. *210 E. Prospect St.*
STAFFORD, W. A., A.M. (Stratford). Head of Dept., Oakland High School, Oakland, Calif. *31 Park Way, Piedmont, Calif.*
STAGER, H. W., Ph.D. (California). Instr., Univ. of Washington, Seattle, Wash.
STAHL, SARAH S., Ph.B. (Oberlin). Wendell Phillips High School, Chicago, Ill.
STANWICK, C. A., B.S. in E.E. (Washington). Penna. Water and Power Co., Baltimore, Md. *2206 Elsinor Ave.*
STARK, MARION E. Grad. Student, Univ. of Chicago, Chicago, Ill. *41 Beecher Hall.*
STEED, Asst. Prof. D. V., Ph.D. (California). Univ. of Southern California, Los Angeles, Calif.
STEED, Prof. J. B., B.S. in Educ. (Missouri). Educ., Miami State Sch. of Mines, Miami, Okla.
STEIMLEY, L. L., Ph.D. (Illinois). Univ. of Illinois, Urbana, Ill.
STEIN, Dr. S. G. *P. O. Box 146*, Muscatine, Ia.
STEIRNAGLE, W. M., A.B. (Indiana). *111 Michigan Ave.*, Sturgis, Mich.
STEPHENS, EUGENE. Instr., Washington Univ., St. Louis, Mo.
STEPHENS, Prof. R. P., Ph.D. (Johns Hopkins). Univ. of Georgia, Athens, Ga.
STETSON, J. M., Ph.D. (Princeton). Instr., Univ. of Illinois, Champaign, Ill.
STEVENS, A. H., A.B. (Georgia). Carlton, Ga.
STEVENS, Prof. W. A., B.S. (Colgate). Virginia Union Univ. *On leave 1924–25, Grad. Student Columbia Univ. 106 Overlook Terr., Leonia, N. J.*

STEWART, MABEL M., A.B. (Greenville), B.S. (Central States Teachers Coll.). *24 W. Campell, Edmond, Okla.*

STOKES, W. B., M.S., M.E. (Ala. Poly. Inst.). Head of Dept., Southwestern La. Inst., Lafayette, La.

STONE, Prof. ORMOND, A.M. (Old Univ. of Chicago). Formerly Prof. of Astr., Univ. of Virginia, Charlottesville, Va. *Clifton Station, Fairfax Co., Va.*

STONE, Asso. Prof. R. B., A.M. (Harvard). Purdue Univ., LaFayette, Ind. *307 Russell St., W. LaFayette, Ind.*

STOUFFER, Prof. E. B., Ph.D. (Illinois). Univ. of Kansas, Lawrence, Kans.

STOUT, C. E., B.S. (Heidelberg). Grad. Student, Univ. of Wisconsin, Madison, Wis. *432 W. Main.*

STOWELL, Prof. C. J., Ph.D. (Illinois). McKendree Coll., Lebanon, Ill.

STRATTON, Prof. W. T., A.M. (Indiana). Kansas State Agric. Coll., Manhattan, Kans.

STROM, C. W., A.B. (Luther Coll.). Rhodes Scholar, Queen's Coll., Oxford, Eng.

STROMQUIST, Prof. C. E., Ph.D. (Yale). *808 Waverly St.*, Palo Alto, Calif.

STRONG, Prof. CORA, A.B. (Cornell). North Carolina Coll. for Women, Greensboro, N. C.

SUBLETTE, G. W., A.B. (N. Missouri State Normal Coll.). Consulting and Construction Eng., Minneapolis, Minn.

SUESMAN, W. P., LL.B. (Lake Forest). Treas., Household Furniture Co., Providence, R. I.

SUFFA, Prof. MARY C., A.M. (Brown). Elmira Coll., Elmira, N. Y.

SUYDAM, Prof. V. A., Ph.D. (Princeton). Physics, Beloit Coll., Beloit, Wis.

SWANSON, EDITH M., A.B. (Minnesota). Instr., Univ. High School, Univ. of South Dakota, Vermilion, S. D.

SWARTZEL, Prof. K. D., M.S. (Ohio State). Univ. of Pittsburgh, Pittsburgh, Pa. *4360 Center Ave.*

SWEAZEY, Dean G. B., A.M. (Wabash). Westminster Coll., Fulton, Mo.

SWEET, H. L. Instr., Phillips Exeter Acad., Exeter, N. H.

SWIFT, Prof. ELIJAH, Ph.D. (Göttingen). Univ. of Vermont, Burlington, Vt. *415 S. Willard St.*

TABER, G. H. *P. O. Box 1214*, Pittsburgh, Pa.

TALIAFERRO, CARRIE B., B.S. (Columbia). Asso., State Normal School, Farmville, Va.

TANNER, Prof. J. H., Ph.D. (New Hampshire). Cornell Univ., Ithaca, N. Y.

TANZOLA, J. J., A.M. (Columbia). Instr., Cooper Union. *127 Parkside Ave., Brooklyn, N. Y.*

TAPPAN, Asso. Prof. ANNA H., Ph.D. (Cornell). Iowa State Coll., Ames, Iowa. *Cranford Apt.*

TAYLOR, Prof. E. H., Ph.D. (Harvard). Eastern Illinois State Teachers Coll., Charleston, Ill. *885 7th St.*

TAYLOR, Prof. EUGENE, A.M. (DePauw). Univ. of Idaho, Moscow, Ida.

TAYLOR, Prof. F. J., A.B. (St. Thomas). Coll. of St. Thomas, St. Paul, Minn.

TAYLOR, Asst. Prof. J. S., Ph.D. (California). Univ. of Pittsburgh, Pittsburgh, Pa.

TAYLOR, Prof. MARY S. N. Y. Training Sch. for Teachers, Leonia, N. J.

TAYLOR, MILDRED E., A.M. (Illinois). Instr., Knox Coll., Galesburg, Ill.

TAYLOR, Prof. W. E., Ph.D. (Syracuse). Syracuse Univ., Syracuse, N. Y.

TEMPLE, MARGARET F., A.B. (Hamline). Stud., Univ. of Southern California, Los Angeles, Calif.

TEMPLIN, R. J. W., A.M. (Bucknell). Supt. School, W. Pittston, Pa.

THEOBALD, Prof. JOHN, A.B. (Columbia), S.T.B. (Catholic Univ.). Columbia Coll., Dubuque, Ia.

THOMAS, Asso. Prof. C. F., A.B. (Amherst). Case Sch. of Appl. Sc., Cleveland, Ohio.

THOMAS, Prof. E. H., A.B. (Indiana). Physics, Southeast Mo. St. Teachers Coll., Cape Girardeau, Mo.

THOMAS, Prof. EVAN. Coll. of Eng., Univ. of Vermont, Burlington, Vt.

THOMAS, Asst. Prof. R. W., M.S. (Washington and Jefferson). Washington and Jefferson Coll., Washington, Pa.

THOME, W. J., C.E. (Mich. Agric.). Brown-Hutchinson Iron Wks., Detroit, Mich. *223 Avalon Ave.*

THOMPSON, E. L. Grad. Stud., Univ. of Chicago, Chicago, Ill. *6032 Ingleside Ave.*

THOMPSON, HELEN, A.M. (Columbia). *106 Morningside Dr.*, New York, N. Y.

THOMPSON, Prof. H. D., Ph.D. (Göttingen). Princeton Univ., Princeton, N. J. *11 Morven St.*

THOMPSON, RUTH, B.S. (New Jersey Coll. for Women). Instr., N. J. Coll. for Women, New Brunswick, N. J.

THOMPSON, W. N., B.S. (Drury). Drury Coll., Springfield, Mo.

THOMSEN, H. I., Ph.D. *1928 Mt. Royal Terrace*, Baltimore, Md.

THORNE, Asst. Prof. P. L., M.S. (New York Univ.). New York Univ., New York, N. Y.

THORNTON, W. M., LL.D. (Hampden-Sidney). Univ. of Virginia, Univ., Va.

THORP, ELLA, A.B. (Minnesota). Instr., Univ. of Minnesota, Minneapolis, Minn. *656 Jefferson St., N.E.*

TIENZO, TELESFORO, A.B. (Philippines), B.S. (Chicago). Instr., Univ. of the Philippines, Manila, P. I. *1276 Sande, Tondo.*

TILLY, ARTHUR, M.E. (New York Univ.). Instr., Washington Square Coll., New York Univ., New York, N. Y.

TINNER, Prof. J. C., M.S. (Chicago). Wilberforce Univ., Wilberforce, Ohio.

TITSWORTH, Prof. A. A., M.S., C.E., D.Sc. (Rutgers). Rutgers Coll., New Brunswick, N. J.

TITSWORTH, Prof. W. A., M.S. (Wisconsin). Alfred Univ., Alfred, N. Y.

TITT, Prof. H. G., A.M. (Michigan). Huron Coll., Huron, S. D.

TITUS, C. M., A.M. (Stanford). Instr., Univ. Farm School, Davis, Calif.

DE TOLEDO, Prof. L. O. Univ. of Madrid, Madrid, Spain. *Velasques 28–3°, Facultad de Ciencias.*

TORREY, MARIAN M., Ph.D. (Cornell). Instr., Univ. of Illinois, Urbana, Ill. *Dept. of Math.*

TORREY, Asso. Prof. ROBERT, Ph.B. (Miss.). Univ. of Mississippi, University, Miss.

TOUCHSTONE, NORMA E., A.B. (La. State). Teacher, High School, Bolton, La. *1902 Polk St., Alexandria, La.*

TOUTON, Prof. F. C., Ph.D. (Columbia). Educ., Univ. of Southern Calif., Los Angeles, Calif.

TOWNES, S. B., A.B. (Oklahoma). *125 W. 2d St.*, Oklahoma City, Okla. (?)

TOWNSEND, Prof. E. J., Ph.D. (Göttingen). Head of Dept., Univ. of Illinois, Urbana, Ill.

TRACEY, Asst. Prof. J. I., Ph.D. (Johns Hopkins). Yale Univ., New Haven, Conn. *314 Norton St.*

TREFETHEN, Prof. H. E. Colby Coll., Waterville, Me. *4 West Court.*

TREMBLAY, Prof. ALTHÉOD. Higher Math., Laval Univ., Quebec, Can.

TRIPP, Prof. M. O., Ph.D. (Columbia). Wittenberg Coll., Springfield, Ohio.

TROTT, Prof. T. E. Mt. Union Coll., Alliance, Ohio. *50 E. College St.*

TRYON, G. M. V. Fenton, Mich.

TRYTTEN, M. H., M.S. (Iowa). Instr., Physics, Univ. of Pittsburgh, Pittsburgh, Pa.

TUCKER, Prin. B. A., A.B. (Mississippi). High School, Elton, La.

TUCKER, Asso. Prof. E. R., B.S. (Military Coll. of S. C.). Texas Christian Univ., Fort Worth, Tex.

TURNER, Prof. A. B., Ph.D. (Penna.). Coll. of the City of New York, New York, N. Y. *245 N. Mountain Ave., Montclair, N. J.*

TURNER, Asst. Prof. BIRD M., Ph.D. (Bryn Mawr). Univ. of W. Va., Morgantown, W. Va.

TURNER, Asst. Prof. J. S., Ph.D. (Chicago). Iowa State Coll., Ames, Ia.

TWISS, J. F., M.A. (McMaster). Instr., Royal Milit. Coll., Kingston, Ont., Can. *1021 Union St.*

TYLER, Prof. H. W., Ph.D. (Erlangen). Mass. Inst. of Tech., Cambridge, Mass.

TYLER, Asst. Prof. JOHN. U. S. Naval Academy, Annapolis, Md.

UHLER, Asso. Prof. H. S., Ph.D. (Johns Hopkins). Physics, Yale Univ., New Haven, Conn.

UNDERHILL, Asso. Prof. A. L., Ph.D. (Chicago). Univ. of Minnesota, Minneapolis, Minn.

UNDERWOOD, P. H. Ball High School, Galveston, Texas. *2527 Ave. I.*

UNDERWOOD, Asso. Prof. R. S., A.M. (Minnesota). Alabama Poly. Inst., Auburn, Ala.

UNSELD, G. P., A.M. (Colorado). Instr., West High School, Salt Lake City, Utah.

UPTON, Asso. Prof. C. B., A.M. (Columbia). Teachers Coll., Columbia Univ., New York, N. Y.

VALLANDINGHAM, J. T., A.B. (Georgetown). Instr., Cumberland Coll., Williamsburg, Ky.

VAN BENSCHOTEN, Prof. ANNA L., Ph.D. (Cornell). Retired, Wells Coll. *230 N. Friends Ave., Whittier, Calif.*

VAN BUSKIRK, Prof. H. C., Ph.B. (Cornell). Calif. Inst. of Tech., Pasadena, Calif. *3400 E. Colorado St.*

VANDIVER, H. S. Instr., Cornell Univ. *On leave 1924–25, Acting Asst. Prof., Univ. of Texas, Austin, Texas. 2612 Guadalupe St.*

VAN HEE, LOUIS, Jesuit Father. Teacher of Math., Liége, Belgium. *66 Quai de Longdoz, 66.*

VAN HORNE, R. N., Ph.B. (Morningside). Morningside Coll., Sioux City, Ia.

VAN ORDEN, E. M., B.S. (Utah). North Cache High School, Richmond, Utah. *Lewiston, Utah.*

VAN ORSTRAND, C. E., M.S. (Michigan). Physical Geologist, U. S. Geol. Surv., Washington, D. C.

VAN VELZER, C. A., Ph.D. (Hillsdale). Carthage Coll., Carthage, Ill.

VAN VLECK, Prof. E. B., Ph.D. (Göttingen). Univ. of Wisconsin, Madison, Wis.

VASS, J. I., A.B. (Cornell). Instr., Northwestern Univ., Evanston, Ill. *1411 Chicago Ave.*

VEBLEN, Prof. OSWALD, Ph.D. (Chicago). Princeton Univ., Princeton, N. J.

VEDDER, Prof. J. N., A.M. Union Coll., Schenectady, N. Y.
VIVIAN, Prof. ROXANA, Ph.D. (Chicago). Wellesley Coll., Wellesley, Mass.

WAGAR, G. L., A.M. (Columbia). Coburg, Ont.
WAGNER, W. J., A.M. (Illinois). Instr., Allegheny Coll., Meadville, Pa.
WAHLIN, Prof. G. E., Ph.D. (Yale). Univ. of Missouri, Columbia, Mo.
WAITS, Prof. B. L., A.M. (Clark). Physics and Dean of Coll. of Liberal Arts, Wilberforce Univ., Wilberforce, Ohio.
WALDO, Prof. C. A., Ph.D. (Syracuse). Emeritus, Washington Univ. *435 W. 119th St., New York, N. Y.*
WALKER, Vice-Pres. B. M., Ph.D. (Chicago). Miss. A. & M. Coll., A. & M. College, Miss.
WALKER, Asst. Prof. EVELYN, A.M. (Columbia). Hunter Coll., New York, N. Y.
WALKER, HELEN M., A.M. (Columbia). Univ. of Kansas, Lawrence, Kans.
WALKER, L. C., A.M. (Stanford). Ceresco, Nebr.
WALLACE, W. W., A.M. (Stanford). Jr. Coll., Sacramento, Calif.
WALSH, Asst. Prof. J. L., Ph.D. (Harvard). Harvard Univ., Cambridge, Mass.
WALTER, ARTHUR, A.M. (Stanford). City Supt. of Schools, Salinas, Calif.
WALTON, A. R. Head of Dept., High School, Waycross, Ga. *Box 401.*
WALTON, B. F. Prin. High School, South Hill, Va.
WALTON, Asst. Prof. T. O., A.B. (Kalamazoo). Kalamazoo Coll., Kalamazoo, Mich. *804 Minor Ave.*
WALTZ, A. K., M.S. (Gettysburg). Instr., Cornell Univ., Ithaca, N. Y. *130 Linden Ave.*
WAPPLE, A. R., A.M. (California). Instr., Colorado Coll., Colorado Springs, Colo. *1504 N. Roger St.*
WARD, L. E., A.M. (Harvard). *381 Mass. Ave.*, Lexington, Mass.
WARNE, W. R., Ph.B., Pd.B. (Syracuse). Instr., Eng., Univ. of Minnesota, Minneapolis, Minn. *602 Essex St., S.E.*
WARNER, I. N., B.S. (Chicago). State Normal School, Platteville, Wis.
WARREN, Asst. Prof. L. A. H., Ph.D. (Chicago). Univ. of Manitoba, Winnipeg, Can.
WASHBURNE, A. C., Graduate U. S. Milit. Acad. Actuary, Berkshire Life Ins. Co., Pittsfield, Mass.
WATKEYS, Prof. C. W., A.M. (Harvard). Univ. of Rochester, Rochester, N. Y. *20 Girton Pl.*
WATT, Asst. Prof. MARTHA W., A.M. (Columbia). Wheaton Coll., Norton, Mass. *2144 Broad St., Providence, R. I.*
WATTS, C. B., A.B. (Indiana). Asso. Astr., Naval Observ., Washington, D. C.
WATTS, Prof. C. W., C.E. (Va. Milit. Inst.). Virginia Milit. Inst., Lexington, Va. *P. O. Box 713.*
WATTS, E. VIRGINIA, A.M. (Chicago). Stanford, Ky.
WEAR, Asso. Prof. L. E., Ph.D. (Johns Hopkins). Calif. Inst. of Tech., Pasadena, Calif. *68 S. Grand Oaks Ave.*
WEAVER, ESTHER M., M.S. (Chicago). Instr., Northwestern Univ., Evanston, Ill. *2020 Sherman Ave., Apt. B-3.*
WEAVER, Asst. Prof. J. H., Ph.D. (Penna.). Ohio State Univ., Columbus, Ohio. *Hilliard, Ohio.*
WEAVER, Asst. Prof. WARREN, Ph.D. (Wisconsin). Univ. of Wisconsin, Madison, Wis. *1852 Summit Ave.*
WEBB, H. E., A.B. (Rochester). Central High School, Newark, N. J. *12 Irving Pl., Summit, N. J.*
WEBB, Prin. R. O., B.S. (Tennessee), A.B. (Oklahoma). High School, Wilson, Okla. *Box 2.*
WEBBER, Prof. W. P., Ph.D. (Cincinnati). La. State Univ., Baton Rouge, La.
WEBER, W. W., A.M. (Georgia). Dean and Prof. of Math., Lander Coll., Greenwood, S. C. *P. O. Box 111.*
WEBSTER, Asso. Prof. LOUISA M., M.S. (New York Univ.). Hunter Coll., New York, N. Y. *Hunter College, E. 68th St.*
WECHSLER, A. L., A.M. (Columbia). Instr., Univ. Extension, Columbia Univ., New York, N. Y. *610 Cathedral P'kway, Apt. 10D.*
WEDDERBURN, Prof. J. H. M., D.Sc. (Edinburgh). Princeton Univ., Princeton, N. J. *P. O. Box 53.*
WEEKS, EULA A., Ph.D. (Missouri). Cleveland High School, St. Louis, Mo. *5793 Kingsbury Pl.*

WEIDA, Asst. Prof. F. M., Ph.D. (Iowa). Montana State Coll., Bozeman, Mont. *P. O. Box 706.*
WEINBERG, Prof. E. F., C.E. (Manhattan). Rollins Coll., Winter Park, Fla. *P. O. Box 1693, Orlando, Fla.*
WEISNER, LOUIS, Ph.D. (Columbia). Instr., Univ. of Rochester, Rochester, N. Y.
WELCH, Asso. Prof. J. F., B.S. (Ohio State). La. State Univ., Baton Rouge, La. *P. O. Box 275.*
WELLING, W. C. Dir., Bur. of Vital Statistics, Dept. of Health, Hartford, Conn. *412 Farmington Ave.*
WELLS, F. A., B.S. (Virginia). Univ. of Virginia, Charlottesville, Va. *414 Park St.*
WELLS, Asso. Prof. MARY EVELYN, Ph.D. (Chicago). Vassar Coll., Poughkeepsie, N. Y. *Box 73.*
WELLS, Prof. R. A., A.M. (Franklin). Park Coll., Parkville, Mo.
WELLS, Asst. Prof. V. H., Ph.D. (Michigan). Williams Coll., Williamstown, Mass. *14 Southworth St.*
WERKMAN, MARY REICHELDERFER, M.S. (Chicago). St. Xavier Coll., Chicago, Ill. *4928 Xavier Pk.*
WEST, Mrs. GRACE R., B.S. (N. Tex. State Teachers Coll.). Teacher, High School, Austin, Tex. *711 W. 22d St.*
WEST, Asst. Prof. J. M., A.M. (Hamilton). Penn. State Coll., State College, Pa. *10 Liberal Arts.*
WESTER, Prof. C. W., A.M. (Univ. of Washington). State Teachers Coll., Cedar Falls, Ia. *2312 Olive St.*
WESTFALL, BERTHA G., A.M. (Columbia). *52 Orchard St.*, Pleasantville, N. Y.
WESTFALL, Prof. W. D. A., Ph.D. (Göttingen). Univ. of Missouri, Columbia, Mo. *802 Richmond.*
WHALEY, MARY RUTH, A.B. (Smith). Registrar, Virginia Coll., Roanoke, Va.
WHEELER, A. H., A.M. (Clark). Teacher, North High School, Worcester, Mass. *16 Bellevue St.*
WHEELER, Asst. Prof. J. J., A.M. (Kansas). Univ. of Kansas, Lawrence, Kans. *1024 Ala. St.*
WHELAN, Prof. A. MARIE, Ph.D. (Johns Hopkins). Olivet Coll., Olivet, Mich.
WHITE, Prof. A. E., M.S. (Purdue). State Agric. Coll., Manhattan, Kans.
WHITE, Prof. C. E., A.M. (Indiana). Muskingum Coll., New Concord, Ohio.
WHITE, E. T., D.Paed. (Toronto). Math. Master, Normal School, London, Can.
WHITE, IDA, A.B. (Coll. of Idaho). Teacher, High School, Caldwell, Ida.
WHITE, Asst. Prof. MARION B., Ph.D. (Chicago). Carleton Coll., Northfield, Minn.
WHITED, WILLIS, M.E. (Ames). Advisory Bridge Engineer, State Department of Highways, Harrisburg, Pa. *202 Reily St.*
WHITFORD, Pres. A. E., A.M. (Wisconsin). Milton Coll., Milton, Wis.
WHITFORD, D. E., A.M. (Brown). Grad. School, Harvard Univ., Cambridge, Mass. *31 Conant Hall.*
WHITFORD, Asst. Prof. E. E., Ph.D. (Columbia). Coll. of the City of New York, New York, N. Y.
WHITING, MABEL G., A.M. (Oberlin). Teacher, Jr. Coll., Santa Ana, Calif. *506 E. Chestnut Ave.*
WHITMAN, Asst. Prof. E. A., A.M. (Pittsburgh). Carnegie Inst. of Tech., Pittsburgh, Pa. *623 Kirtland St.*
WHITNEY, Prof. C. S., A.B. (Oklahoma). N. E. Okla. Junior Coll., Miami, Okla. *Box 553.*
WHITNEY, J. D., A.B. (Oklahoma). Fellow, Univ. of Oklahoma, Norman, Okla. *P. O. Box 692.*
WHITON, Asso. Prof. EMMA K., A.M. (California). Mills Coll., Mills College, Calif.
WHITTED, Prof. J. A., A.M. (Southwestern). Ohio Northern Univ., Ada, Ohio. *927 S. Main St.*
WHITTEMORE, Asst. Prof. J. K., A.M. (Harvard). Yale Univ., New Haven, Conn. *45 Lincoln St.*
WHYBURN, Asst. Prof. W. M., A.M. (Texas). Pure Math., Texas A. & M. Coll., College Station, Texas.
WIFVAT, S. J. A., S.B. (Drake). Instr., Drake Univ., Des Moines, Ia.
WILBUR, W. E., M.S. (Maine). Instr., Univ. of New Hampshire, Durham, N. H.
WILCOX, Asst. Prof. H. B., M.S. (Minnesota). Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.
WILCZEWSKI, Prof. JOSEPH, A.M. (St. Louis Univ.). St. Xavier Coll., Cincinnati, Ohio.
WILCZYNSKI, Prof. E. J., Ph.D. (Berlin). Univ. of Chicago, Chicago, Ill.
WILDER, Asst. Prof. C. E., Ph.D. (Harvard). Dartmouth Coll., Hanover, N. H.
WILDER, G. F. Teacher, Erasmus Hall High Sch., Brooklyn, N. Y.
WILDER, Asst. Prof. R. L., Ph.D. (Texas). Ohio State Univ., Columbus, Ohio. *440 E. Maynard Ave.*
WILDERMUTH, Asso. Prof. R. B., A.M. (Ohio State). Capital Univ., Columbus, Ohio.
WILEY, Prof. F. B., Ph.D. (Chicago). Denison Univ., Granville, Ohio. *On leave 1924–25, Robert Coll., Constantinople, Turkey.*

WILEY, W. O. Treas., John Wiley & Sons, Inc., New York, N. Y. *440 Fourth Ave.*
WILKINS, P. D., A.B. (Bowdoin). Instr., Tufts Coll., Tufts College, 57, Mass.
WILLETT, Prof. H. C., A.M. (So. Calif.). Univ. of Southern California, Los Angeles, Calif.
WILLIAMS, A. R., Ph.D. (California). Instr., Univ. of California, Berkeley, Calif. *416 Y.M.C.A.*
WILLIAMS, Prof. F. B., Ph.D. (Clark). Clark Univ., Worcester, Mass.
WILLIAMS, F. G., A.M. (Penn. State). Penn. State Coll., State Coll., Pa. *Box 574.*
WILLIAMS, Prof. J. E., Ph.D. (Virginia). Virginia Poly. Inst., Blacksburg, Va.
WILLIAMS, Prof. K. P., Ph.D. (Princeton). Indiana Univ., Bloomington, Ind. *523 E. 3d St.*
WILLIAMS, Dean L. E., A.M. (Chicago). Woman's Coll. of Alabama, Montgomery, Ala.
WILLIAMS, Adj. Prof. W. L., A.M. (South Carolina). Univ. of South Carolina, Columbia, S. C. *1012 Marion St.*
WILLIAMSON, Asso. Prof. C. O., M.S. (Ohio Univ.). Applied Math., Coll. of Wooster, Wooster, Ohio. *1141 Beall Ave.*
WILLIS, Miss RUBY, A.B. (Wellesley). Instr., Wellesley Coll., Wellesley, Mass.
WILLSON, Prof. F. N., C.E. (Rensselaer). Princeton Univ., Princeton, N. J.
WILMER, F. L. *1104 N. 16th St.*, Omaha, Nebr.
WILSON, Asso. Prof. A. H., Ph.D. (Chicago). Haverford Coll., Haverford, Pa.
WILSON, EDWARD B., B.S. (Chicago). Ripley, Tenn.
WILSON, Prof. EDWIN B., Ph.D. (Yale). Vital Statistics, Harvard School of Public Health, Boston, 17, Mass.
WILSON, ELIZABETH W., A.M. (Radcliffe). Teacher, Central High School, Washington, D. C. *1619 R St., N. W.*
WILSON, Asso. Prof. W. A., Ph.D. (Yale). Yale Univ., New Haven, Conn. *1960 Chapel St.*
WILSON, Asst. Prof. W. H., Ph.D. Univ. of Iowa. *On leave, 1198 Front St. N., Niles, Mich.*
WINBIGLER, Prof. ALICE, A.M. (Monmouth). Monmouth Coll., Monmouth, Ill.
WINKELMAN, Rev. G. L., A.M. (St. John's Univ.). Head of Dept., St. John's Univ., Collegeville, Minn.
WINTERS, Asst. Prof. F. W., A.M. (Harvard). Miami Univ., Oxford, Ohio. *111 N. Univ. St.*
WITT, R. E., C.E. (W. and L.). Instr. in Washington and Lee Univ., and Civil Engr., Lexington, Va.
WOLFE, Asst. Prof. CLYDE, Ph.D. (California). California Inst. of Tech., Pasadena, Calif.
WOLFE, Asst. Prof. H. E., Ph.D. (Indiana). Indiana Univ., Bloomington, Ind. *314 N. Washington St.*
WOLFE, S. MILDRED, B.S. (Penna.). Grad. Student, Univ. of Pennsylvania. *838 Old Lancaster Rd., Bryn Mawr, Pa.*
WONG, B. C., Ph.D. (California). Asst., Univ. of California, Berkeley, Calif. *1933 Grant St.*
WOOD, Prof. FREDRICK, Ph.D. (Wisconsin). Lake Forest Coll., Lake Forest, Ill.
WOOD, Asst. Prof. F. E., Ph.D. (Chicago). Northwestern Univ., Evanston, Ill.
WOOD, META A., A.M. (Wisconsin). Instr., Synodical Coll., Fulton, Mo.
WOOD, P. W., M.A. (Cambridge Univ.). Fellow, Tutor, Libr. and Math. Lecturer, Emmanuel Coll., Cambridge, Eng.
WOOD, ROSE B., A.B. (Barnard). Greenville Woman's Coll., Greenville, S. C.
WOOD, Prof. RUTH G., Ph.D. (Yale). Smith Coll., Northampton, Mass. *249 Crescent St.*
WOOD, R. R., B.S. (Haverford). Riverton, N. J.
WOODARD, Prof. D. W., M.S. (Chicago). Howard Univ., Washington, D. C.
WOODMANSEE, Prof. W. R., A.M. (Wisconsin). Ripon Coll., Ripon, Wis.
WOODS, Prof. F. S., Ph.D. (Göttingen). Mass. Inst. of Tech., Cambridge, Mass.
WOODS, ROSCOE, Ph.D. (Illinois). Asso., State Univ. of Iowa, Iowa City, Ia.
WOODYARD, ELLA, A.B., B.Pd. (Baker). Teachers Coll., Columbia Univ., New York, N. Y.
WOOLARD, E. W., A.B. (George Washington). Meteorologist, U. S. Weather Bureau, Washington, D. C.
WORDEN, ORPHA E., A.M. (Columbia). Head of Dept., Detroit Teachers Coll., Detroit, Mich.
WORK, ISABEL, A.B. (Oklahoma). Asst. Instr., S. E. State Teachers Coll., Durant, Okla.
WORTHINGTON, EUPHEMIA R., Ph.D. (Yale). So. Branch, Univ. of California, Los Angeles, Calif. *855 N. Vermont Ave.*
WRIGHT, FRANCES W., A.M. (Brown). Instr., Elmira Coll., Elmira, N. Y.
WRIGHT, Dean H. N., Ph.D. (California). Earlham Coll., Richmond, Ind. *103 S. W. Fifth St.*
WRIGHT, VERA L., A.M. (Minnesota). Instr., Martha Washington Coll., Abingdon, Va.
WRIGHT, Prof. W. L., A.M. (Princeton). Lincoln Univ., Lincoln University, Pa.

WUNDER, C. N., Ph.D. (Virginia). Dean and Prof. of Math., Southwestern Univ., Georgetown, Texas.

WYLIE, C. C., Ph.D. (Illinois). Asso. in Astronomy, Univ. of Illinois Observatory, Urbana, Ill.

YANCEY, Prof. W. P., A.M. (Woodstock). St. Ambrose Coll., Davenport, Ia.

YANNEY, Prof. B. F., Ph.D. (Chicago). The Coll. of Wooster, Wooster, Ohio. *666 N. Bever St.*

YANOSIK, G. A., C.E. (New York Univ.). Instr., New York Univ. *167 Woodland Ave., Yonkers, N. Y.*

YEATON, Asst. Prof. C. H., Ph.D. (Chicago). Oberlin Coll., Oberlin, Ohio. *106 South Cedar Ave.*

YODER, Prof. P. R., A.M. (Kansas). Blue Ridge Coll., New Windsor, Md.

YOTHERS, J. L., A.M. (Otterbein). Head of Dept., Coe Coll., Cedar Rapids, Ia.

YOUNG, A. L., M.S. (Ames). Instr., Univ. of Illinois, Urbana, Ill. *709 Indiana Ave.*

YOUNG, Asst. Prof. JESSICA M., Ph.D. (California). Washington Univ., St. Louis, Mo. *4511 N. 20th St.*

YOUNG, Prof. J. W., Ph.D. (Cornell). Dartmouth Coll., Hanover, N. H.

YOUNG, Asso. Prof. J. W. A., Ph.D. (Clark). Univ. of Chicago, Chicago, Ill. *5422 Blackstone Ave.*

YOUNG, Asso. Prof. MABEL M., Ph.D. (Johns Hopkins). Wellesley Coll., Wellesley, Mass. *6 Norfolk Terr.*

YOUNG, VIVIAN, A.B. (Williamette Univ.). *301 W. 5th St.*, Aberdeen, Wash.

YOWELL, E. I., C.E., Ph.D. (Cincinnati). Astronomer, Univ. of Cincinnati, Cincinnati, Ohio.

ZEHRING, Asso. Prof. W. A., A.M. (Otterbein). Purdue Univ., W. LaFayette, Ind. *303 Russell St.*

ZELDIN, S. D., Ph.D. Instr., Mass. Inst. of Tech., Cambridge, Mass.

ZIMMERMAN, Prof. JOHN, B.S. (Princeton). Univ. of Dubuque, Dubuque, Ia. *1089 Glen Oak Ave.*

ZIWET, Prof. A., C.E. (Karlsruhe). Univ. of Mich., Ann Arbor, Mich. *532 Packard St.*

INSTITUTIONAL MEMBERS.

The College of the Ozarks, Clarksville, Ark.
University of Arkansas, Fayetteville, Ark.
Little Rock College, Little Rock, Ark.
Occidental College, Los Angeles, Calif.
Southern Branch, University of California, Los Angeles, Calif.
University of Southern California, Los Angeles, Calif.
California Institute of Technology, Pasadena, Calif.
University of Colorado, Boulder, Colo.
Wesleyan University, Middletown, Conn.
The University of Delaware, Newark, Del.
George Washington University, Washington, D. C.
University of Georgia, Athens, Ga.
Georgia School of Technology, Atlanta, Ga.
Illinois Wesleyan University, Bloomington, Ill.
St. Viator College, Bourbonnais, Ill.
Armour Institute of Technology, Chicago, Ill.
De Paul University, Chicago, Ill.
St. Xavier College, Chicago, Ill.
University of Chicago, Chicago, Ill.
Northwestern University, Evanston, Ill.
Knox College, Galesburg, Ill.
University of Illinois, Urbana, Ill.
Indiana University, Bloomington, Ind.
Purdue University, LaFayette, Ind.
University of Notre Dame, Notre Dame, Ind.
St. Mary-of-the-Woods College, St. Mary-of-the-Woods, Ind.
Iowa State College, Ames, Ia.
Drake University, Des Moines, Ia.
Columbia College, Dubuque, Ia.
State University of Iowa, Iowa City, Ia.
St. Benedict's College, Atchison, Kans.
University of Kansas, Lawrence, Kans.
State Agricultural College, Manhattan, Kans.
Louisiana State Normal College, Natchitoches, La.
Loyola University, New Orleans, La.
Boston University, Boston, Mass.
Harvard College, Cambridge, Mass.
Wellesley College, Wellesley, Mass.
Worcester Polytechnic Institute, Worcester, Mass.
Albion College, Albion, Mich.
University of Michigan, Ann Arbor, Mich.
Michigan Agricultural College, East Lansing, Mich.
Kalamazoo College, Kalamazoo, Mich.
Michigan State Normal College, Ypsilanti, Mich.
St. Olaf College, Northfield, Minn.
St. Benedict's College, St. Joseph, Minn.
College of St. Teresa, Winona, Minn.
Blue Mountain College, Blue Mountain, Miss.
Culver-Stockton College, Canton, Mo.
Stephens Junior College, Columbia, Mo.
University of Missouri, Columbia, Mo.
Central College, Fayette, Mo.
Northwest Missouri State Teachers College, Maryville, Mo.
Washington University, St. Louis, Mo.
University of Montana, Missoula, Mont.

INSTITUTIONAL MEMBERS (*continued*).

University of Nebraska, Lincoln, Nebr.
Creighton University, Omaha, Nebr.
New Hampshire College, Durham, N. H.
Dartmouth College, Hanover, N. H.
The New Jersey College for Women, New Brunswick, N. J.
Rutgers College, New Brunswick, N. J.
Princeton University, Princeton, N. J.
New Mexico Normal University, Las Vegas, N. M.
New York State College for Teachers, Albany, N. Y.
Adelphi College, Brooklyn, N. Y.
Canisius College, Buffalo, N. Y.
University of Buffalo, Buffalo, N. Y.
Hamilton College, Clinton, N. Y.
Elmira College, Elmira, N. Y.
College of Mount St. Vincent, Mount St. Vincent-on-Hudson, N. Y.
College of the City of New York, New York, N. Y.
Columbia University, New York, N. Y.
The Cooper Union, New York, N. Y.
New York University, New York, N. Y.
University of Rochester, Rochester, N. Y.
Union College, Schenectady, N. Y.
United States Military Academy, West Point, N. Y.
University of Cincinnati, Cincinnati, Ohio.
Case School of Applied Science, Cleveland, Ohio.
Western Reserve University, Cleveland, Ohio.
Ohio Wesleyan University, Delaware, Ohio.
Kenyon College, Gambier, Ohio.
Denison University, Granville, Ohio.
Oberlin College, Oberlin, Ohio.
The Western College for Women, Oxford, Ohio.
Otterbein College, Westerville, Ohio.
The College of Wooster, Wooster, Ohio.
Antioch College, Yellow Springs, Ohio.
University of Oklahoma, Norman, Okla.
Lafayette College, Easton, Pa.
Seton Hill College, Greensburg, Pa.
Allegheny College, Meadville, Pa.
The Drexel Institute, Philadelphia, Pa.
Swarthmore College, Swarthmore, Pa.
Washington and Jefferson College, Washington, Pa.
Providence College, Providence, R. I.
Northern Normal and Industrial School, Aberdeen, S. Dakota.
Vanderbilt University, Nashville, Tenn.
University of Texas, Austin, Tex.
East Texas Normal College, Commerce, Tex.
Southern Methodist University, Dallas, Tex.
Baylor University, Waco, Tex.
University of Utah, Salt Lake City, Utah.
Middlebury College, Middlebury, Vt.
Virginia Polytechnic Institute, Blacksburg, Va.
University of Virginia, University, Va.
College of William and Mary, Williamsburg, Va.
University of Washington, Seattle, Wash.
Bethany College, Bethany, W. Va.
University of Wyoming, Laramie, Wyo.

GEOGRAPHICAL DISTRIBUTION OF INDIVIDUAL MEMBERS.

UNITED STATES AND CANADA.

ALABAMA. (11)

AUBURN. Crenshaw, Killebrew, R. S. Underwood.
BIRMINGHAM. Eagles, W. A. Moore, Ordway.
FLORENCE. Culmer.
MARION. Lee.
MONTGOMERY. L. E. Williams.
ROANOKE. Mickle.
UNIVERSITY. Dahlene.

ARIZONA. (4)

FLAGSTAFF. Lampland.
TUCSON. Cresse, Cruse, H. B. Leonard.

ARKANSAS. (9)

ARKADELPHIA. Mooney.
CLARKSVILLE. Hillard.
FAYETTEVILLE. A. D. Campbell, Droke, A. M. Harding, J. C. Hughes.
LITTLE ROCK. Bigbee, T. F. Smith.
SEARCY. B. R. Allen.

CALIFORNIA. (81)

ATASCADERO. H. E. Anderson.
BAKERSFIELD. Powelson, Sagen.
BERKELEY. Bernstein, Cajori, M. W. Haskell, Irwin, Lehmer, A. L. McCarty, E. J. McFarland, Noble, T. M. Putnam, Saund, Schmiedel, P. Sperry, A. R. Williams, Wong.
BUENA PARK. Schofield.
CHICO. Iloff.
CLAREMONT. Belcher, G. E. Berry, Brackett, W. P. Russell.
DAVIS. Titus.
DEEP SPRINGS. Donnelly.
FRESNO. F. R. Morris.
FRUITVALE. F. M. Gibson.
FULLERTON. L. E. Reynolds.
HOLLYWOOD. J. R. Campbell.
LA JOLLA. G. F. McEwen.
LOS ANGELES. Basoco, A. C. Clark, Collier, Daus, Gilbert, Glazier, Hedrick, G. H. Hunt, James, Parsons, Sherwood, Showman, D. V. Steed, Temple, Touton, Willett, Worthington.
MARE ISLAND. See.
MILLS COLLEGE. Alderton, Whiton.
MODESTA. Pobanz.
MT. HAMILTON. Jeffers.
OAKLAND. G. E. Allen.
ONTARIO. P. E. Clark.
PALO ALTO. Hoskins, Stromquist.
PASADENA. Bateman, Birchby, Demaree, H. C. Hicks, Van Buskirk, Wear, C. Wolfe.
PIEDMONT. Stafford.
POINT LOMA. Dick.
REDLANDS. Albert.
SACRAMENTO. Wallace.
SALINAS. Walter.
SAN DIEGO. Klauber, Livingston.
SAN FRANCISCO. Libby.
SANTA ANA. Whiting.
SANTA MARIA. Funk.
SOUTH PASADENA. K. M. Meek.
STANFORD UNIVERSITY. Blichfeldt, Green, Moreno.
STOCKTON. Corbin.
VENICE. W. A. Austin.
WHITTIER. Skarstedt, Van Benschoten.

CANADA. (42)

COBURG. Wagar.
EDMONTON SOUTH. J. W. Campbell, Sheldon.
HALIFAX. F. H. Murray.
KINGSTON. Gummer, K. P. Johnston, Matheson, N. Miller, Twiss.
KITCHENER. D. Forsyth.
LENNOXVILLE. A. V. Richardson.
LONDON. Kingston, Patterson, E. T. White.
MONTREAL. Beaupré, Léveillé, D. A. Murray, Pelletier.
OTTAWA. Dube, Henroteau.
POINT GREY. E. E. Jordan.
PORT COLBORNE. Kinnear.
QUEBEC. Tremblay.
SASKATOON. Ling.
STRATHCONA. R. S. Sheppard.
TORONTO. Beatty, De Lury, Fields, Findlay, Keith, Mackenzie, Pounder, Rosebrugh, N. E. Sheppard, Skirrow.
VANCOUVER. D. Buchanan, Dederick, G. E. Robinson.
WINNIPEG. Darracott, MacLean, Warren.
WOLFVILLE. Coit.

COLORADO. (28)

BOULDER. De Long, Edgecomb, Hutchinson, Kendall, Lester, Light, W. K. Nelson.
CANON CITY. McNatt.

Colorado Springs. Albright, Loud, Lovitt, Sisam, Wapple.
Denver. E. L. Brown, F. L. Carmichael, P. Fitch, Gorrell, H. A. Howe, Odell, H. E. Russell.
Fort Collins. S. L. Macdonald.
Golden. J. R. Everett, Martinson, Risley.
Greeley. Finley, R. D. Perry.
Loretto. Bernadita.
Pueblo. Deesz.

Connecticut. (32)

Greenwich. R. H. Hall.
Hartford. A. H. Andrews, Burkett, Dadourian, C. H. Davis, Elston, Flynn, Welling.
Lakeville. Creelman.
Middletown. H. E. Arnold, B. H. Camp, Howland, Raynor.
Milford. Rosenbaum.
New Haven. Barney, E. W. Brown, H. T. Burgess, Kovarik, Longley, Mikesh, Miles, E. S. Palmer, C. A. Shook, P. F. Smith, Tracey, Uhler, Whittemore, W. A. Wilson.
New London. Carlen, Leib.
Sales Ferry. Dimick.
Stamford. T. H. Brown.

Delaware (3)

Newark. Harter, C. J. Rees, C. A. Short.

District of Columbia. (40)

Brookland. Landry.
Chevy Chase. Heal.
Washington. O. S. Adams, Archer, Arnaud, K. S. Arnold, Ashmun, Avers, L. A. Bauer, Beall, Bigelow, Bixby, Borden, Breit, Cromwell, Darling, H. English, Erwin, Federico, Ficken, J. F. Fox, Garner, W. M. Hamilton, J. P. Hill, Hodgkins, Ingels, Keulegan, W. D. Lambert, Mangold, McKnight, Mourhess, Ramler, J. N. Rice, Roeser, Ross, Van Orstrand, C. B. Watts, E. W. Wilson, Woodard, Woolard.

Florida. (7)

Apopka. Larson.
Gainesville. Chandler, Hale, Higgins, T. M. Simpson.
Tallahassee. Elmer R. Smith.
Winter Park. Weinberg.

Georgia. (24)

Athens. Barrow, R. P. Stephens.
Atlanta. P. L. Armstrong, Fulmer, Morton, Skiles, D. M. Smith.
Augusta. H. A. Robinson.
Carlton. A. H. Stevens.
Decatur. F. Field, Gaylord, W. W. Rankin.
Emory University. Messick, Peed, Rumble.
Forsyth. Baten.
Gainesville. Allison, H. W. Smith.
La Grange. E. A. Bailey.
Macon. Hinton, Holder.
Rome. Hightower.
Surrency. Brindle.
Waycross. A. R. Walton.

Hawaii. (1)

Honolulu. Donaghho.

Idaho. (3)

Caldwell. J. M. Rankin, I. White.
Moscow. E. Taylor.

Illinois. (128)

Bellwood. M. Hildebrandt.
Bloomington. M. Hunt, Pettit.
Bluffs. C. C. Carter.
Carlinville. McMichael, Renner.
Carthage. Van Velzer.
Champaign. B. M. Armstrong, Stetson.
Charleston. Modesitt, E. H. Taylor.
Chicago. Barnard, W. S. Beckwith, Bliss, P. Butler, E. L. Carr, Cobb, Dickson, Z. Ferguson, Georges, Granville, Jarrett, J. H. Jones, Kinney, Krathwohl, Kurzin, Lane, Laves, Logsdon, A. C. Lunn, MacMillan, Miser, E. H. Moore, F. R. Moulton, Muehlman, G. W. Myers, Nicolet, Nyberg, C. I. Palmer, Pettersen, Phalen, Plapp, Riley, Scharf, Schottenfels, Schweitzer, Simmons, Slaught, Stahl, Stark, E. L. Thompson, Werkman, Wilczynski, J. W. A. Young.
Decatur. Kiefer.
De Kalb. Parson.
Eureka. Newson.
Evanston. D. F. Campbell, Connelly, Curtis, Curtiss, Garabedian, Holgate, R. L. Jackson, E. J. Moulton, Newell, Vass, E. M. Weaver, F. E. Wood.
Freeport. Eichelberger, Mensenkamp.
Galesburg. Calkins, Heren, Sellew, M. E. Taylor.
Greenville. M. G. Smith.
Jacksonville. M. Anderson, H. W. English, G. H. Scott.
Johnston City. Giesecke.
Kenilworth. Furrey, C. H. Jones.
Lake Forest. M. M. Johnson, F. Wood.
La Salle. Carus.
Lebanon. Stowell.
Lincoln. Denny.
Lisle. Fleisig.
Macomb. Ginnings.
Maywood. Schreiber.

MONMOUTH. Winbigler.
MOUNT MORRIS. L. S. Shively.
NAPERVILLE. Coultrap.
OAK PARK. Escott.
ORLEANS. J. T. Holmes.
PEORIA. Comstock, Gault.
RIVER FOREST. Dobbin.
ROCKFORD. McGavock, B. I. Miller.
ROCK ISLAND. Cederberg.
TAYLORVILLE. Dappert.
URBANA. Bagby, H. W. Bailey, Bender, Blumberg, C. C. Camp, R. D. Carmichael, Coble, Crathorne, Emch, Graesser, M. G. Haseman, Hoover, F. E. Johnston, Kempner, Libman, Lytle, R. M. Mathews, G. A. Miller, Morril, Mouzon, Steimley, M. M. Torrey, Townsend, Wylie, A. L. Young.
WHEATON. Garlough.

INDIANA. (44)

BLOOMINGTON. H. T. Davis, Davisson, Hanna, Hennel, Rothrock, K. P. Williams, H. E. Wolfe.
COLUMBIA CITY. Knisely.
CRAWFORDSVILLE. Carscallen, Cragwall.
DANVILLE. L. R. Cole.
EARLHAM. F. Long.
GOSHEN. Lehman.
GREENCASTLE. Arnold, Greenleaf.
HANOVER. G. E. Moore.
HARTFORD CITY. Clevenger.
INDIANAPOLIS. Aley, Banes, E. N. Johnson, Lutz.
LAFAYETTE. Marshall, Mason.
NORTH MANCHESTER. Dotterer.
NOTRE DAME. Caparo, Hull, McCue, Maurus.
RICHMOND. E. D. Grant, H. N. Wright.
TERRE HAUTE. Kennedy, Sousley.
WEST LAFAYETTE. J. C. Bennett, E. M. Berry, Doan, Edington, Graves, Hadley, Hazard, Hodge, Knox, Robbins, R. B. Stone, Zehring.

IOWA. (56)

AMES. Agnew, Brandner, Colpitts, Daniells, Ernsberger, Fleming, Gouwens, J. V. McKelvey, M. M. McKelvey, Pattengill, Roberts, Rogers, Sage, E. R. Smith, Snedecor, Tappan, J. S. Turner.
CEDAR FALLS. Condit, Daugherty, Wester.
CEDAR RAPIDS. L. M. Coffin, Yothers.
DAVENPORT. Yancey.
DES MOINES. Blom, Corey, Moon, Neff, Wifvat.
DUBUQUE. Theobald, Zimmerman.
FAYETTE. Deming.
GRINNELL. McClenon, Rusk.
HOPKINTON. Earhart.
HUBBARD. Reese.
INDIANOLA. C. W. Emmons.
IOWA CITY. Baker, Chittenden, Gwinn, H. K. Hughes, Reilly, Rietz, R. Woods.
LAMONI. Mortimore.
LE MARS. Kime.
MASON CITY. George.
MT. PLEASANT. King.
MOUNT VERNON. McGaw, Moots.
MUSCATINE. Stein.
OSKALOOSA. S. M. Hadley.
SAC CITY. Dickey.
SIOUX CITY. Graber, Van Horne.
UNIVERSITY PARK. Overstreet.
WELLMAN. Kreth.

KANSAS. (44)

BALDWIN CITY. Garrett.
EMPORIA. Rumney.
HAYS. Colyer.
HIGHLAND. T. B. Henry.
IOLA. R. H. Carpenter, F. L. Marshall.
KANSAS CITY. Dougherty, Helwig.
LAWRENCE. Ashton, Babcock, Black, H. E. Jordan, Lefschetz, Marm, U. G. Mitchell, Reagan, G. W. Smith, Stouffer, H. M. Walker, J. J. Wheeler.
MANHATTAN. W. H. Andrews, Hyde, Janes, C. F. Lewis, Lyons, Mossman, Remick, Stratton, A. E. White.
NEWTON. Richert.
OTTAWA. J. L. Bennett, Loewen.
PARSONS. Farner.
PITTSBURGH. W. H. Hill, Hunter, Shirk.
ST. MARY'S. Case.
STERLING. T. Bell.
TOPEKA. Harshbarger, McLatchey.
WELLINGTON. Lynch.
WICHITA. Hoare, W. O. Mendenhall.
WINFIELD. H. S. Myers.

KENTUCKY. (15)

BEREA. Guilliams, Peck.
DANVILLE. Crooks, Fehn.
GEORGETOWN. B. Hall, C. H. Richardson.
LEXINGTON. Andrew, Boyd, J. M. Davis, Downing, Le Stourgeon, E. L. Rees.
PAINTSVILLE. Backer.
STANFORD. E. V. Watts.
WILLIAMSBURG. Vallandingham.

LOUISIANA. (33)

ALEXANDRIA. Touchstone.
BATON ROUGE. J. P. Cole, Daspit, Droz, Guilbeau, Nichols, O'Quinn, S. T. Sanders, Shoptaugh, Webber, Welch.
CLINTON. Petty.
ELTON. B. A. Tucker.

HOMER. Z. Allen.
LAFAYETTE. Stokes.
NAPOLEONVILLE. Lastrapes.
NATCHITOCHES. Frary, Maddox, Sparks.
NEW ORLEANS. H. E. Buchanan, Cope, Daly, Dinwiddie, A. M. Howe, Latimer, Many, J. H. Meyer, Spencer.
PINEVILLE. C. D. Smith.
RUSTON. Pugh.
SHREVEPORT. Hardin.
VIVIAN. J. H. Manning.
ZACHARY. Peters.

MAINE. (10)

BRUNSWICK. Hammond, Moody.
LEWISTON. Ramsdell.
ORONO. Bryan, Hadlock, J. N. Hart.
WATERVILLE. Ashcraft, B. E. Carter, Trefethen.
WOODLAND. A. S. Adams.

MARYLAND. (44)

ANNAPOLIS. Bingley, Bramble, J. A. Bullard, Capron, Clements, Dillingham, Eppes, Garrison, Gaver, Mayer, Rawlins, Robert, Root, J. B. Scarborough, Shenton, J. Tyler.
BALTIMORE. Bacon, Cohen, Harry, Hulburt, F. P. Lewis, L. T. Moore, F. Morley, Murnaghan, Musselman, C. A. Nelson, L. J. Reed, A. C. Robinson, Stanwick, Thomsen.
BRUNSWICK. C. W. R. Crum.
CHESTERTOWN. J. S. W. Jones.
COLLEGE PARK. Gwinner.
EMMITSBURG. Burke.
FREDERICK. L. O. Brown, Packer.
GLENCOE. A. A. Grant.
NEW WINDSOR. Yoder.
RIVERDALE. Le Lacheur, Somers.
ROLAND PARK. Morrow.
WESTMINSTER. B. I. Hart.
WOODSTOCK. E. C. Phillips, J. P. Smith.

MASSACHUSETTS. (90)

AMHERST. Esty, F. C. Moore, W. J. Newlin, Olds, C. S. Porter, W. R. Porter.
BELMONT. Douglass, Downey, Rutledge.
BOSTON. C. A. Benander, Brigham, Bruce, G. W. Evans, Gould, Laurentine, Mode, Osborne, M. M. Smith, Spear, Edwin B. Wilson.
BROOKLINE. A. L. Miller.
CAMBRIDGE. F. H. Bailey, Beatley, Birkhoff, Bradley, Burrows, Coolidge, W. L. Crum, Franklin, Graustein, E. V. Huntington, M. F. Jordan, Kellogg, Kennelly, C. L. E. Moore, Osgood, Passano, F. W. Perkins, H. B. Phillips, L. H. Rice, H. W. Tyler, Walsh, D. E. Whitford, F. S. Woods, Zeldin.
EVERETT. Bryant.
HOLYOKE. Moriarty.
HYDE PARK. Norwood.
LAWRENCE. Lord.
LEXINGTON. Ward.
LYNN. Maizlish.
NORTHAMPTON. S. R. Benedict, Munroe, Rambo, R. G. Wood.
NORTON. G. A. Shook.
PITTSFIELD. Washburne.
SOUTH HADLEY. Doak, Hazlett, E. N. Martin, S. E. Smith.
SPRINGFIELD. Hickox, Marsh.
TUFTS COLLEGE. Bush, Mergendahl, Ransom, Wilkins.
WATERTOWN. Haigler.
WELLESLEY. Anderton, Copeland, Merrill, C. E. Smith, Vivian, Willis, M. M. Young.
WILLIAMSTOWN. Agard, Hardy, V. H. Wells.
WOLLASTON. Gardner.
WORCESTER. E. C. Brown, Gay, Hollis, Kimball, Lazott, Melville, R. K. Morley, Phinney, H. Rice, A. H. Wheeler, F. B. Williams.

MICHIGAN. (66)

ALBION. H. Barton, F. E. Field, Sleight.
ALMA. Clack.
ANN ARBOR. Anning, Blessing, Blumenthal, Bradshaw, Coe, Denton, Dostal, P. Field, S. E. Field, W. B. Ford, J. W. Glover, A. G. Hall, T. H. Hildebrandt, Hopkins, M. F. Johnson, Karpinski, Kazarinoff, Love, Markley, A. L. Nelson, O. J. Peterson, Poor, C. Reid, Rood, Rouse, Running, Schorling, Shohat, Ziwet.
BAY CITY. Shellenberger.
DETROIT. Baldwin, Barlow, Darnell, Mullen, Worden.
EAST LANSING. Crowe, L. C. Emmons, V. G. Grove, R. N. Haskell, H. L. Olson, Plant, Speeker.
FENTON. Tryon.
HIGHLAND PARK. Thome.
HILLSDALE. Herron.
HOLLAND. Lampen.
KALAMAZOO. H. Blair, J. P. Everett, T. O. Walton.
MARQUETTE. Spooner.
MONROE. Paula.
MOUNT PLEASANT. Pearce.
NILES. W. H. Wilson.
OLIVET. Whelan.
ROYAL OAK. Schoonover.
STURGIS. Steirnagle.
YPSILANTI. Barnhill, Gee, Lindquist, Lyman, Matteson, Norton.

MINNESOTA. (42)

COLLEGEVILLE. Winklemann.

DULUTH. Brigetta.
HERON LAKE. L. E. Lunn.
MANKATO. Chapman, A. V. Robbins.
MINNEAPOLIS. Beal, Brink, Brooke, Bussey, Dalaker, Gibbens, W. L. Hart, Hartig, Herrick, D. Jackson, Kirchner, W. H. McEwen, McGuire, C. A. V. Peterson, Priester, Shuman, Shumway, H. L. Smith, Sublette, Thorp, Underhill, Warne, Wilcox.
MOOREHEAD. K. Leonard.
NORTHFIELD. Clement, Gingrich, Nordgaard, M. B. White.
ST. PAUL. R. A. Johnson, Kingery, Mickelson, Reuterdahl, Rupp, F. J. Taylor.
VIRGINIA. C. L. Hancock.
WINONA. Bogard, French.

MISSISSIPPI. (13)

A. AND M. COLLEGE. H. Fox, B. M. Walker.
BLUE MOUNTAIN. Hutchins.
CLINTON. Hitt.
GRENADA. Duncan.
GULFPORT. Mauldin.
HATTIESBURG. Sharp.
JACKSON. Babbitt, B. E. Mitchell.
JONESTOWN. Barr.
UNIVERSITY. Hume, R. Torrey.
WASHINGTON. Gwaltney.

MISSOURI. (43)

CAPE GIRARDEAU. B. F. Johnson, E. H. Thomas.
COLUMBIA. E. F. Allen, Callaway, Haynes, Ingold, Jaeger, P. H. Pepper, Wahlin, W. D. A. Westfall.
FULTON. Sweazey, M. A. Wood.
KANSAS CITY. Cutting, Epperson, Luby, Pierson.
KIRKSVILLE. Cosby, Jamison.
LIBERTY. Fleet.
PARKVILLE. R. A. Wells.
SPRINGFIELD. Finkel, W. N. Thompson.
ST. CHARLES. Karr.
ST. LOUIS. Ammerman, Brennan, A. Davis, Dunkel, Gerst, A. H. Huntington, Nauer, Osborn, Rider, P. Robertson, Roever, Ryan, Shannon, E. Stephens, Weeks, J. M. Young.
TARKIO. Jenison.
WARRENSBURG. J. H. Scarborough.
WARRENTON Knorr.
WEBSTER GROVES. M. B. Clarke.

MONTANA. (6)

BOZEMAN. Weida.
GLASGOW. Calderwood.
HELENA. O'Neill.
MISSOULA. Carey, Lennes, A. S. Merrill.

NEBRASKA. (22)

BETHANY. Osterhout, Sherer.
CERESCO. L. C. Walker.
CRETE. J. N. Bennett.
GIBBON. N. M. Johnston.
GRAND ISLAND. H. Anderson.
HASTINGS. McDill.
KEARNEY. Hanthorn.
LINCOLN. Brenke, Candy, Congdon, Gaba, M. F. Jackson, Kirker, Opp, T. A. Pierce.
OMAHA. H. A. Campbell, T. I. Porter, Wilmer.
PERU. A. L. Hill.
UNIVERSITY PLACE. Howie.
YORK. Feemster.

NEVADA. (2)

RENO. C. Haseman, Shirley.

NEW HAMPSHIRE. (15)

DOVER. Huntley.
DURHAM. G. N. Bauer, Slobin, Wilbur.
EXETER. Sweet.
HANOVER. Beetle, Bill, B. H. Brown, Cook, C. H. Forsyth, Mathewson, Morgan, Silverman, C. E. Wilder, J. W. Young.

NEW JERSEY. (32)

CLINTON. G. H. Hill.
CONVENT STATION. Maskell.
EAST ORANGE. Koch, Mallory, F. H. Robinson.
LAWRENCEVILLE. Durell.
LEONIA. Gafafer, M. S. Taylor.
MORRIS PLAINS. A. E. Johnson.
NEWARK. Conkling.
NEW BRUNSWICK. Carlson, Garretson, R. Morris, R. Thompson, A. A. Titsworth.
PATERSON. Caster.
PRINCETON. E. P. Adams, Alexander, Eisenhart, Fine, W. Gillespie, Hille, H. D. Thompson, Veblen, Wedderburn, Willson.
RIDGEWOOD. Phelps.
RIVERTON. R. R. Wood.
RUTHERFORD. McMackin.
SUMMIT. H. E. Webb.
TRENTON. Colliton.
WOODBURY. DoBell.

NEW MEXICO. (3)

ALBUQUERQUE. Barnhart.
EAST LAS VEGAS. Rodgers.
SOCORRO. Reece.

NEW YORK. (168)

ALBANY. Birchenough, G. M. Conwell.
ALFRED. Seidlin, W. A. Titsworth.
ANNANDALE-ON-HUDSON. Packard.

AURORA. Hollcroft.
BALDWIN. C. C. Grove.
BEECHHURST. E. Berger.
BROOKLYN. Angelica, Bergstresser, W. J. Berry, Bowden, Emery, Kreines, Lehmann, Locke, Rosanoff, Schuyler, Tanzola, G. F. Wilder.
BUFFALO. Harrington, T. H. Milne, Pound, Sherk.
CANTON. Earl.
CLINTON. H. S. Brown, Carruth, Ferry, A. L. Fitch.
DE KALB JUNCTION. H. M. Phillips.
DUNKIRK. Lufkin.
EAST ELMHURST. Hanson.
ELMHURST. Harper.
ELMIRA. Suffa, F. W. Wright.
FLUSHING. P. H. Graham, Oglesby.
GENEVA. W. H. Durfee, W. P. Durfee, Hubbs.
HAMILTON. A. W. Smith.
HEMPSTEAD. Coffin.
ITHACA. Boothroyd, Carver, Farnum, D. C. Gillespie, Hurwitz, MacCreadie, H. B. Meek, H. M. Morse, F. W. Owens, H. B. Owens, Poritsky, Ranum, Shaub, V. Snyder, Tanner, Waltz.
MOUNT VERNON. Breckenridge.
NEW YORK. J. Allen, Auerbach, Autenrieth, Ballantine, Berkeley, V. Blair, W. M. Bond, Brahdy, Brewster, Burdick, R. W. Burgess, G. A. Campbell, Chellborg, R. F. Clark, Dantzcher, C. H. Douglas, Eckersley, Edmonson, Fiske, Fite, Fort, Foster, Frankel, Fry, Hawkes, R. Henderson, Himwich, Hirsch, Hodgdon, Jablonower, Joffe, M. I. Johnson, Kasner, Kunte, Langellotti, Langman, Linehan, J. J. McCarthy, Meder, Merriman, Mirick, H. B. Mitchell, Molina, Mullins, Paaswell, Pedersen, Penn, Pooler, Post, Pride, R. G. Putnam, Reddick, Reeve, Ritt, Saurel, Schmall, Schub, Siceloff, Simons, D. E. Smith, R. F. Smith, R. R. Smith, Spies, W. A. Stevens, H. Thompson, Thorne, Tilly, A. B. Turner, Upton, Waldo, E. Walker, Webster, Wechsler, E. E. Whitford, W. O. Wiley, Woodyard.
OLEAN. Lowry.
PARISH. Church.
PLEASANTVILLE. B. G. Westfall.
POUGHKEEPSIE. Cowley, Cummings, M. E. Wells.
ROCHESTER. Betz, Gale, E. L. Hall, H. Harding, T. R. Long, Silberstein, Watkeys, Weisner.
SCARSDALE. MacNeish.
SCHENECTADY. Hussey, D. S. Morse, Newkirk, A. D. Snyder, Vedder.
SYRACUSE. W. G. Bullard, Carroll, Decker, Lindsey, Secy. Pi Mu Epsilon Frat., Roe, M. Sperry, W. E. Taylor.
TROY. Crockett.
WEST POINT. C. P. Echols.
YONKERS. Hubert, F. W. John, Yanosik.

NORTH CAROLINA. (22)

CHAPEL HILL. Browne, Cain, A. Henderson, M. A. Hill, Hobbs, Lasley.
DAVIDSON. J. L. Douglas.
DURHAM. M. R. Richardson, Robison.
ELON COLLEGE. Amick.
GREENSBORO. G. W. Mendenhall, F. S. Mitchell, Pegram, Strong.
GREENVILLE. M. D. Graham.
GUILFORD COLLEGE. R. L. Newlin, Pancoast.
JAMESTOWN. Ragsdale.
MURFREESBORO. Caldwell.
RALEIGH. Prosser.
WILMINGTON. H. B. Smith.
WINSTON-SALEM. F. H. Jackson.

NORTH DAKOTA. (9)

FARGO. Atwood, Duerner, Householder, I. W. Smith.
JAMESTOWN. T. W. Jackson.
MINOT. De La.
UNIVERSITY. Bibb, Hitchcock.
VALLEY CITY. J. B. Meyer.

OHIO. (95)

ADA. Fairchild, Whitted.
ALLIANCE. Trott.
ATHENS. Borger, F. W. Reed.
BEREA. Dustheimer.
BOWLING GREEN. Overman.
BLUFFTON. Hirschler.
CEDARVILLE. Diederich.
CHAUNCEY. Minister.
CHILLICOTHE. Cornetet.
CINCINNATI. I. A. Barnett, Brand, H. Hancock, Kindle, Lubin, C. N. Moore, E. S. Smith, Wilczewski, Yowell.
CLEVELAND. E. R. Beckwith, V. I. Benander, Focke, W. W. Johnson, B. W. Jones, Justin, McBane, J. E. Merrill, M. Morris, Nassau, Palmié, Simon, C. F. Thomas.
COLUMBUS. C. L. Arnold, Bareis, Bohannan, V. B. Caris, Cottingham, Harmount, Kuhn, MacDuffee, McCoy, Manson, C. C. Morris, Preston, Rasor, Rickard, Singer, R. L. Wilder, Wildermuth.
DAYTON. Hofmann.
DEFIANCE. A. G. Caris.
DELAWARE. G. N. Armstrong, Crane.
GAMBIER. R. B. Allen, Denston.
GRANVILLE. Lemon, F. B. Wiley.
HARRIETTSVILLE. G. S. Jones.
HILLIARD. J. H. Weaver.
HIRAM. E. H. Clarke.

KENT. Manchester.
MARIETTA. Coar, Rea.
NEWARK. Bumer.
NEW CONCORD. C. E. White.
OBERLIN. Cairns, Carr, Sinclair, Yeaton.
OXFORD. W. E. Anderson, Baudin, M. B. Carter, Lange, Schoonmaker, Sheets, Spenceley, Winters.
PAINESVILLE. Freas, A. D. Lewis.
ROSS. Haldeman.
SPRINGFIELD. Tripp.
STEUBENVILLE. Horn.
TIFFIN. J. Pierce.
TOLEDO. Brandeberry, Dancer, Mercedes.
WESTERVILLE. B. C. Glover.
WILBERFORCE. B. Sanders, Tinner, Waits.
WILMINGTON. Spinks.
WOOSTER. Williamson, Yanney.
YELLOW SPRINGS. W. A. Hamilton.

OKLAHOMA. (29)

CHICKASHA. Hawkins.
COALGATE. Ralls.
DALE. R. L. Hicks.
DUNCAN. McBee.
DURANT. A. Berger, Work.
EDMOND. Stewart.
HENRYETTA. Begley.
MIAMI. J. B. Steed, C. S. Whitney.
NORMAN. Court, Barbour, Hassler, D. McFarland, McGilvray, S. W. Reaves, Rohrbaugh, F. L. Smith, J. D. Whitney.
OKLAHOMA CITY. U. Butler, Cornell, Demaud, Meador, L. V. Robinson, Townes.
SHAWNEE. W. T. Short.
STILLWATER. Gundersen.
WEATHERFORD. McCormick.
WILSON. R. O. Webb.

OREGON. (11)

ALBANY. L. W. Moore.
ASTORIA. H. M. Manning.
CORVALLIS. Beaty, C. L. Johnson.
EUGENE. De Cou, McAlister, W. E. Milne, Smail.
PORTLAND. Griffin, Merriss, J. M. Short.

PENNSYLVANIA. (91)

ALLENTOWN. Bauman.
ANNVILLE. Redditt.
BEAVER FALLS. Cleland.
BETHLEHEM. Lyle, Rau, J. B. Reynolds.
BRYN MAWR. Pell, C. A. Scott, S. M. Wolfe.
CAMP HILL. Foberg.
CARLISLE. Landis.
COLLEGEVILLE. Clawson.
CYNWYD. Sensenig.
DEVON. J. A. Clarke.
EASTON. Benner, Doushkess, W. S. Hall, Hatch, W. M. Smith.
GETTYSBURG. Arms.
GROVE CITY. Ramsey.
HARRISBURG. Whited.
HAVERFORD. L. W. Reid, A. H. Wilson.
HUNTINGTON. C. S. Shively.
IRWIN. A. A. Jones.
LANCASTER. R. L. Charles, W. F. Long.
LANSDOWNE. Chambers, Glenn, Gummere.
LATROBE. Seubert.
LEWISBURG. Bartol, H. S. Everett, Gold, Lindemann.
LINCOLN UNIVERSITY. W. L. Wright.
LOCK HAVEN. High.
MEADVILLE. Akers, Wagner.
MECHANICSBURG. N. B. Freeman.
MILLERSVILLE. Seiverling.
MYERSTOWN. Kiess.
NEW WILMINGTON. McCain.
PHILADELPHIA. P. A. Caris, Crawley, J. E. Davis, Eshleman, H. B. Evans, Gehman, F. John, Kline, Levita, Linton, Partridge, Rittenhouse, Rosengarten, Safford.
PITTSBURGH. Baird, Barrett, Bishop, Burley, Geckeler, R. P. Johnson, J. H. Mathews, Riggs, Rosenbach, Simester, Swartzel, Taber, J. S. Taylor, Trytten, Whitman.
SOUTH BETHLEHEM. P. A. Lambert, MacNutt.
STATE COLLEGE. Bushyager, Gravatt, L. S. Johnston, E. D. McCarthy, Shibli, J. M. West, F. G. Williams.
SWARTHMORE. Marriott, J. A. Miller.
SWISSVALE. Foraker.
WASHINGTON. Atchison, Bert, Cardin, R. W. Thomas.
WEST PHILADELPHIA. Latshaw.
WEST PITTSTON. Templin.

PHILIPPINE ISLANDS. (2)

MANILA. Gokhale, Tienzo.

PORTO RICO. (1)

MAYAGUEZ. C. E. Horne.

RHODE ISLAND. (13)

PROVIDENCE. C. R. Adams, Archibald, Batchelder, Burwell, Chace, Currier, Gilman, Hickson, H. P. Manning, R. G. D. Richardson, Sauté, Suesman, Watt.

SOUTH CAROLINA. (14)

CHARLESTON. O. J. Bond, R. H. Coleman.
CLINTON. A. V. Martin.
COLUMBIA. Coker, J. B. Coleman, J. B. Jackson, W. L. Williams.
DOVESVILLE. P. K. Smith.

GREENVILLE. Earle, R. B. Wood.
GREENWOOD. Weber.
HARTSVILLE. C. M. Reaves.
ROCK HILL. Pope.
SALUDA. Ramage.

SOUTH DAKOTA. (10)

ABERDEEN. Mills.
BROOKINGS. I. L. Miller.
CANTON. E. J. Olson.
HURON. Titt.
RAPID CITY. Bowles, McLaury.
SIOUX FALLS. Hacker.
VERMILION. McKinney, Swanson.
YANKTON. Faught.

TENNESSEE. (15)

CHATTANOOGA. Hooper.
FAYETTEVILLE. Boyce.
JACKSON. Hess.
KNOXVILLE. J. D. Bond, Brezler, Ghormley.
MARYVILLE. Knapp.
NASHVILLE. R. V. Blair, S. I. Jones, Roman.
PULASKI. Feagan, Mize.
RIPLEY. Edward B. Wilson.
SEWANEE. S. M. Barton, Claytor.

TEXAS. (71)

ABILENE. Burnam.
AUSTIN. W. L. Ayres, H. Y. Benedict, A. A. Bennett, Cooper, Decherd, Dodd, Ettlinger, H. L. Holmes, Horton, Hulse, Jacobs, Lubben, Mayne, D. E. Mitchell, R. L. Moore, Phenix, M. B. Porter, P. K. Rees, W. A. Rees, Vandiver, G. R. West.
BEAUMONT. M. A. Campbell.
BOERNE. Hathaway.
BROWNSVILLE. de la Garza.
BROWNWOOD. Gayden, McClelland.
CANYON. L. G. Allen.
COLLEGE STATION. F. Ayres, A. A. Blumberg, Halperin, McKee, Whyburn.
COMMERCE. Cowling.
DALLAS. Dice, Hammer, Hartsfield, E. H. Jones, Mahoney, Seale.
DENTON. M. C. Brown, H. Porter.
FORNEY. Benson.
FORT WORTH. Estes, Hargett, Howard, E. R. Tucker.
GALVESTON. Burrell, P. H. Underwood.
GEORGETOWN. Wunder.
HOUSTON. Bray, Dean, G. C. Evans, L. R. Ford, E. O. Lovett, Michal.
LOCKNEY. Ewing.
MILFORD. Durham.
NACOGDOCHES. C. E. Ferguson.
PORT ARTHUR. G. S. Smith.
SAN ANGELO. Hagelstein.
SAN ANTONIO. McNelly, Roach.
SAN MARCOS. J. S. Brown, Sewell.
SHERMAN. May.
STEPHENVILLE. McSweeny, Redden.
WACO. Harrell, W. A. Nelson.
WICHITA FALLS. B. T. Adams.

UTAH. (8)

LEWISTON. Van Orden.
MORONI. Olsen.
PROVO. O. P. Barnett.
SALT LAKE CITY. J. L. Gibson, Horsfall, Marthakis, Pehrson, Unseld.

VERMONT. (7)

BURLINGTON. Donahue, Millington, Swift, E. Thomas.
MIDDLEBURY. Hazeltine, L. R. Perkins.
NORTHFIELD. Flanders.

VIRGINIA. (33)

ABINGDON. V. L. Wright.
ASHLAND. T. McN. Simpson.
BLACKSBURG. Brodie, Gudheim, O'Shaughnessy, J. E. Williams.
BRIDGEWATER. Shull.
CHARLOTTESVILLE. F. A. Wells.
CLIFTON STATION. O. Stone.
EAST RADFORD. Bowers.
EMORY. J. S. Miller.
FARMVILLE. Taliaferro.
HOLLINS. Dickinson.
LANGLEY FIELD. Hemke.
LEXINGTON. L. W. Smith, C. W. Watts, Witt.
LYNCHBURG. Larew, Pattillo, B. Russell.
MARION. Henna.
MONTEREY. Colaw.
RICHMOND COLLEGE. Gaines.
ROANOKE. Whaley.
SALEM. Carpenter.
SOUTH HILL. B. F. Walton.
SWEET BRIAR. Morenus, Searle.
UNIVERSITY. W. H. Echols, Luck, Thornton.
UNIVERSITY OF RICHMOND. I. Harris.
WILLIAMSBURG. Rowe.

WASHINGTON. (18)

ABERDEEN. V. Young.
CONNELL. Hays.
EVERETT. Robb.
LA CONNER. L. G. Butler.
PULLMAN. G. H. Freeman, Isaacs.
SEATTLE. E. T. Bell, Biggerstaff, Caffrey, Cramlet, Jerbert, Moritz, Mullemeister, Neikirk, Stager.
TACOMA. Hanawalt.
WALLA WALLA. Bratton, Eells.

West Virginia. (11)

Bethany. Cramblet.
Bridgeport. Slawter.
Huntington. Hackney.
Keyser. Fisher.
Morgantown. M. Buchanan, Colwell, H. A. Davis, Eiesland, C. N. Reynolds, B. M. Turner.
Williamson. Romig.

Wisconsin. (33)

Ashland. Kendrigan.
Beloit. H. H. Conwell, Huffer, Suydam.
Madison. F. E. Allen, Bunyan, Dowling, Dresden, Feltges, W. W. Hart, Ingraham, E. B. Miller, Pollard, Skinner, Slichter, Stout, Van Vleck, W. Weaver.
Milton. A. E. Whitford.
Milwaukee. Atwater, Ericson, P. H. Evans, Frumveller, Morrissy, Quarles, C. G. Simpson.
Platteville. Warner.
Ripon. Woodmansee.
River Falls. McMillan.
South Milwaukee. Hoar.
Superior. C. W. Smith.
West Allis. Roth.
West De Pere. DeCleene.

Wyoming. (4)

Laramie. Bellamy, Fitterer, Gossard, Rechard.

FOREIGN MEMBERS. (Other than Canada.)

Argentine. (2)

Buenos Aires. Baidaff, Broggi.

Belgium. (1)

Liege. Van Hee.

Bulgaria. (1)

Lovetch. E. M. Perry.

China. (7)

Canton. W. E. MacDonald.
Changsha. Leavens.
Peking. Heinz, Konantz.
Shanghai. Ely.
Tangshan. Patten.
Tientsin. Chin.

France. (5)

Besançon. Lebeuf.
Nancy. Gérardin.
Paris. Borel, Hadamard.
Strasbourg. Fréchet.

Germany. (1)

Göttingen. Coates.

Great Britain. (7)

Cambridge. Ball, P. W. Wood.
Edinburgh. Horsburgh.
Hove. Chepmell.
Oxford. Hardy, W. E. Robertson, Strom.

India. (4)

Allahabad City. Mitra.
Calcutta. Bose, Dey.
Vedaraniam. Ramana-Sastrin.

Italy. (6)

Bologna. Bortolotti, Enriques, Pincherle.
Catania. Cipolla.
Pisa. Bianchi.
Turin. Fubini.

Japan. (3)

Pyengyang. Parker.
Tokyo. Mikami, Ono.

New Zealand. (1)

Dunedin. Martyn.

Poland. (1)

Warsaw. Dickstein.

Portugal. (1)

Lisbon. da Cunha.

South Africa. (3)

Bloemfontein. Arndt.
Johannesburg. Dalton.
Rondebosch. Muir.

Spain. (1)

Madrid. de Toledo.

Sweden. (1)

Lund. Craig.

Switzerland. (3)

Fribourg. Bays.
Geneva. Fehr.
Neuchatel. DuPasquier.

Syria. (1)

Beirut. Jurdak.

Turkey. (2)

Constantinople. A. F. Johnson, Mourad.

Ukraine. (1)

Kieff. Kryloff.

RECAPITULATION OF MEMBERSHIP.

Individual members November 15, 1924	1,712	
Institutional members November 15, 1924	110	
Total membership November 15, 1924		1,822
Total membership November 1, 1922		1,476

CHARTER MEMBERSHIP.

Individual charter members	1,045	
Institutional charter members	52	
Total charter membership		1,097
Net gain in individual members	667	
Net gain in institutional members	58	
Total net gain over charter membership		725
Total net gain since November 1, 1922		346

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

(As amended and adopted by unanimous vote at a special meeting of members called for the purpose at Rochester, N. Y., September 7, 1922, a quorum being present.)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL.

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF TRUSTEES AND OFFICERS.

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President and Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief, the Manager and one other member, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be

adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-Presidents such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safe-keeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificate shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

Article IV—Meetings.

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for such meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections.

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings.

The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues.

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the listl after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to McClintock's Male Annuitant Table based upon four (4) per cent. interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shal, be made under the McClintock Male Four (4) Per Cent. Table; and the reserve thus computed shall be held as a liability.

Article VIII—Amendments to the Articles of Association and By-Laws.

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds (2/3) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION.

Presidents.

E. R. Hedrick	1916	G. A. Miller	1921
Florian Cajori	1917	R. C. Archibald	1922
E. V. Huntington	1918	R. D. Carmichael	1923
H. E. Slaught	1919	H. L. Rietz	1924
D. E. Smith	1920		

Vice-Presidents.

E. V. Huntington	1916	E. J. Wilczynski	1920
D. A. Miller	1916	R. C. Archibald	1921
G. N. Lehmer	1917, 1918	R. D. Carmichael	1921, 1922
Oswald Veblen	1917	B. F. Finkel	1922
J. W. Young	1918	A. B. Chace	1923
R. G. D. Richardson	1919	L. P. Eisenhart	1923
H. L. Rietz	1919	J. L. Coolidge	1924
Helen A. Merrill	1920	Dunham Jackson	1924

Secretary-Treasurer.

(Appointed by the Council or Board after 1918.)

W. D. Cairns..........1916–

Committee on Publications.

(Appointed by the Council or Board.)

H. E. Slaught	1916–
R. D. Carmichael	1916–1918
W. H. Bussey	1916–1918
R. C. Archibald	1919–1921
W. A. Hurwitz	1919–1921
A. A. Bennett	1922
H. P. Manning	1922
W. B. Ford	1923–
J. L. Coolidge	1923
A. J. Kempner	1924

Elected Members of the Council or Board.

D. N. Lehmer	1916–1918, 1922–	Helen A. Merrill	1917–1919
R. E. Moritz	1916–1918	D. E. Smith	1917–1919, 1921–
K. D. Swartzel	1916	Elizabeth B. Cowley	1918–1920
Oswald Veblen	1916, 1920–1922	G. A. Miller	1918–1920, 1922–
R. C. Archibald	1916–1917, 1923–	E. J. Wilczynski	1918–1919, 1922–
Florian Cajori	1916, 1918–1923	L. P. Eisenhart	1919–1922
M. B. Porter	1916–1917	E. V. Huntington	1917, 1919–
J. W. Young	1916–1917, 1920–1922	E. L. Dodd	1920
B. F. Finkel	1916–1921	R. D. Carmichael	1920, 1924–
E. H. Moore	1916–1921, 1923–	A. A. Bennett	1921
J. N. Van der Vries	1916–1918	H. L. Rietz	1921–1923
Alexander Ziwet	1916–1918	C. F. Gummer	1921–
E. R. Hedrick	1917–1922, 1924–	A. B. Chace	1924–
		Dunham Jackson	1923
		Clara E. Smith	1923–

INDEX TO VOLUME XXXI, 1924.

By H. S. EVERETT, Bucknell University.

Misprinted names in the text are corrected in this index.

PAPERS, REPORTS OF MEETINGS.

QUESTIONS AND DISCUSSIONS—DISCUSSIONS.

QUESTIONS AND DISCUSSIONS—QUESTIONS.

RECENT PUBLICATIONS—REVIEWS.

RECENT PUBLICATIONS—NOTES—BOOKS.

RECENT PUBLICATIONS—PERIODICALS.

MATHEMATICAL CLUBS—ACTIVITIES.

PROBLEMS—AUTHORS.

Numbers refer to pages, black-face type indicating a problem solved and solution published, italics a problem solved but solution not published, ordinary type a problem proposed.

PROBLEMS—SOLUTIONS.

Numbers in black-face type refer to problems, those in light-face to pages.

NOTES AND NEWS.

PERSONAL MENTION.

PERSONAL MENTION—NECROLOGY.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924
NUMBER 10, DECEMBER

PUBLISHED BY THE ASSOCIATION
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

$$\begin{aligned}\textstyle\sum[(\xi_2-\xi_1)a_k+(\eta_2-\eta_1)b_k+(\zeta_2-\zeta_1)c_k]^2 \\ &= (\xi_2-\xi_1)^2\textstyle\sum a_k^2+(\eta_2-\eta_1)^2\textstyle\sum b_k^2+(\zeta_2-\zeta_1)^2\textstyle\sum c_k^2 \\ &\quad +2(\xi_2-\xi_1)(\eta_2-\eta_1)\textstyle\sum a_kb_k+2(\xi_2-\xi_1)(\zeta_2-\zeta_1)\textstyle\sum a_kc_k \\ &\quad +2(\eta_2-\eta_1)(\zeta_2-\zeta_1)\textstyle\sum b_kc_k \\ &= (\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2,\end{aligned}$$

and so is the square of the distance P_1P_2. The other distances in the figure can be calculated similarly. By the law of cosines in the triangle P_1OP_2 (or by a familiar formula of solid analytic geometry), the cosine of the angle $\theta = P_1OP_2$ is found to be

$$\cos\theta = \frac{\overline{OP}_1^2+\overline{OP}_2^2-\overline{P_1P}_2^2}{2\overline{OP}_1\cdot\overline{OP}_2} = \frac{\sum x_k^2+\sum y_k^2-\sum(y_k-x_k)^2}{2\sqrt{\sum x_k^2\sum y_k^2}} = \frac{\sum x_ky_k}{\sqrt{\sum x_k^2\sum y_k^2}},$$

which, in the particular case that $\sum x_k = \sum y_k = 0$, is the coefficient of correlation of the sets of numbers $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$.

When distance and angle have been established, other geometrical measures can be worked out *ad libitum;* it would be superfluous to dwell on the details, beyond a reference to the discussion of the corresponding integral formulas in the preceding pages, and the paper T already cited. With the present definitions, a coefficient of partial correlation not merely resembles, but *is* the cosine of an angle between two planes, a coefficient of double correlation *is* the cosine of an angle between a line and a plane,[1] and, a little outside the ordinary notation of statistics, a Gramian determinant is found by actual calculation to be equal to the square of the volume of a parallelepiped. The geometrical basis of the statistical relations is thus concretely realized.

A RAPID METHOD OF APPROXIMATING ARITHMETIC ROOTS.

By GLENN JAMES, University of California, Southern Branch.

A method sometimes used to approximate the square root of a number consists of dividing the number by an estimated root and taking, for the root, the arithmetic mean between the divisor and the quotient,[2] for example,

$$\sqrt{6} = \tfrac{1}{2}(2.5 + 6/2.5) = 2.45.$$

This method is a special case of the following rule:

---

[1] When the words *coefficient of correlation* are used, it is to be understood throughout that $\Sigma x_k = \Sigma y_k = \Sigma z_k = 0$; this means merely that the mathematical concepts are more general than is necessary for their immediate statistical application.

[2] See M. A. Nordgaard, "*A Historical Survey of Algebraic Methods of Approximating the Roots of Numerical Higher Equations,*" art. 3. New York, Columbia University, 1922.

RULE. *To extract the n-th integral root of a number divide an estimate of the root into the number, then into the quotient and so on until n — 1 divisions have been made, and take for the root the arithmetic mean between the n — 1 divisors and the final quotient; e.g.*

$$\sqrt[3]{20} = (3 + 3 + 20/9)/3 = 2.74.$$

Using the approximation to the root obtained in one step for the estimate in the next, one can repeat this process as many times as desired. Obviously the sequence of approximations thus obtained would converge to the desired root. However, the usefulness of such a method depends upon the establishment of measures of the errors in the successive approximations.

1. The Remainders in the Approximation of the nth Root of a Number.[1] Let E^n be the number whose nth root is to be extracted and let E_0 be the estimated root, and denote $E_0 - E$ by ϵ_0. Then

$$E^n = (E_0 - \epsilon_0)^n \tag{1}$$

and according to the rule the first approximation to the nth root of E^n would be

$$1/n[E^n/E_0^{n-1} + (n - 1)E_0]. \tag{2}$$

By means of (1) this can be simplified into

$$E + \frac{(n-1)\epsilon_0^2}{2E_0} - \frac{(n-1)(n-2)\epsilon_0^3}{3!E_0^2} + \cdots. \tag{3}$$

The error in this approximation, namely,

$$\frac{(n-1)\epsilon_0^2}{2E_0} - \frac{(n-1)(n-2)\epsilon_0^3}{3!E_0^2} + \cdots \tag{4}$$

is positive for ϵ_0 is small relatively to E_0 and n. And, for ϵ_0 thus chosen the error in the ith approximation is less than

$$\frac{(n-1)\epsilon_{i-1}^2}{2E_{i-1}}, \qquad i > 1, \tag{5}$$

where ϵ_{i-1} and E_{i-1}, respectively, denote the error and the approximated root of the previous step. Moreover, when ϵ_0 is negative and small relatively to n and E_0, the second term of (4) is small relatively to the first. Besides, we are not so much interested in the first digit in the error as we are in its position relative to the decimal point. In the light of these considerations we are justified in determining the measure of the errors from the sequence,

$$\epsilon_0, \epsilon_1', \epsilon_2', \epsilon_3', \cdots, \tag{6}$$

[1] This paper considers only numbers greater than unity since the application of its result to fractions requires only the proper placing of the decimal point in the roots.

where

$$\epsilon_1' = \frac{(n-1)\epsilon_0^2}{2E_0}, \tag{7}$$

$$\epsilon_2' = \frac{(n-1)(\epsilon_1')^2}{2E_1} = \frac{(n-1)^3\epsilon_0^4}{2^3E_0^2E_1}, \tag{8}$$

$$\epsilon_3' = \frac{(n-1)(\epsilon_2')^2}{2E_2} = \frac{(n-1)^7\epsilon_0^8}{2^7E_0^4E_1^2E_2}, \tag{9}$$

and in general

$$\epsilon_i' = \frac{(n-1)^{2^i-1}\epsilon_0^{2^i}}{2^{2^i-1}E_0^{2^{i-1}}E_1^{2^{i-2}}\cdots E_{i-1}}. \tag{10}$$

But

$$E_j \gtreqless E_{j-1}, \qquad j = 2, 3, \cdots, i, \tag{11}$$

for

$$E_j = \frac{1}{n}\left[(n-1)E_{j-1} + \frac{E^n}{E_{j-1}^{n-1}}\right]$$
$$= \frac{nE_{j-1}^n + E^n - E_{j-1}^n}{nE_{j-1}^{n-1}},$$

which is equal to or less than E_{j-1} provided $E_{j-1} \geqq E$, and this is true whenever $j > 1$. Moreover, E_0 will differ from E_{i-1} by an additive error which is small relatively to E_0. Hence we shall use for our measure of error in the ith approximation, M_i, where

$$M_i = \frac{(n-1)^{2^i-1}\epsilon_0^{2^i}}{2^{2^i-1}E_{i-1}^{2^i-1}}.$$

This can be written in the form

$$M_i = \left[\frac{(n-1)\epsilon_0}{2E_{i-1}}\right]^{2^i-1}\epsilon_0. \tag{12}$$

It remains now to indicate rules for making estimates of roots, which will assure certain upper limits for ϵ_0. These upper limits should be as small as possible without requiring more labor to make the estimates than would be required to secure as small errors from easier estimates by additional approximations.

For any value of n, ϵ_0 can be made less than .5 by choosing for E_0 the arithmetic mean between the two integers nearest the correct root. For example, if the estimated fifth root of any number between 1 and 32 be 1.5, neither it nor any of the sequence of approximations could differ from the correct root by more than .5.

Whence we have for the measure of the error in the ith approximation to the nth root,

$$M_i = \frac{1}{2}\left[\frac{n-1}{4E_{i-1}}\right]^{2^i-1}. \tag{13}$$

3. Illustrative Example. To evaluate $\sqrt[3]{52.75}$, choose 4 for the estimated cube root.

Then

$$E_1 = \left(\frac{52.75}{16} + 8\right)\Big/3 = 3.7448.$$

$$M_1 = .028 -.$$

$$E_2 = \left(\frac{52.75}{3.75^2} + 7.5\right)\Big/3 = 3.75106 -.$$

$$M_2 = .0003 -.$$

$$E_3 = 3.7510770 +.$$

$$M_3 = .000{,}000{,}02 -.$$

A SIMPLE DISCUSSION OF THE REPRESENTATION OF FUNCTIONS BY FOURIER SERIES.

By PHILIP FRANKLIN, Massachusetts Institute of Technology.

1. Introduction. Some time ago Professor Birkhoff (*1921*, 200) gave a brief treatment of the Fourier expansion of functions which were periodic and had three continuous derivatives. As most of the elementary examples of Fourier expansions correspond to functions with discontinuities, at least at the end points of the fundamental period interval, it is thought that the following discussion, which covers such cases, may be of interest. While our proof reduces everything to first principles, it is much briefer than those usually given.

2. Theorem. Having defined the Fourier coefficients of a function $f(x)$ by the equations

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt; \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt,$$

we shall now establish the following theorem:

THEOREM. *Let $f(x)$ be a periodic function, period 2π, continuous with a continuous first derivative, except for a finite number of points in each period interval, and let it have forward and backward derivatives at the points of discontinuity. Moreover, let it be defined as equal to the average value of its right and left hand limit values at those points. Then $f(x)$ is represented by the series*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

PROOF. We wish to prove that

$$f(x) = \lim_{n=\infty} S_n,$$

3. Illustrative Example. To evaluate $\sqrt[3]{52.75}$, choose 4 for the estimated cube root.

Then

$$E_1 = \left(\frac{52.75}{16} + 8\right)\bigg/3 = 3.7448.$$

$$M_1 = .028 -.$$

$$E_2 = \left(\frac{52.75}{3.75^2} + 7.5\right)\bigg/3 = 3.75106 -.$$

$$M_2 = .0003 -.$$

$$E_3 = 3.7510770 +.$$

$$M_3 = .000{,}000{,}02 -.$$

A SIMPLE DISCUSSION OF THE REPRESENTATION OF FUNCTIONS BY FOURIER SERIES.

By PHILIP FRANKLIN, Massachusetts Institute of Technology.

1. Introduction. Some time ago Professor Birkhoff (*1921*, 200) gave a brief treatment of the Fourier expansion of functions which were periodic and had three continuous derivatives. As most of the elementary examples of Fourier expansions correspond to functions with discontinuities, at least at the end points of the fundamental period interval, it is thought that the following discussion, which covers such cases, may be of interest. While our proof reduces everything to first principles, it is much briefer than those usually given.

2. Theorem. Having defined the Fourier coefficients of a function $f(x)$ by the equations

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt dt; \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt dt,$$

we shall now establish the following theorem:

Theorem. *Let $f(x)$ be a periodic function, period 2π, continuous with a continuous first derivative, except for a finite number of points in each period interval, and let it have forward and backward derivatives at the points of discontinuity. Moreover, let it be defined as equal to the average value of its right and left hand limit values at those points. Then $f(x)$ is represented by the series*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Proof. We wish to prove that

$$f(x) = \lim_{n=\infty} S_n,$$

where

$$S_n = \frac{1}{2}a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{m=1}^{n}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos mtdt \cos mx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin mtdt \sin mx\right)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{m=1}^{n} \cos m(t-x)\right]dt.$$

We first change the variable of integration from t to u where

$$t = u + x.$$

Since the integrand is periodic, we may integrate over any complete period, and hence keep the limits unchanged, obtaining

$$S_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\left[\frac{1}{2} + \sum_{m=1}^{n} \cos mu\right]du = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)s_n du.$$

We may evaluate the trigonometric sum s_n by the following device:

$$(2 \sin u/2)s_n = \sin u/2 + \sum_{m=1}^{n} 2 \cos mu \sin u/2$$

$$= \sin u/2 + \sum_{m=1}^{n} [\sin (m + \tfrac{1}{2})u - \sin (m - \tfrac{1}{2})u]$$

$$= \sin (n + \tfrac{1}{2})u,$$

since the remaining terms appear twice with opposite signs. This gives:

$$s_n = \frac{1}{2} + \sum_{m=1}^{n} \cos mu = \frac{\sin (n + \frac{1}{2})u}{2 \sin u/2}.$$

Our problem now is to show that

$$f(x) = \lim_{n=\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)s_n du.$$

From the definition of $f(x)$ at the points of discontinuity, at all points,

$$f(x) = \tfrac{1}{2}f(x+) + \tfrac{1}{2}f(x-).$$

Since

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)s_n du = \frac{1}{\pi}\int_{0}^{\pi} f(u+x)s_n du + \frac{1}{\pi}\int_{-\pi}^{0} f(u+x)s_n du,$$

the conclusion will follow if

$$\frac{1}{2}f(x+) = \lim_{n=\infty} \frac{1}{\pi}\int_{0}^{\pi} f(u+x)s_n du; \quad \text{and} \quad \frac{1}{2}f(x-) = \lim_{n=\infty} \frac{1}{\pi}\int_{-\pi}^{0} f(u+x)s_n du.$$

We shall merely prove the first in full, as the second can be proved either by similar reasoning, or by applying the first result to the function $F(y) = f(2x - y)$.

Since

$$\frac{1}{\pi}\int_0^{\pi} s_n du = \frac{1}{\pi}\int_0^{\pi}\left[\frac{1}{2} + \sum_{m=1}^{n} \cos mu\right]du = \frac{1}{2},$$

$$\frac{1}{2} f(x+) = \frac{1}{\pi}\int_0^{\pi} f(x+)s_n du = \lim_{n=\infty}\frac{1}{\pi}\int_0^{\pi} f(x+)s_n du.$$

Thus the equation in question will follow from

$$\lim_{n=\infty}\frac{1}{\pi}\int_0^{\pi} f(x+)s_n du = \lim_{n=\infty}\frac{1}{\pi}\int_0^{\pi} f(u+x)s_n du,$$

or

$$\lim_{n=\infty}\int_0^{\pi} [f(u+x) - f(x+)]s_n du = 0.$$

To discuss this equation, we break up the interval of integration $(0, \pi)$ into smaller intervals: $(0, p_0)$, (p_0, p_1), $\cdots$, (p_{D-1}, π); where $p_1, \cdots, p_{D-1}$ are the values of u for which $f(u + x)$ is discontinuous (exclusive of 0, which may be such a value), and where p_0 is a value between 0 and p_1 to be specified presently.

The integral in the first interval is:

$$\int_0^{p_0} [f(u+x) - f(x+)]s_n du = \int_0^{p_0} \frac{f(u+x) - f(x+)}{u}\,\frac{u/2}{\sin u/2}\sin\left(n + \frac{1}{2}\right)u du,$$

as appears upon replacing s_n by its value, and inserting the factor u in both numerator and denominator. We may now obtain upper bounds for each factor in succession. Thus,

$$\left|\frac{f(x+u) - f(x+)}{u}\right| = f'(x + \theta u) \leq D, \qquad (|\theta| < 1),$$

as appears upon using the law of the mean, letting D represent an upper bound for the derivative of the given function $f(x)$. Moreover,

$$\frac{u/2}{\sin u/2} \leq \frac{\pi}{2},$$

since its derivative $\dfrac{(\cos u/2)(\tan u/2 - u/2)}{2\sin^2 u/2}$ is positive in the interval $(0, \pi)$, showing that its maximum value in that interval is that for $u = \pi$, or $\pi/2$.

Finally,

$$|\sin (n + \tfrac{1}{2})u| \leq 1.$$

Consequently,

$$\left|\int_0^{p_0} [f(u+x) - f(x+)]s_n du\right| \leq \left|\int_0^{p_0} D\cdot\frac{\pi}{2}\cdot 1 du\right| \leq \frac{D\pi p_0}{2}.$$

This can be made small, say less than $\epsilon/2$, by taking p_0 small.

The remaining integrals may now be treated by integrating by parts. We have:

$$I_k = \int_{p_{k-1}}^{p_k} [f(u+x) - f(x+)]s_n du$$

$$= \int_{p_{k-1}}^{p_k} \frac{f(u+x) - f(x+)}{2\sin u/2} \sin\left(n+\frac{1}{2}\right) u du$$

$$= -\frac{f(u+x) - f(x+)}{2\sin u/2} \frac{\cos(n+\frac{1}{2})u}{n+\frac{1}{2}}\Bigg|_{p_{k-1}}^{p_k}$$

$$+ \frac{1}{n+\frac{1}{2}} \int_{p_{k-1}}^{p_k} \left[\frac{f'(u+x)\cos(n+\frac{1}{2})u}{2\sin u/2} - \frac{[f(u+x) - f(x+)]\cos(n+\frac{1}{2})u\cos u/2}{4\sin^2 u/2}\right] du.$$

But, if F is an upper bound for $f(x)$,

$$|f(u+x) - f(x+)| \leq 2F,$$

while, since all the p_i are greater than p_0,

$$\sin u/2 > \sin p_0/2.$$

It follows from these inequalities that

$$|I_k| \leq \frac{1}{n+\frac{1}{2}}\left[\frac{2F}{\sin p_0/2} + (p_{k+1} - p_k)\left(\frac{D}{2\sin p_0/2} + \frac{F}{2\sin^2 p_0/2}\right)\right].$$

As the quantity in the braces is independent of n, the sum of all the integrals I_k will be numerically less than some expression independent of n (though involving ϵ through p_0), divided by $n + \frac{1}{2}$. Consequently, by taking n sufficiently large ($\geq N$), it can be made small, say less than $\epsilon/2$.

We have shown that, for any ϵ, an n can be found for which

$$\int_0^\pi [f(u+x) - f(x+)]s_n du < \epsilon \qquad \text{if} \qquad n \geq N.$$

Consequently, when $n = \infty$, the limit of the left member must be zero, thus completing the proof.

AN ELEMENTARY ANALYSIS OF THE GENERAL EQUATION OF SECOND DEGREE.[1]

By E. S. ALLEN, Iowa State College.

1. Introduction. Almost all elementary books on analytic geometry analyze the general quadratic equation in x and y by means of two changes of axes. It is, so far as I know, only in Smith's *Conic Sections* (p. 207) that a method of finding the foci of a conic section without change of coördinates is described. This method makes use of the polar properties of the foci and directrices. While making a class acquainted with conics recently, I was asked how it was possible to find the eccentricity, directrices, and foci of such a curve directly from its equation. The following answer to the question, which uses only "elementary" mathematical ideas, is, I believe, new.

2. General considerations. If a conic has the focus (α, β), the directrix

$$x \cos \theta + y \sin \theta - p = 0$$

and the eccentricity e, its equation is

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e(x \cos \theta + y \sin \theta - p);$$

that is,

$$\begin{aligned} x^2(1 - e^2 \cos^2 \theta) - 2xy(e^2 \sin \theta \cos \theta) + y^2(1 - e^2 \sin^2 \theta) \\ - 2x(\alpha - e^2 p \cos \theta) - 2y(\beta - e^2 p \sin \theta) + (\alpha^2 + \beta^2 - e^2 p^2) = 0. \end{aligned} \tag{1}$$

In order that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{2}$$

may have the same coefficients as an equation of the form (1), it is necessary that

$$(a-1)(b-1) = h^2.$$

This results from the first three of the following equations:

$$\begin{aligned} a &= 1 - e^2 \cos^2 \theta, & g &= -(\alpha - e^2 p \cos \theta), \\ h &= -e^2 \sin \theta \cos \theta, & f &= -(\beta - e^2 p \sin \theta), \\ b &= 1 - e^2 \sin^2 \theta, & c &= \alpha^2 + \beta^2 - e^2 p^2. \end{aligned} \tag{3}$$

Consequently, if we have any equation of second degree, with real coefficients,

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0, \tag{4}$$

we must multiply the first member by such a number k that

$$(ka' - 1)(kb' - 1) = (kh')^2. \tag{5}$$

[1] Read at the meeting of the Iowa Section of the Mathematical Association of America, May 2, 1924.

consequently

$$B = -k^3 \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}.$$

In view of the fact that $k_1 k_2 = \dfrac{1}{a'b' - h'^2} < 0$, the two values of B will have opposite signs, and that one of the two k's must be chosen whose sign is opposite that of the determinant. k having been found, the solution is identical with that for the ellipse.

7. Perpendicularity of axes. Since

$$\tan\theta_1 \tan\theta_2 = \frac{k_1 h'}{k_1 a' - 1} \cdot \frac{k_2 h'}{k_2 a' - 1} = \frac{\dfrac{h'^2}{a'b' - h'^2}}{\dfrac{a'^2}{a'b' - h'^2} - \dfrac{(a' + b')a'}{a'b' - h'^2} + 1} = -1,$$

we find, of course, that the two values of θ differ by $\dfrac{\pi}{2}$.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. ON CHECKING THE SOLUTION OF A TRIANGLE.

BY C. N. MILLS, Teachers College, Aberdeen, S. Dak.

In the June, 1924, issue of the MONTHLY, page 292, Professor E. J. Moulton gives a discussion relative to using the Sine Theorem for checking the solution of a triangle. For a complete and dependable check it is much better to use a formula which involves all the sides and angles of the triangle. Besides the usual relation that the sum of the angles must equal 180°, use the formula

$$(a - b)\cos(C/2) = c \sin[(A - B)/2].$$

When using logarithms the products are readily found, since the logarithms of some of the terms have been used in finding the required parts.

This equation is known as Mollweide's equation, but it was known before his time. (See Wilczynski, *Plane Trigonometry and Applications*, pages 104 and 105.)

consequently

$$B = -k^3 \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}.$$

In view of the fact that $k_1 k_2 = \dfrac{1}{a'b' - h'^2} < 0$, the two values of B will have opposite signs, and that one of the two k's must be chosen whose sign is opposite that of the determinant. k having been found, the solution is identical with that for the ellipse.

7. Perpendicularity of axes. Since

$$\tan\theta_1 \tan\theta_2 = \frac{k_1 h'}{k_1 a' - 1} \cdot \frac{k_2 h'}{k_2 a' - 1} = \frac{\dfrac{h'^2}{a'b' - h'^2}}{\dfrac{a'^2}{a'b' - h'^2} - \dfrac{(a' + b')a'}{a'b' - h'^2} + 1} = -1,$$

we find, of course, that the two values of θ differ by $\dfrac{\pi}{2}$.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. ON CHECKING THE SOLUTION OF A TRIANGLE.

BY C. N. MILLS, Teachers College, Aberdeen, S. Dak.

In the June, 1924, issue of the MONTHLY, page 292, Professor E. J. Moulton gives a discussion relative to using the Sine Theorem for checking the solution of a triangle. For a complete and dependable check it is much better to use a formula which involves all the sides and angles of the triangle. Besides the usual relation that the sum of the angles must equal 180°, use the formula

$$(a - b)\cos(C/2) = c \sin[(A - B)/2].$$

When using logarithms the products are readily found, since the logarithms of some of the terms have been used in finding the required parts.

This equation is known as Mollweide's equation, but it was known before his time. (See Wilczynski, *Plane Trigonometry and Applications*, pages 104 and 105.)

II. On Taking Square Roots of Integers.

By D. F. Barrow, University of Georgia.

The Diophantine equation

$$y^2 - nx^2 = 1 \tag{1}$$

has integral solutions in x and y when n is not a perfect square and they have the following property:[1] If x_0 and y_0 satisfy (1), then

$$\begin{aligned} x_1 &= 2x_0y_0, \\ y_1 &= 2y_0^2 - 1 \end{aligned} \tag{2}$$

will also satisfy (1).[2]

Now it is evident that if x and y are large numbers satisfying (1), the fraction y/x will be a close approximation to the square root of n. Furthermore, if we can find any solution of (1), then successive applications of (2) will give us as large a solution as we please. We can thus find the square root of n as accurately as we please as a common fraction; and, of course, reduce it to a decimal if we wish. The method seems to be of most use in case very accurate results are desired.

As an example we will find the square root of 2 to fifty decimals. By inspecting the equation

$$y^2 - 2x^2 = 1,$$

we see that $x_0 = 2$ and $y_0 = 3$ furnish an initial solution. Then, by successive applications of (2), we find:

$$\begin{aligned}
x_1 &= 2\cdot 2\cdot 3 & y_1 &= 17 \\
x_2 &= 2\cdot 2\cdot 2\cdot 3\cdot 17 & y_2 &= 577 \\
x_3 &= 2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 17\cdot 577 & y_3 &= 665857 \\
x_4 &= 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 17\cdot 577\cdot 665857 & y_4 &= 866731088897.
\end{aligned}$$

If we divide the largest value of y by the corresponding value of x, we obtain the square root of two to about twenty-four decimals. However, the following method, which is evidently applicable to all such problems, is more convenient and increases the accuracy without using any larger numbers: From (2) we find

$$y_1/x_1 = y_0/x_0 - 1/2x_0y_0. \tag{3}$$

Now we will use subscripts to denote the successive values of x and y obtained from an initial pair of values by use of (2); and remember that the accuracy increases with the size of the numbers. We are led to the following series:

$$\sqrt{n} = y_0/x_0 - 1/2x_0y_0 - 1/4x_0y_0y_1 - 1/8x_0y_0y_1y_2 - \cdots. \tag{4}$$

[1] R. D. Carmichael, *Diophantine Analysis*, pp. 26–29. John Wiley & Sons.

[2] This fact was noted by Wallis. See L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 351. Editor.

are as follows:

1.50000	00000	00000	00000	00000	00000	00000	00000	00000	00000	0000
−.08333	33333	33333	33333	33333	33333	33333	33333	33333	33333	3333
−.00245	09803	92156	86274	50980	39215	68627	45098	03921	56862	7450
−.00000	21238	99819	89329	52730	48560	84548	20430	03122	98229	5171
−.00000	00000	01594	86182	46059	55387	60466	65015	77696	33954	6616
−.00000	00000	00000	00000	00008	99292	83216	50453	10050	39896	3426
−.00000	00000	00000	00000	00000	00000	00000	00000	00000	00028	5928.

When the negative terms are added together and subtracted from the positive term, the following value is obtained for the square root of two:

1.41421 35623 73095 04880 16887 24209 69807 85696 71875 37694 807 $\cdots$.

III. Real Roots of Equations with Complex Coefficients.

By I. A. Barnett, University of Cincinnati.

The purpose of this note is to give necessary and sufficient conditions that algebraic equations of the second and third degrees with complex coefficients shall have an assigned number of real roots. These criteria follow almost immediately as corollaries of a theorem given in Dickson's *Elementary Theory of Equations*, First Edition, page 164. This theorem is as follows:

Necessary and sufficient conditions that

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m, \qquad a_0 \neq 0,$$

and

$$g(x) = b_0x^n + b_1x^{n-1} + \cdots + b_n, \qquad b_0 \neq 0,$$

shall have a common divisor of degree d, but none of higher degree, are $R = 0$, $R_1 = 0, \cdots, R_{d-1} = 0$, $R_d \neq 0$, where R is given by the expression

$$R = \left|\begin{array}{cccccccccccc} a_0 & a_1 & a_2 & \cdots & a_m & 0 & & \cdots & & 0 \\ 0 & a_0 & a_1 & a_2 & \cdots & a_m & 0 & \cdots & & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots & a_m & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & a_0 & a_1 & a_2 & & \cdots & & a_m \\ b_0 & b_1 & & \cdots & & b_n & 0 & \cdots & & 0 \\ 0 & b_0 & b_1 & & \cdots & & b_n & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & b_0 & b_1 & & \cdots & & & b_n \end{array}\right| \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} n \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} m \text{ rows} \end{array}$$

and R_k is the determinant derived from R by deleting the last k rows of a's, the last k rows of b's, and the last $2k$ columns.

This will be referred to as Theorem A.

are as follows:

1.50000 00000 00000 00000 00000 00000 00000 00000 00000 00000 0000
−.08333 33333 33333 33333 33333 33333 33333 33333 33333 33333 3333
−.00245 09803 92156 86274 50980 39215 68627 45098 03921 56862 7450
−.00000 21238 99819 89329 52730 48560 84548 20430 03122 98229 5171
−.00000 00000 01594 86182 46059 55387 60466 65015 77696 33954 6616
−.00000 00000 00000 00000 00008 99292 83216 50453 10050 39896 3426
−.00000 00000 00000 00000 00000 00000 00000 00000 00000 00028 5928.

When the negative terms are added together and subtracted from the positive term, the following value is obtained for the square root of two:

1.41421 35623 73095 04880 16887 24209 69807 85696 71875 37694 807 ⋯.

III. Real Roots of Equations with Complex Coefficients.

By I. A. Barnett, University of Cincinnati.

The purpose of this note is to give necessary and sufficient conditions that algebraic equations of the second and third degrees with complex coefficients shall have an assigned number of real roots. These criteria follow almost immediately as corollaries of a theorem given in Dickson's *Elementary Theory of Equations*, First Edition, page 164. This theorem is as follows:

Necessary and sufficient conditions that

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m, \qquad a_0 \neq 0,$$

and

$$g(x) = b_0x^n + b_1x^{n-1} + \cdots + b_n, \qquad b_0 \neq 0,$$

shall have a common divisor of degree d, but none of higher degree, are $R = 0$, $R_1 = 0, \cdots, R_{d-1} = 0$, $R_d \neq 0$, where R is given by the expression

$$R = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_m & 0 & & \cdots & 0 \\ 0 & a_0 & a_1 & a_2 & \cdots & a_m & 0 & \cdots & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots & a_m & 0 \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & a_0 & a_1 & a_2 & & \cdots & a_m \\ b_0 & b_1 & & \cdots & b_n & 0 & & \cdots & 0 \\ 0 & b_0 & b_1 & & \cdots & b_n & & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & b_0 & b_1 & & \cdots & & b_n \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\end{array}}\right\} n \text{ rows} \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} m \text{ rows} \end{array}$$

and R_k is the determinant derived from R by deleting the last k rows of a's, the last k rows of b's, and the last $2k$ columns.

This will be referred to as Theorem A.

In case $m = n$, one may write the determinant R in the form

$$R = (-1)^{[n(n-1)]/2} \begin{vmatrix} d_{11} & \cdots & d_{1n} \\ d_{21} & \cdots & d_{2n} \\ \vdots & & \\ d_{n1} & \cdots & d_{nn} \end{vmatrix},$$

where

$$d_{ij} = (a_0 b_{i+j-1}) + (a_1 b_{i+j-2}) + \cdots + (a_{i-1} b_j)$$

and

$$(a_i b_k) = a_i b_k - a_k b_i.$$

1. Consider first the quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0, \tag{1}$$

where

$$a = \alpha_1 + i\alpha_2, \qquad b = \beta_1 + i\beta_2, \qquad c = \gamma_1 + i\gamma_2.$$

Equation (1) may be written in the form

$$\alpha_1 x^2 + \beta_1 x + \gamma_1 + i(\alpha_2 x^2 + \beta_2 x + \gamma_2) = 0. \tag{2}$$

Case 1. One and only one real root. Suppose it is required that (1) shall have one and only one real root x_1. From (2) it is seen that a necessary and sufficient condition for this to be so is that the equations

$$\alpha_1 x^2 + \beta_1 x + \gamma_1 = 0 \tag{3}$$

and

$$\alpha_2 x^2 + \beta_2 x + \gamma_2 = 0 \tag{4}$$

shall have one and only one real root x_1 in common. In other words, the left hand members of (3) and (4) must have a linear factor in common and no factor of higher degree. Therefore, upon application of Theorem A, one may state the following result:

Necessary and sufficient conditions that equation (1) *have one and only one real root are that*

$$Q_1 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 \\ 0 & \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 \\ 0 & \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0, \qquad Q_2 = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \neq 0,$$

which may also be written in the form

$$Q_1 \equiv (\alpha_1\gamma_2)^2 - (\alpha_1\beta_2)(\beta_1\gamma_2) = 0, \qquad Q_2 \equiv (\alpha_1\beta_2) \neq 0.$$

That is, $(\alpha_1\gamma_2)$ is a mean proportional between $(\alpha_1\beta_2)$ and $(\beta_1\gamma_2)$.

It follows of course as a corollary that $Q_1 \neq 0$ is a necessary and sufficient condition that the quadratic equation (1) have no real root.

Another corollary is that a quadratic equation with real coefficients ($\alpha_2 = \beta_2 = \gamma_2 = 0$) cannot have one real root.

Case 2. Two real roots. Let it now be required that both roots of (1) be real. Then (3) and (4) must have two real roots in common. Now, according to Theorem A, $Q_1 = 0$, $Q_2 = 0$ would be necessary and sufficient conditions that the left hand members of (3) and (4) have a quadratic factor in common, but this would be the case even if equations (3) and (4) had a complex root in common. To rule out this case one must have the condition $\beta_1^2 - 4\alpha_1\gamma_1 \geqq 0$. Hence, necessary and sufficient conditions that equation (1) have two real roots are:

$$Q_1 = 0, \qquad Q_2 = 0, \qquad \beta_1^2 - 4\alpha_1\gamma_1 \geqq 0.$$

This may be put more simply. For, the condition $Q_2 \equiv (\alpha_1\beta_2) = 0$ with $Q_1 = 0$ implies $(\alpha_1\gamma_2) = 0$; and hence,

$$\alpha_2 = \rho\alpha_1, \qquad \beta_2 = \rho\beta_1, \qquad \gamma_2 = \rho\gamma_1.$$

Thus equation (1) reduces to

$$\alpha_1 x^2 + \beta_1 x + \gamma_1 = 0$$

with the restriction $\beta_1^2 - 4\alpha_1\gamma_1 \geqq 0$, as was to be expected.

One may note in passing that when the two roots of equation (1) are equal but not necessarily real, one must have the two conditions

$$\begin{cases} \beta_1^2 - \beta_2^2 - 4\alpha_1\gamma_1 + 4\alpha_2\gamma_2 = 0 \\ \beta_1\beta_2 - 2\alpha_1\gamma_2 - 2\alpha_2\gamma_1 = 0 \end{cases}$$

obtained by setting the discriminant of (1) equal to zero.

2. Consider next the cubic equation

$$ax^3 + bx^2 + cx + d = 0, \qquad a \neq 0, \tag{5}$$

which may be written more explicitly

$$(\alpha_1 x^3 + \beta_1 x^2 + \gamma_1 x + \delta_1) + i(\alpha_2 x^3 + \beta_2 x^2 + \gamma_2 x + \delta_2) = 0.$$

Case 1. One and only one real root. The equations

$$\alpha_1 x^3 + \beta_1 x^2 + \gamma_1 x + \delta_1 = 0, \tag{6}$$

$$\alpha_2 x^3 + \beta_2 x^2 + \gamma_2 x + \delta_2 = 0 \tag{7}$$

must have one and only one real root in common. Hence, by Theorem A, one has that necessary and sufficient conditions that equation (5) has one and only one real root are that

$$C_1 \equiv \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & 0 \\ 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & 0 & 0 \\ 0 & \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{vmatrix} = 0, \qquad C_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ 0 & \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ 0 & \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} \neq 0.$$

The expansions of these determinants may be put in the following form (cf. Dickson's Theory of Equations, p. 156):

$$C_1 \equiv (\alpha_1\delta_2)^3 - 2(\alpha_1\beta_2)(\alpha_1\delta_2)(\gamma_1\delta_2) - (\alpha_1\gamma_2)(\alpha_1\delta_2)(\beta_1\delta_2) + (\alpha_1\gamma_2)^2(\gamma_1\delta_2) + (\alpha_1\beta_2)(\beta_1\delta_2)^2 - (\alpha_1\beta_2)(\beta_1\gamma_2)(\gamma_1\delta_2),$$

$$C_2 \equiv (\alpha_1\gamma_2)^2 - (\alpha_1\beta_2)[(\beta_1\gamma_2) + (\alpha_1\delta_2)].$$

Case 2. Two and only two real roots. The left hand members of (6) and (7) must now have a quadratic factor in common but since it is further required that the common roots be real, one must have, moreover, that the discriminant of (6) be non-negative. Hence *necessary and sufficient conditions that equation* (5) *shall have two and only two real roots are that*

$$C_1 = 0, \qquad C_2 = 0, \qquad C_3 \equiv \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \neq 0,$$

and

$$\Delta \equiv 18\alpha_1\beta_1\gamma_1\delta_1 - 4\beta_1^3\delta_1 + \beta_1^2\gamma_1 - 4\alpha_1\gamma_1^3 - 27\alpha_1^2\delta_1^2 \geqq 0.$$

One may readily verify that the conditions $C_1 = 0$, $C_2 = 0$ may be replaced by the simpler conditions

$$C_2 = 0, \qquad (\alpha_1\delta_2)^3 - (\alpha_1\beta_2)(\alpha_1\delta_2)(\gamma_1\delta_2) - (\alpha_1\gamma_2)(\alpha_1\delta_2)(\beta_1\delta_2) + (\alpha_1\beta_2)(\beta_1\delta_2)^2 = 0.$$

Case 3. Three real roots. Applying Theorem A, one finds immediately that *necessary and sufficient conditions for equation* (5) *to have three real roots are that*

$$C_1 = 0, \qquad C_2 = 0, \qquad C_3 = 0, \qquad \Delta \geqq 0.$$

Since $C_3 \equiv (\alpha_1\beta_2) = 0$, one sees from the expanded form of C_2 that $(\alpha_1\gamma_2) = 0$, and hence, from the expression for C_1, that $(\alpha_1\delta_2) = 0$. Hence,

$$\alpha_2 = \rho\alpha_1, \qquad \beta_2 = \rho\beta_1, \qquad \gamma_2 = \rho\gamma_1, \qquad \delta_2 = \rho\delta_1,$$

and equation (5) reduces to

$$\alpha_1x^3 + \beta_1x^2 + \gamma_1x + \delta_1 = 0$$

with the condition $\Delta \geqq 0$.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Projective Geometry. By R. M. WINGER. New York, D. C. Heath and Co., 1923. xiii + 443 pages. Price, $4.40.

The task which Professor Winger has set himself in this book and the notable contribution which it makes to mathematical text-book literature are in no way adequately suggested by the title. It bears little resemblance to the text-book with which we are familiar under this name, but represents pioneer work in the spirit of modern advances in algebra and geometry. The author has chosen the analytic method of attack because, as he says in the preface, it "makes it possible to capitalize the student's collegiate training in algebra, analytic geometry, and calculus, and at the same time to articulate the subject with his future mathematical work."

The emphasis and spirit of the work are even more strongly algebraic than this would suggest. In fact the applications to geometry of one and two dimensions of the invariant theory of binary and ternary forms is the field to which Professor Winger would introduce the student. It would not be easy to find a satisfying short title for these chapters on the geometrical interpretation of modern algebra, and perhaps the author was wise in not trying to disclose in the title the exact nature of his undertaking. In the following outline the reviewer has attempted not merely to give the author's selection of material but also to suggest the emphasis.

The definition of projective coördinates in one and two dimentions and their transformation discussed in Chapter V may be said to be the goal reached in the first quarter of the book. The projective coördinate of a point on a line is defined as a double ratio expressed in terms of abscissas; homogeneous projective coördinates of a point in a plane are defined formally in terms of the Cartesian equations of the sides of the triangle of reference; while for homogeneous projective coördinates of a line the Plücker equations of the vertices are similarly used.

Leading up to this chapter we have a discussion of essential constants; duality in the plane based on a comparison of Cartesian and Plücker coördinates; classification of the conic defined as the locus of the general equation of the second degree in three homogeneous variables with real coefficients, and of the circle with complex coefficients, as exhibiting the use of the line at infinity and the circular points; and the harmonic properties of the quadrangle and quadrilateral.

In Chapter VI we find a scant forty pages devoted to the "beautiful synthetic treatment of the conic, including Pascal's theorem and its consequences," Brianchon's theorem, poles and polars, and polar reciprocation. Even here our conic is introduced as the result of eliminating the parameter between two projective pencils expressed analytically.

Chapters VII, VIII, and IX, the remaining eighty pages of the second quarter, are occupied with binary geometry: collineations and involutions, polar forms derived by means of a differential operator, and algebraic invariants culminating in the complete system of the binary quartic.

Chapter X, sixty pages, the analytic treatment of the conic, the author says in the preface, may follow Chapter V; but let him who thinks to avoid the algebraic details of the intervening chapters beware. Much of this chapter will be wholly unintelligible without them. He will find, in fact, many explicit references to those omitted pages. The chapter is divided into two parts: the first considers the conic as a rational curve, the three coördinates of a point or line being expressed as quadratic functions of a parameter, and is largely devoted to binary forms on the curve; the second handles the conic as the locus of a ternary equation in point and line coördinates, considering especially polar properties.

Chapter XI opens with a classification of non-singular planar collineations which it follows with a discussion of certain collineation groups and absolute coördinates. This furnishes an introduction to the work of Morley on Reflexive Geometry.

Chapter XII treats of cubic and higher involutions and their applications to rational curves and makes contact with the author's investigations on self-projective curves.

The concluding Chapter XIII, an introduction to non-Euclidean geometry, is based largely on the work of Klein.

The reviewer gladly expresses his thanks to Professor Winger for his book as a treatment of the subject-matter it deals with. Preceded or accompanied by a thorough training in synthetic geometry with due emphasis on construction problems, it will perform a valuable service. But if the title of the book and the author's description of it as "an introductory account for senior college and beginning graduate students" imply an invitation to replace our time-honored courses in synthetic geometry, the reviewer must emphatically express his opinion that such a step would be a sad mistake. While acknowledging the important contributions of the analytic method, he is firmly convinced that the synthetic procedure must not be discarded, but must be given a large place in introductory courses if that power of visualization which Reye claimed projective geometry could develop is to be attained.

Of course any adequate appreciation of the historical development of the subject cannot neglect the synthetic. The footnote on page 116, "Pascal's theorem marks the climax of the classical theory of projective geometry," if it means that development along synthetic lines has been of minor importance since the days of Pascal, conveys a wholly mistaken conception of the origin and growth of modern geometry. One cannot do justice to the achievements on which our present knowledge rests without citing von Staudt, whose name nowhere appears in our text, not to mention a host of others who have advanced synthetic geometry. For the cultivation of the geometrical imagination a little work with axial pencils and the regulus is worth far more to students of the grade of advancement

But in spite of this criticism of Professor Winger's writing, it should again be asserted that he has produced a valuable book, one with which every teacher of advanced geometry must be familiar. The numerous exercises, nearly nine hundred in all, and the very full index covering eighteen pages are features which will especially commend its use.

J. W. BRADSHAW.

The Teaching of Mathematics in the Elementary and Secondary School. By J. W. A. YOUNG. New Edition. New York, Longmans, Green and Co., 1924. xviii + 451 pages. Price, $2.20.

It is now about nineteen years since the first edition of this work appeared. These years have brought many changes in the teaching of mathematics, many innovations in text-books, and many new opinions as to the purposes that should characterize the selection of the material that is best suited to the needs of American pupils. Indeed, as one considers the mathematical offering in our schools twenty years ago, and compares it with the offering of today, and as he contrasts present methods of measuring the accomplishments of pupils with those of the early years of the twentieth century, he may be led to feel that the changes have been almost revolutionary. It is with some such ideas in mind that Professor Young has given us the revision of a book that has long been recognized as a standard work for teachers, and that ranks as one of the most sane and scholarly productions of its kind to be found in any language.

The new edition is in large part a reprint of the original one. The first 345 pages are identical, or substantially so, with the first edition; but upwards of a hundred pages have been added, setting forth the changes that the author feels to be the most significant in the last ten years. These changes center about the work of the National Committee on Mathematical Requirements, and in particular the study made by Miss Blair of the question of the disciplinary value of the science, and a consideration of the purposes of the study of mathematics, the modern curriculum, the possibilities of introducing an elementary course in the calculus, the use of the function concept as a basis for curriculum making, and the simplification of the language of mathematics. Professor Young also includes in the added pages a bibliography of various important works on the teaching of mathematics that have appeared since the first edition was printed.

In an article in *Die Rundschau* for September, 1923, George Brandes sets forth in these words Ibsen's method of judging new books: "He judged them by a single quality: the only thing that mattered in his mind was whether they were intimately connected with the soul-life of the author. If so, they were good. If not, they were merely mechanical."

This is the opinion of a great mind, and it is a noble view to take. Professor Young's book is intimately connected with his "soul-life"; it is a good book; it is filled with scholarship of the type that our teachers in America need; it is expressed in a readable and felicitous style; and it fills a need on the part of teachers that no other book so seriously attempts to recognize.

The book will be criticized by the ultraradical school. It has little to say of the modern types of psychological tests, for example, and there are those who see little else in the teaching of anything,—just as a man who insures a house is likely to see nothing in the artistic work of the architect or in the fine details due to the master builder. It assumes that the well-educated man or woman today needs to know what mathematics signifies, just as each needs to know what geography and history and literature and science and the fine arts signify, and it is little concerned with those who talk endlessly about project methods and curricula which shall omit all mathematics and language and science except as these have a direct bearing, as slight as possible, upon some narrow vocation. Against all this educational froth of narrow-minded theorists Professor Young wages silent but effective combat, and his book deserves to be read and reread as one of our best exponents of common-sense mathematics that, freed from many of its early imperfections, should find place in our American schools.

DAVID EUGENE SMITH.

Kreisevolventen und Ganze Algebraische Funktionen. By DR. H. ONNEN. Leipzig, Teubner (Vol. 51, Mathematisch-Physikalische Bibliothek), 1923. 49 pages. Price M. 1.40.

This pamphlet is one of a collection which has received little or no notice hitherto in the pages of the MONTHLY devoted to reviews. Its editors, W. Lietzmann and A. Witting, characterize its volumes as having two purposes:—first to go more deeply into certain elementary problems that have general cultural significance or especial scientific importance; next, without supposing too much prerequisite knowledge on the part of the reader, to introduce him to new fields in pure and applied mathematics. In many cases this prerequisite knowledge does not exceed that furnished by our freshman mathematics, but it might be well to bring to the reading somewhat more than average freshman intelligence. This collection would help solve many a difficulty in the preparation of programs for mathematical clubs, unless, indeed, German has become for the time being a dead language for our undergraduates.

The present volume gives, with considerable detail and with the help of a number of figures, a most interesting geometrical interpretation of the problem of real root separation and approximation for real algebraic equations in one variable. No new algebraic results are obtained,—on this side the author merely obtains from his geometric considerations such well-known elementary theorems as those of Descartes, Budan-Fourier, and Sturm, and illustrates Newton's method of approximation. The interest is in the way he does it.

When a stretched string with a knot on it is unwound from the circumference of a circle, the knot traces a curve called a *first involute* of the circle. If the radius OP_0 of the circle revolves through an angle w, the tracing point P_1 of the unwinding string determines on the tangent at P_0 a segment P_0P_1 of varying length ρ_1, which in all its positions is a radius of curvature of the first involute.

It readily follows that if ρ_0 is the radius of the circle, then

$$\rho_1 = \rho_0 w + \text{constant},$$

the constant depending on the initial value of ρ_1. Similarly we obtain a *second involute* by unwinding a stretched string from the first involute in such a way that its point of tangency is at P_1. From the *second involute* we generate a *third involute* and so proceed to the *nth involute,* whose radius of curvature ρ_n is expressible as a polynomial $f(w)$ of the nth degree in w. The ρ's are so related that each is the derivative with respect to w of its successor.

Conversely, to every polynomial $f(w)$ and its successive derivatives corresponds a chain of involutes. The real roots of $f(w)$ are values of w for which the nth involute has simple or coincident cusps. We can think of the whole chain of involutes corresponding to $f(w)$ as traced by the unwinding process as w increases, and we have thus a picture of the simultaneous variation of the polynomial and its derivatives.

From these fundamental ideas the author proceeds to rediscover, or at least reillustrate, certain familiar chapters of the elementary theory of equations. The style is clear, interesting, graceful.

D. R. Curtiss.

NOTES ON NEW PUBLICATIONS.

Volume one of the *Exterior Ballistic Tables,* planned and initiated by Professor A. A. Bennett, of the University of Texas, while with the Ordnance Department, has just appeared in print as Ordnance Department Publication No. 1107. It is the most extensive table of ballistic functional values ever published. Volume two, computed by refined methods of triple interpolation in the present volume and without new trajectory integration, will present the same data in a form available for range table construction, and while analogous to the Gavre tables will be vastly more comprehensive.

ARTICLES IN CURRENT PERIODICALS.

American Journal of Mathematics, volume 46, no. 3, July, 1924: "Geometrical aspects of the Abelian modular functions of genus four" by A. B. Coble, 143–192; "The curve of ambiance" by F. Morley, 193–200; "On a class of invariant subgroups of the conformal and projective groups in function space" by I. A. Barnett, 201–214.

Annals of Mathematics, second series, volume 25, no. 2, December, 1923: "A contribution to the theory of closed chains" by A. Arwin, 91–117; "On the characterization of the set of points of λ-continuity" by H. Blumberg, 118–122; "On the Ampere-Cauchy derived functions" by H. L. Smith, 123–136; "Note on certain semi-invariants of n-lines" by L. P. Copeland, 137–141; "The Fredholm theory of the Stieltjes integral equations" by C. A. Fischer, 142–158; "Rectilinear congruences referred to special surfaces" by M. C. Foster, 159–180; "Parallel propagation of a vector around an infinitesimal circuit in an affine-connected manifold" by J. L. Synge, 181–184.

Bulletin of the American Mathematical Society, volume 30, no. 7, July, 1924: "The equation of the eighth degree" by A. B. Coble, 301–313; "Group of a set of simultaneous algebraic equations" by L. Weisner, 314–316; "On a type of plane unicursal curve" by H. Hilton, 317–321; "Surfaces with orthogonal loci of the centers of geodesic curvature of an orthogonal

It readily follows that if ρ_0 is the radius of the circle, then

$$\rho_1 = \rho_0 w + \text{constant},$$

the constant depending on the initial value of ρ_1. Similarly we obtain a *second involute* by unwinding a stretched string from the first involute in such a way that its point of tangency is at P_1. From the *second involute* we generate a *third involute* and so proceed to the *nth involute,* whose radius of curvature ρ_n is expressible as a polynomial $f(w)$ of the nth degree in w. The ρ's are so related that each is the derivative with respect to w of its successor.

Conversely, to every polynomial $f(w)$ and its successive derivatives corresponds a chain of involutes. The real roots of $f(w)$ are values of w for which the nth involute has simple or coincident cusps. We can think of the whole chain of involutes corresponding to $f(w)$ as traced by the unwinding process as w increases, and we have thus a picture of the simultaneous variation of the polynomial and its derivatives.

From these fundamental ideas the author proceeds to rediscover, or at least reillustrate, certain familiar chapters of the elementary theory of equations. The style is clear, interesting, graceful.

D. R. CURTISS.

NOTES ON NEW PUBLICATIONS.

Volume one of the *Exterior Ballistic Tables,* planned and initiated by Professor A. A. Bennett, of the University of Texas, while with the Ordnance Department, has just appeared in print as Ordnance Department Publication No. 1107. It is the most extensive table of ballistic functional values ever published. Volume two, computed by refined methods of triple interpolation in the present volume and without new trajectory integration, will present the same data in a form available for range table construction, and while analogous to the Gavre tables will be vastly more comprehensive.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 46, no. 3, July, 1924: "Geometrical aspects of the Abelian modular functions of genus four" by A. B. Coble, 143–192; "The curve of ambiance" by F. Morley, 193–200; "On a class of invariant subgroups of the conformal and projective groups in function space" by I. A. Barnett, 201–214.

ANNALS OF MATHEMATICS, second series, volume 25, no. 2, December, 1923: "A contribution to the theory of closed chains" by A. Arwin, 91–117; "On the characterization of the set of points of λ-continuity" by H. Blumberg, 118–122; "On the Ampere-Cauchy derived functions" by H. L. Smith, 123–136; "Note on certain semi-invariants of n-lines" by L. P. Copeland, 137–141; "The Fredholm theory of the Stieltjes integral equations" by C. A. Fischer, 142–158; "Rectilinear congruences referred to special surfaces" by M. C. Foster, 159–180; "Parallel propagation of a vector around an infinitesimal circuit in an affine-connected manifold" by J. L. Synge, 181–184.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 30, no. 7, July, 1924: "The equation of the eighth degree" by A. B. Coble, 301–313; "Group of a set of simultaneous algebraic equations" by L. Weisner, 314–316; "On a type of plane unicursal curve" by H. Hilton, 317–321; "Surfaces with orthogonal loci of the centers of geodesic curvature of an orthogonal

system" by M. Foster, 322–327; "Quadratic fields in which factorization is always unique" by L. E. Dickson, 324–334; "The Jacobian of a contact transformation" by E. F. Allen, 335–337; "Integro-differential invariants of one-parameter groups of Fredholm transformations" by A. D. Michal, 338–344; "Reductions of enumerations in homogeneous forms" by E. T. Bell; "The scientific work of Joseph Lipka" by W. C. Graustein, 352–356; "Extension of Bernstein's theorem to Sturm-Liouville sums" by E. Carlson, 230–240; "An existence theorem" by E. Hille, 241–248; "On the complete independence of the postulates for betweenness" by W. E. Van de Walle, 249–256; "A new set of postulates for betweenness, with proof of complete independence" by E. V. Huntington, 257–282.

L'Enseignement Mathématique, volume 23, no. 5–6, July, 1923: "Sur l'inversion des produits arithmétiques" by E. T. Bell, 305–308.

Proceedings of the National Academy of Sciences, volume 10, no. 7, August, 1924: "Concerning upper semi-continuous collections of continua which do not separate a given continuum" by R. L. Moore, 356–360.

Quarterly Journal of Pure and Applied Mathematics, volume 50, no. 2, August, 1924: "On surfaces whose asymptotic curves are cubics" by C. H. Sisam, 149–153; "Some two-dimensional loci" by J. L. Walsh, 154–164.

Transactions of the American Mathematical Society, volume 26, no. 2, April, 1924: "On the expansion of analytic functions in series of polynomials" by J. L. Walsh, 155–170; "Operations with respect to which the elements of a boölian algebra form a group" by B. A. Bernstein, 171–175; "Isometric *W*-surfaces" by W. C. Graustein, 176–204; "Space-time continua of perfect fluids in the general relativity" by L. P. Eisenhart, 205–220; "Equivalent rational substitutions" by J. F. Ritt, 221–229.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

Pi Mu Epsilon of Syracuse University, Syracuse, N. Y. [*1923*, 40.]

In the academic year 1922–1923 twenty-two members were elected, one instructor, two graduate students and nineteen undergraduates, among them two advanced students in the College of Business Administration. In the year 1923–1924 thirty-six members were elected, classified as follows: faculty three, graduate students two, undergraduates thirty-one; according to colleges, graduate school two, liberal arts twenty-three, applied science nine, business administration two.

The officers for the year 1922–1923 were: director, Otto Gelormini, faculty; vice director, I. S. Carroll, faculty; secretary, Olive Jakway '23; treasurer, Otis P. Hendershot '23; librarian, Helen Franklin '23. The executive committee consisted of the five above-named officers and four additional members: Olga Pfau '23, Georgia Mason '23, Margaret Smith '23, Vivian B. Peckham '23. The scholarship committee was F. F. Decker and I. S. Carroll, faculty members. The officers for 1923–1924 were: director, I. S. Carroll, faculty; vice director, May J. Sperry, faculty; secretary, Eleanor A. Carpenter '24; treasurer, Vivian B. Peckham '24; librarian, Julia W. Bower '25. The executive committee consisted of the five above officers and the following four additional members: Otis P. Hendershot, Gr., Ruth Lyons '24, Adele Armstrong '24, George Howell '24. The scholarship committee was composed of W. G. Bullard, May N. Harwood, faculty, Freda Jones, Helene Aldrich, Vivian B. Peckham '24. The officers elected to serve during the year 1924–1925 are: director, May N. Harwood, faculty; vice director, May J. Sperry, faculty; secretary, Vera Keeney '25; treasurer, Reginald Steel '25; librarian, Dorothy Park '25. The additional members of the executive committee are Helen O'Donnell '26, Warren Lyon, Gr., Louis Rees, Gr., Kenneth Robertson '26.

3104. Proposed by OTTO DUNKEL, Washington University.

If $f(x)$ is a single-valued and continuous function of x in the interval $a \leq x \leq b$ which is not identically zero and which satisfies the inequality $0 \leq f(x) \leq M$, show that

$$0 < \left[\int_a^b f(x)dx\right]^2 - \left[\int_a^b f(x) \cos x dx\right]^2 - \left[\int_a^b f(x) \sin x dx\right]^2 \leq M^2(b-a)^4/12.$$

3105. Proposed by W. A. GRANVILLE, Chicago, Illinois.

Find the length of the arc of the cissoid, $\rho = 2a \tan \theta \sin \theta$, from $\theta = 0$ to $\theta = (\pi/4)$.

3106. Proposed by EDWARD CONDON, University of California.

A spool of thread rolls without slipping on a horizontal plane. It is of radius r_1 and if it rolls from left to right the thread unwinds from the drum part of the spool radius r_2 ($< r_1$). Initially the spool is at rest and the thread, leaving the spool in a direction perpendicular to the axis of the spool, passes over a pulley (considered of zero radius) which is at h above the plane and at horizontal distance l to the left of the point of contact of spool and plane. A weight w is fixed to the thread after it has passed over the pulley.

Determine the motion of the spool, assuming that there is no loss by friction or otherwise of the total energy of the system.

SOLUTIONS.

2949 [1922, 29], [Corrected; 1922, 420]. Proposed by J. B. REYNOLDS, Lehigh University.

Find the lateral area of the cone with vertex at $(0, 0, h)$ and whose base is the epicycloid

$$2x = a(3 \cos \theta - \cos 3\theta), \qquad 2y = a(3 \sin \theta - \sin 3\theta).$$

SOLUTION BY OTTO DUNKEL, Washington University.

The given epicycloid is the locus of the point P fixed on the circumference of a circle of radius $a/2$, center C, which rolls on a fixed circle with center O and radius a. At the beginning of the rolling the point P is at A on the circumference of the circle O. Consider the position when the point of contact of the two circles is at M, and produce the diameter NCM to O, and let $\angle NOA = \theta$. Then NP is the tangent to the epicycloid. Draw the diameter PCP' and let $P'O$ cut PM produced in E. Then EP is the radius of curvature, and the figure gives $EP = (3/2)MP = (3/2)a \sin \theta$. Thus $ds = 3a \sin \theta d\theta$. The element of area, dA, of the lateral surface of the cone, with the epicycloid as base and vertex, V, vertically above O at the height h, is the triangle with base ds and vertex at V. Draw the perpendicular OK from O to NP produced; then VK, the altitude of the little triangle, is $\sqrt{h^2 + \overline{OK}^2} = \sqrt{h^2 + 4a^2 \sin^2 \theta}$. Therefore

$$\begin{aligned} A &= 12a^2 \int_0^{\pi/2} \sqrt{n^2 - \cos^2 \theta} \sin \theta d\theta, \qquad n^2 = (h^2 + 4a^2)/4a^2, \\ &= 6a^2 \left[\sqrt{n^2 - 1} + n^2 \sin^{-1}\left(\frac{1}{n}\right)\right], \\ &= 3ah + \frac{3(h^2 + 4a^2)}{2} \tan^{-1}\left(\frac{2a}{h}\right). \end{aligned}$$

3043 [1923, 403 and 1924, 358]. Proposed by O. D. KELLOGG, Harvard University.

(*Corrected statement.*) Let T denote an open continuum of the xy-plane, say the interior of a smooth simple closed curve. Then if U is continuous in T, and is such that

$$\iint_T U\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) dxdy = 0 \tag{1}$$

for all functions V with continuous second derivatives in T, and such that (a) the normal derivatives of V vanish on the boundary of T, or, (b) the boundary not being a smooth curve, the first derivatives of V at a point P approach 0 as P approaches any boundary point of T, U is harmonic in T, *i.e.*, U satisfies Laplace's equation $(\partial^2 U/\partial x^2) + (\partial^2 U/\partial y^2) = 0$.

NOTE BY THE EDITORS: A solution, by DR. H. E. BRAY, of the problem as now correctly stated has already appeared in the MONTHLY (*1924*, 358). It is believed, however, that the interest and importance attached to this problem warrants the publication of a second solution.

Solution by the Proposer.

The original statement of this problem contained an oversight. The condition there given was that V should vanish on the boundary of T.

A well-known theorem in potential theory states that the value of a function which is harmonic in a region T is, at any point P, the arithmetic mean of its values on any circle in T with center at P. Koebe has proved a converse of this theorem (*Sitzungsberichte d. Berliner Math. Gesellschaft*, vol. 5 (1906), p. 39), namely, if U is continuous in T, and assumes at each point, P, of T the arithmetic mean of its values on any circle in T with center at P, then U is harmonic.

If the hypothesis of Koebe's theorem were not fulfilled by the function U of the present problem, there would be in T a point, P, and a circle, C, entirely in T, with P as center, such that

$$U(P) \neq \frac{1}{2\pi}\int_0^{2\pi} U(B)d\vartheta, \tag{2}$$

where B is the point on C with vectorial angle θ. We shall show that the relation (2) is incorrect, so that U must satisfy the conditions of Koebe's theorem, and hence be harmonic.

With the use of polar coördinates, with P as pole, the condition (1) on U may be written

$$\iint\limits_T U\left(\frac{\partial}{\partial r} r\frac{\partial V}{\partial r}\right) drd\vartheta = 0 \tag{3}$$

for all allowable V. Among the allowable V is one which we proceed to form. Let c denote the radius of C, and let α be a positive parameter, restricted to be less than $c/3$, and less than the minimum distance between C and the boundary of T. We then equate $\frac{\partial}{\partial r} r \frac{\partial V}{\partial r}$ to a continuous function $f(r)$, which is positive for $0 \leqq r \leqq \alpha$, negative for $c - \alpha \leqq r \leqq c + \alpha$, and zero elsewhere, and such that $\int_0^{c+\alpha} f(r)dr = 0$. The differential equation thus obtained has a solution, V, which is continuous, together with its partial derivatives of the first two orders, and whose first derivatives vanish outside the circle $r = c + \alpha$. This solution is an allowable V.

We now carry out the integration with respect to ϑ in (3), in which V is to be the function just determined, writing $\overline{U}(r) = \frac{1}{2\pi}\int_0^{2\pi} U(r, \vartheta)d\vartheta$:

$$\int_0^{\alpha} \overline{U}(r)f(r)dr + \int_{c-\alpha}^{c+\alpha} \overline{U}(r)f(r)dr = 0.$$

If the law of the mean be used, this equation leads to

$$\overline{U}(\theta'\alpha) = \overline{U}(c + \theta''\alpha), \qquad (0 < \theta' < 1, \quad -1 < \theta'' < 1).$$

Since U is continuous, and since this last equation holds for all positive α near 0, we conclude that

$$\overline{U}(0) = \overline{U}(c)$$

so that the relation (2) is disproved, as desired.

The problem was also solved by Dr. Wiener. He too used Koebe's theorem, but was able to give to the rest of the treatment a somewhat more elegant, if less elementary form, by use of a theorem to the effect that if a function is orthogonal on a given interval to all continuous functions themselves orthogonal to 1, that function is a constant.

3053 [1924, 49]. Proposed by J. Lense, Vienna University.

Let

$$a_1 = a, \qquad a_2 = a^{a_1}, \qquad a_3 = a^{a_2}, \quad \cdots \quad a_{n+1} = a^{a_n}, \quad \cdots.$$

Discuss $\lim\limits_{n=\infty} a_n$ as a function of a. (The limit exists also for some values of $a > 1$.)

Solution by the Proposer.

First, we suppose $0 < a < 1$. Then we have $1 > a_2 > a_1$, $a_1 < a_3 < a_2$, $a_2 > a_4 > a_3$, $a_3 < a_5 < a_4$, $\cdots$, and generally,

$$0 < a_1 < a_3 < a_5 < \cdots < a_{2n-1} < \cdots;$$
$$1 > a_2 > a_4 > a_6 > \cdots > a_{2n} > \cdots;$$

and $a_{2i-1} < a_{2k}$ for i, $k = 1, 2, \cdots$ to inf. Therefore, there are two limits: $\lim_{n=\infty} a_{2n-1} = x$ and $\lim_{n=\infty} a_{2n} = y$ and $x < y$. The limits are solutions of the following equations:

$$a^x = y, \qquad a^y = x.$$

Both curves intersect at one point for all values of a in the interval $\frac{1}{e^e} \leqq a \leqq 1$; in this point $x = y$.

They intersect at three points for all values of a in the interval $0 < a < \frac{1}{e^e}$; these points have the coördinates (x_1, y_1), (x_2, y_2), (x_3, y_3) and we have $x_1 < y_1$, $x_2 = y_2$, $x_3 > y_3$, $x_3 = y_1$, $x_1 = y_3$.

For all values of a in the interval $1 < a < \sqrt[e]{e}$ the curves intersect at two points (x_1, y_1), (x_2, y_2), for which we have $x_1 = y_1$, $x_2 = y_2$, $x_1 < x_2$. The curves do not intersect when $a > \sqrt[e]{e}$. Therefore there is one limit $x = y$ for $\frac{1}{e^e} \leqq a \leqq 1$. Because $\lim_{a=0} a_{2n-1} = 0$ and $\lim_{a=0} a_{2n} = 1$, there are two different limits $x < y$ for $0 \leqq a \leqq \frac{1}{e^e}$ and we have $\lim_{a=0} x = 0$ and $\lim_{a=0} y = 1$.

Now let $1 < a < \sqrt[e]{e}$. Then we have

$$1 < a_2 < \sqrt[e]{e}^{\sqrt[e]{e}}$$

$$1 < a_3 < \sqrt[e]{e}^{\sqrt[e]{e}^{\sqrt[e]{e}}}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$1 < a < a_2 < a_3 < \cdots < a_n < \cdots.$$

Therefore there exists $\lim_{n=\infty} a_n$, if $\lim_{n=\infty} a_n$ exists for $a = \sqrt[e]{e}$. But we have

$$\sqrt[e]{e} < e, \ \sqrt[e]{e}^{\sqrt[e]{e}} < e, \ \sqrt[e]{e}^{\sqrt[e]{e}^{\sqrt[e]{e}}} < e, \ \cdots \qquad \text{and} \qquad \sqrt[e]{e} < \sqrt[e]{e}^{\sqrt[e]{e}} < \sqrt[e]{e}^{\sqrt[e]{e}^{\sqrt[e]{e}}} < \cdots,$$

therefore, $\lim_{n=\infty} a_n \leqq e$ for $a = \sqrt[e]{e}$. The limits $\lim_{n=\infty} a_n$ and $\lim_{n=\infty} a_n \Big|_{a=\sqrt[e]{e}}$ existing for $1 < a < \sqrt[e]{e}$, are given by the solution $x_1 = y_1$ of the two equations mentioned above in the interval $1 < a < \sqrt[e]{e}$. For $a = \sqrt[e]{e}$ we have $\lim_{n=\infty} a_n = e$. Because the curves do not intersect for all values of $a > \sqrt[e]{e}$, we have no limit in this interval.

Conclusion. $\lim_{n=\infty} a_n$ exists only in the interval $\frac{1}{e^e} \leqq a \leqq \sqrt[e]{e}$. If we denote a by x and $\lim_{n=\infty} a_n$ by y, we shall get it as solution of the equation $y = x^y$. Therefore, we have to discuss this curve in the interval $\frac{1}{e^e} \leqq x \leqq \sqrt[e]{e}$. It is easily shown that y is a continuous function of x always increasing from $x = e^{1/e}$ till $x = \sqrt[e]{e}$. All its derivatives exist and are continuous functions of x, the point $x = \sqrt[e]{e}$ being excepted. The curve has a point of inflexion for $x = 0.396$, $y = 0.582$. In the first part of the interval, it is concave; in the second convex quâ the x-axis. In the following table are the coördinates of some points of the curve:

$x = \frac{1}{e^e} = 0.0763$	0.396	1	$\sqrt[e]{e} = 1.44$
$y = \frac{1}{e} = 0.368$	0.582	1	$e = 2.72.$

3054 [1924, 49]. Proposed by S. A. Corey, Des Moines, Iowa.

Construct three triangles with vector sides a, b, c; d, e, f; g, h, k, such that $d = a$, $g = 2a$, $e = 2b$, $h = b$. Prove that $a^2 + b^2 + 4c^2 = f^2 + k^2$.

SOLUTION BY J. A. BULLARD, U. S. N. A.

Construct the triangles as shown in the figure and also draw the vector $2c$ as shown. Then

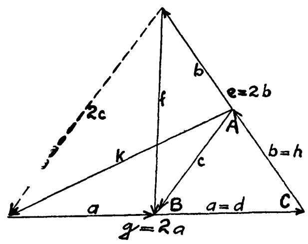


FIG. 1.

$$b^2 + (2c)^2 - 4bc \cos A = k^2,$$
$$a^2 + (2c)^2 - 4ac \cos B = f^2,$$

adding

$$a^2 + b^2 + 8c^2 - 4c(b \cos A + a \cos B) = f^2 + k^2;$$

but $b \cos A + a \cos B = c$, and thus $a^2 + b^2 + 4c^2 = f^2 + k^2$.

Also solved by R. P. AGNEW, H. C. BRADLEY, L. BAGLEY, A. G. CLARK, PHILIP FITCH, J. Q. MCNATT, A. PELLETIER, H. A. SIMMONS, and A. S. WIENER.

3059 [1924, 101]. Proposed by DANIEL KRETH, Wellman, Iowa.

Given the perimeter and the radii of the inscribed and circumscribed circles, to construct the triangle and calculate the lengths of its sides.

SOLUTION BY LOUIS WEISNER, University of Rochester.

The proposed construction is generally impossible as the following analysis shows:

Denoting the sides of the triangle by a, b, c, the perimeter by $2s$, and the radii of the inscribed and circumscribed circles by r and R, respectively, we have

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \qquad 4R = \frac{abc}{\sqrt{s(s-a)(s-b)(s-c)}};$$

whence $abc = 4rsR$. From the first equation,

$$r^2 s = s^3 - (a + b + c)s^2 + (ab + ac + bc)s - abc.$$

Substituting, $a + b + c = 2s$, $abc = 4rsR$, we obtain $ab + ac + bc = r^2 + s^2 + 4rR$. It follows that a, b, c are the roots of the cubic

$$x^3 - 2sx^2 + (r^2 + s^2 + 4rR)x - 4rsR = 0.$$

It is easy to see that this cubic is a general cubic equation, whose roots cannot be expressed in terms of square roots alone. For example, take $s = 6$, $r = \frac{1}{2}\sqrt{2}$, $R = \frac{17}{8}\sqrt{2}$. Then a, b, c are the roots of $x^3 - 12x^2 + 45x - 51 = 0$, which has three real, irrational, positive roots. Hence a, b, c cannot be expressed in terms of square roots alone and the construction of the triangle is impossible with ruler and compasses.

REMARKS ON THE ABOVE SOLUTION BY OTTO DUNKEL.

The equation for x above apparently does not alone determine the triangle in all cases. If we take $R = 2r$, then geometry shows that for a real triangle the sides must be equal. Now the equation will not have three equal roots unless we impose the further condition $r = s/3\sqrt{3}$. It would be interesting to study the nature of the roots with the single condition $R = 2r$.

Another way of determining the equation for the sides is to eliminate A from the equations $\sin A = a/2R$, $\tan(A/2) = r/(s - a)$. This would give at once the equation above, or with a slight change before reduction

$$y^3 - sy^2 + (r^2 + 4rR)y - sr^2 = 0,$$

where the roots are $s - a$, $s - b$, $s - c$. Expressing the coefficients of these two equations in terms of the roots we would have simple proofs of the two formulæ used above, and also a proof of the formula for the tangent of the half angle in any triangle.

Also solved by WILLIAM HOOVER, W. B. PIERCE, and C. K. ROBBINS.

SOLUTION BY J. A. BULLARD, U. S. N. A.

Construct the triangles as shown in the figure and also draw the vector $2c$ as shown. Then

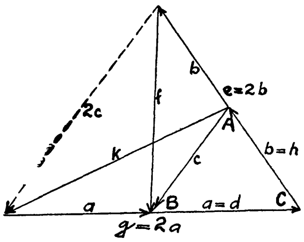


FIG. 1.

$$b^2 + (2c)^2 - 4bc \cos A = k^2,$$
$$a^2 + (2c)^2 - 4ac \cos B = f^2,$$

adding

$$a^2 + b^2 + 8c^2 - 4c(b \cos A + a \cos B) = f^2 + k^2;$$

but $b \cos A + a \cos B = c$, and thus $a^2 + b^2 + 4c^2 = f^2 + k^2$.

Also solved by R. P. AGNEW, H. C. BRADLEY, L. BAGLEY, A. G. CLARK, PHILIP FITCH, J. Q. MCNATT, A. PELLETIER, H. A. SIMMONS, and A. S. WIENER.

3059 [1924, 101]. Proposed by DANIEL KRETH, Wellman, Iowa.

Given the perimeter and the radii of the inscribed and circumscribed circles, to construct the triangle and calculate the lengths of its sides.

SOLUTION BY LOUIS WEISNER, University of Rochester.

The proposed construction is generally impossible as the following analysis shows:

Denoting the sides of the triangle by a, b, c, the perimeter by $2s$, and the radii of the inscribed and circumscribed circles by r and R, respectively, we have

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \qquad 4R = \frac{abc}{\sqrt{s(s-a)(s-b)(s-c)}};$$

whence $abc = 4rsR$. From the first equation,

$$r^2 s = s^3 - (a + b + c)s^2 + (ab + ac + bc)s - abc.$$

Substituting, $a + b + c = 2s$, $abc = 4rsR$, we obtain $ab + ac + bc = r^2 + s^2 + 4rR$. It follows that a, b, c are the roots of the cubic

$$x^3 - 2sx^2 + (r^2 + s^2 + 4rR)x - 4rsR = 0.$$

It is easy to see that this cubic is a general cubic equation, whose roots cannot be expressed in terms of square roots alone. For example, take $s = 6$, $r = \frac{1}{2}\sqrt{2}$, $R = \frac{17}{8}\sqrt{2}$. Then a, b, c are the roots of $x^3 - 12x^2 + 45x - 51 = 0$, which has three real, irrational, positive roots. Hence a, b, c cannot be expressed in terms of square roots alone and the construction of the triangle is impossible with ruler and compasses.

REMARKS ON THE ABOVE SOLUTION BY OTTO DUNKEL.

The equation for x above apparently does not alone determine the triangle in all cases. If we take $R = 2r$, then geometry shows that for a real triangle the sides must be equal. Now the equation will not have three equal roots unless we impose the further condition $r = s/3\sqrt{3}$. It would be interesting to study the nature of the roots with the single condition $R = 2r$.

Another way of determining the equation for the sides is to eliminate A from the equations $\sin A = a/2R$, $\tan (A/2) = r/(s - a)$. This would give at once the equation above, or with a slight change before reduction

$$y^3 - sy^2 + (r^2 + 4rR)y - sr^2 = 0,$$

where the roots are $s - a$, $s - b$, $s - c$. Expressing the coefficients of these two equations in terms of the roots we would have simple proofs of the two formulæ used above, and also a proof of the formula for the tangent of the half angle in any triangle.

Also solved by WILLIAM HOOVER, W. B. PIERCE, and C. K. ROBBINS.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

At the College of William and Mary, Professor J. E. ROWE has been made director of extension, and Miss ELIZABETH MERCER has been appointed instructor of mathematics.

Professor L. L. SILVERMAN, of Dartmouth College, who has just returned from a year's leave of absence in Germany and Russia, has been promoted to a full professorship.

Mrs. ETHELWYN R. BECKWITH has been appointed professor of mathematics at the College for Women, Western Reserve University, for 1924–1925, to serve during the sabbatical leave of Professor ANNA M. PALMIE.

Assistant Professor LACHLAN GILCHRIST, of the University of Toronto, has been promoted to an associate professorship of physics.

Assistant Professor C. C. MAC DUFFEE of Princeton University has been appointed assistant professor of mathematics at Ohio State University.

Dr. J. S. TAYLOR, of the Massachusetts Institute of Technology, has been appointed assistant professor of mathematics at the University of Pittsburgh.

Mr. J. C. TINNER has been appointed professor of mathematics at Wilberforce University.

Mr. H. K. FULMER, of the Georgia School of Technology, has been promoted to an assistant professorship of mathematics.

Assistant Professor A. R. CRATHORNE, of the University of Illinois, has been promoted to an associate professorship of mathematics.

Associate Professor F. R. MOULTON, of the University of Chicago, has been promoted to full professorship.

At St. Olaf College, Northfield, Minnesota, Professor MARTIN NORDGAARD, formerly head of the department of mathematics at Antioch College, has been appointed professor and head of the department of mathematics, and Mr. ARTHUR SOLUM of the University of Minnesota, has been appointed assistant professor.

Mr. E. L. BROWN, for the past 24 years principal of the North High School, Denver, and a charter member of the Association, has been promoted to the position of assistant superintendent of schools, in charge of high schools.

Associate Professor A. S. MERRILL, of the University of Montana, has been promoted to a full professorship of mathematics.

Dr. H. E BRAY, of Rice Institute, has been promoted to an assistant professorship of mathematics.

Dr. J. H. JEANS, of Cambridge University, has been appointed research associate at Mount Wilson Observatory.

The following appointments to instructorships are announced:

Amherst College, Mr. C. S. PORTER.

Bryn Mawr College, Miss ANNA M. LEHR.

Colorado School of Mines, Mr. E. P. MARTINSON.

Cornell University, Mr. H. A. HOOVER.

Georgia School of Technology, Messrs. G. S. BRUTON, G. W. NICHOLSON, L. K. PATTON, H. H. PIXSLEY, D. P. RICHARDSON.

University of Michigan, E. H. WAGNER, L. M. BLUMENTHAL, H. F. SCHIEFER.

University of Oklahoma, Mr. A. E. ANDERSON.

Professor C. B. AUSTIN, of Ohio Wesleyan University, died September 9, 1924, at the age of 73. Professor Austin had for some years been vice-president of the University as well as professor of mathematics.

Associate Professor H. S. EVERETT, of Bucknell University, Lewisburg, Penn., has been promoted to a full professorship.

On November 22, 1924, forty-one mathematics teachers of Southern California gathered in the dining room of the Los Angeles City Club, upon the invitation of Professor E. R. Hedrick. After partaking of a bountiful lunch, the meeting was organized by electing Professor Hedrick Temporary Chairman, and Professor Russell, of Pomona College, Temporary Secretary. Following an informal discussion, it was voted unanimously that it would be desirable to form a Southern California section of The American Mathematical Association and the proper committees were appointed to carry out this plan.

Of the forty-one present, seventeen are members of the Association and most of the others expressed a desire to become members.

CORRECTION: The price of *Charts and Graphs* by K. G. Karsten should be $6.00 and not $4.00 as announced (*1924*, 449).

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.
BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.
BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

The following are dates of Section meetings of the Association in 1924 (unless otherwise specified):

ILLINOIS, Elgin, May 2–3
IOWA, Iowa State College, Ames, May 2–3
KANSAS, Topeka, February 2
KENTUCKY, Center College, April
MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA Annapolis, December 8, 1923
MICHIGAN, Ann Arbor, April 3
MINNESOTA, Hamline University, St. Paul, May 24
MISSOURI, Kansas City, November or December
OHIO, Ohio State University, Columbus, April 4–5
ROCKY MOUNTAIN, Laramie, April, 1925
SOUTHEASTERN, University of Georgia, Athens March 7–8
TEXAS, San Antonio, November 29–30

DESIRED COPIES OF THE MONTHLY

THROUGH an oversight the Association has not reserved a sufficient number of copies of the MONTHLY for October, 1920, and for August, September, and October, 1921, also May, September, and October, 1924. Cash, or credit toward future dues, will be given by the Secretary for such copies at the rate of forty-five cents per copy, up to a limited number of copies.

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary,

W. D. Cairns, Oberlin, Ohio